## Lecture 2-9/13/04

## Evolution of Market Concentration

- Examine concentration in a structure with long-run constant returns to scale
- Static Cournot Duopoly
- Dynamic Duopoly. How calculate
- Dominant Firm Model (use to talk about mergers)


## Technology

- $K_{i}$ capital of firm $i$
- $Q_{i}$ output of firm $i$
- $q=\frac{K}{Q}$ output per unit of capital
- $c(q)$ cost per unit of capital when output intensity is $q . c^{\prime}>0$, $c^{\prime \prime}>0$.
- $C(q)=K c(q)$ is total cost


## Example:

- Cobb-Douglas $Q=L^{\alpha} K^{1-\alpha}$.
- Suppose $L$ is $\$ 1$ per unit.

$$
\begin{aligned}
C(Q) & =\left[\frac{Q}{K^{1-\alpha}}\right]^{\frac{1}{\alpha}} \\
c(q) & =q^{\frac{1}{\alpha}}
\end{aligned}
$$

## Static Cournot

- $K_{i}$ fixed
- $P(Q)$ industry demand where $P^{\prime}(Q)<0$.
- Cournot problem. Firm 1 takes $q_{2}$ as fixed. Maximize profits per unit of capital

$$
\max _{q_{1}} P\left(K_{1} q_{1}+K_{2} q_{2}\right) q_{1}-c\left(q_{1}\right)
$$

- FONC

$$
P\left(K_{1} q_{1}+K_{2} q_{2}\right)+P^{\prime}\left(K_{1} q_{1}+K_{2} q_{2}\right) K_{1} q_{1}-c^{\prime}\left(q_{1}\right)=0
$$

- SOC

$$
2 P^{\prime}\left(K_{1} q_{1}+K_{2} q_{2}\right) K_{1}+P^{\prime \prime}\left(K_{1} q_{1}+K_{2} q_{2}\right) K_{1}^{2} q_{1}-c^{\prime \prime}\left(q_{1}\right)<0
$$

- Reaction function $q_{1}=R\left(q_{2}\right)$ solves above.
- If $K_{1}=K_{1}$, then weak conditions get existence of symmetric equilibrium (if reaction function continuous. ( $P^{\prime \prime} \leq 0$ is sufficient)
- Let $q^{c}$ solve $q^{c}=R\left(q^{c}\right)$.


## Infintely Repeated Game (supergame)

- $K_{1}=K_{2}=1$ fixed over time.
- $\beta$ discount factor
- Can collusion be supported?

$$
\begin{aligned}
& \max _{q_{1}, q_{2}} P\left(q_{1}+q_{2}\right)\left(q_{1}+q_{2}\right)-c\left(q_{1}\right)-c\left(q_{2}\right) \\
F O N C: & P+P^{\prime}-c^{\prime}\left(q_{i}\right)=0
\end{aligned}
$$

- Let $q^{m}$ solve the above

$$
\begin{aligned}
\pi^{c} & =P\left(q^{c}\right) q^{c}-c\left(q^{c}\right) \\
\pi^{m} & =P\left(q^{m}\right) q^{m}-c\left(q^{m}\right)
\end{aligned}
$$

- Can show $\pi^{c}<\pi^{m}$. So have standard prisoner's dilemma.
- Can collusive solution be supported?


## Trigger Strategies

- If deviate play Cournot forever, otherwise $q^{m}$
- Return to cooperation

$$
\frac{1}{1-\beta} \pi^{m}
$$

- Return to deviating

$$
\begin{aligned}
& \max _{q_{1}} P\left(q_{1}+q^{m}\right) q_{1}-c\left(q_{1}\right)+\frac{\beta}{1-\beta} \pi^{c} \\
= & \pi^{d e v}+\frac{\beta}{1-\beta} \pi^{c}
\end{aligned}
$$

- Won't deviate iff

$$
\pi^{d e v}-\pi^{m} \leq \frac{\beta}{1-\beta}\left(\pi^{m}-\pi^{c}\right)
$$

so get cooperation for sufficiently high $\beta$.

- More complicated solutions if there is uncertainty, imperfect monitoring, etc. (Abreu, Pearce, and Staccetti).


## Markov Perfect Equilibria (Maskin and Tirole)

- Equilibrium policy functions depend only on payoff relevant states. Let $s$ be a vector of such states.
- $\pi_{i}\left(a_{1}, a_{2}, s\right)$ current period payoff to player $i$ given actions $a_{1}$ and $a_{2}$ in the current period and state $s . \pi_{1}$
- $s^{\prime}=f\left(a_{1}, a_{2}, s\right)$ be transition function
- Let $\tilde{a}_{i}(s)$ be policy function and suppose $\tilde{v}_{i}(s)$ satisfies

$$
\tilde{v}_{1}(s)=\max _{a_{1}} \pi\left(a_{1}, \tilde{a}_{2}(s), s\right)+\beta \tilde{v}_{1}\left(f\left(a_{1}, \tilde{a}_{2}(s), s\right)\right)
$$

and let $\tilde{a}_{1}$ be the solution Suppose $\tilde{v}_{2}(s)$ and $\tilde{a}_{2}(s)$ satisfy the analogous relationships. Then $\left(\tilde{a}_{1}, \tilde{a}_{2}, \tilde{v}_{1}, \tilde{v}_{2}\right)$ is a Markovperfect equilibrium.

## Cournot Duopoly

- Suppose

$$
K_{1}=K_{2}=1
$$

fixed over time.
-What is the set of Markov-perfect equilibria?
-What is the set of payoff-relevant states?

- Suppose

$$
K_{i, t}=Q_{i, t-1}(1-\delta)
$$

-Intepretation: use capital to make new capital.
—Adjustment costs (Lucas 1967, Prescott and Visscher (1980))

- Can separate output and investment. Add an output stage after the investment state. Assume $Q_{i}$ is capital and $Y_{i}$ is output. Suppose $Y_{i} \leq Q_{i}$ and zero marginal cost up to capacity. Suppose demand is elastic. Then firms always produce up to capacity.
- Define a Markov-perfect equilibrium
- What is a steady state?


## Dynamics with $\beta=0$

- Given ( $K_{1}, K_{2}$ ), solve the (asymmetric) Cournot duopoly problem
- Claim: if $K_{1}>K_{2}$ then $q_{1}<q_{2}$, but $q_{1} K_{1}>q_{2} K_{2}$.
-FONC for two firms

$$
\begin{aligned}
& P+P^{\prime} q_{1} K_{1}-c^{\prime}\left(q_{1}\right)=0 \\
& P+P^{\prime} q_{2} K_{2}-c^{\prime}\left(q_{2}\right)=0
\end{aligned}
$$

Suppose instead that $q_{1} \geq q_{2}$.
$\Rightarrow c^{\prime}\left(q_{1}\right) \geq c^{\prime}\left(q_{2}\right)$
$\Rightarrow P^{\prime} q_{1} K_{1} \geq P^{\prime} q_{2} K_{2}$
$\Rightarrow K_{1} \leq K_{2}$, a contradiction.

- Claim market shares converge to equality.

$$
\begin{aligned}
\frac{K_{1}^{\prime}}{K_{2}^{\prime}} & =\frac{q_{1} K_{1}(1-\delta)}{q_{2} K_{2}(1-\delta)} \\
& =\frac{q_{1} K_{1}}{q_{2} K_{2}} \\
& <\frac{K_{1}}{K_{2}}
\end{aligned}
$$

But

$$
1<\frac{K_{1}^{\prime}}{K_{2}^{\prime}}
$$

- So converge to 50-50 monotonically.
—Kydland, Dominant firm literature
- Intuition?
- Suppose $\beta>0$
—analytic results difficult
-will go to computer and work this out
-Suppose commit to sequence of outputs. Does this matter?
Look at $T=2$ case.


## Benchmark Case of Perfect Competition Steady State

- Suppose agents take as given a constant price p. .
- Let $v$ be the discounted value of owning one unit of capital at the beginning of a period

$$
v=\max _{q} p q-c(q)+\beta \sigma q v
$$

where

$$
\sigma=1-\delta
$$

- FONC

$$
\begin{equation*}
p-c^{\prime}(q)+\beta \sigma v=0 \tag{1}
\end{equation*}
$$

- In a stationary equilibrium,

$$
\begin{aligned}
\sigma q & =1 \\
q^{*} & =\frac{1}{\sigma}
\end{aligned}
$$

- $v^{*}$ solves

$$
\begin{aligned}
v^{*} & =p q^{*}-c\left(q^{*}\right)+\beta \sigma q^{*} v^{*} \\
& =p q^{*}-c\left(q^{*}\right)+\beta v^{*}
\end{aligned}
$$

so

$$
v^{*}=\frac{p q^{*}-c\left(q^{*}\right)}{1-\beta}
$$

- From the FONC

$$
p=c^{\prime}\left(q^{*}\right)-\beta \sigma v^{*}
$$

- Plugging in the formula for $v^{*}$ yields

$$
p=c^{\prime}\left(q^{*}\right)-\beta \sigma \frac{p q^{*}-c\left(q^{*}\right)}{1-\beta}
$$

Solving for $p$ yields the stationary competitive price

$$
p_{C}^{*}=(1-\beta) c^{\prime}\left(q^{*}\right)+\beta \sigma c\left(q^{*}\right)
$$

- $Q_{C}^{*}$ be the stationary competitive output
- $x_{C}^{*}=\sigma Q_{C}^{*}$ be the stationary competitive capital level.


## Pure Monopoly.

- The state variable is $K$ at the beginning of period capital. Let $w(K)$ be discounted maximized monopoly profit. This solves

$$
w(K)=\max _{q} P(K q) K q-K c(q)+\beta w(\sigma K q)
$$

- The FONC is

$$
P K+P^{\prime} K^{2} q-K c^{\prime}+\beta \sigma K \frac{d w}{d K}=0
$$

- Dividing by $x$,

$$
P+P^{\prime} K q-c^{\prime}+\beta \sigma \frac{d w}{d K}=0
$$

- Use the envelope theorem to verify that

$$
\frac{d w}{d K}=q c^{\prime}(q)-c(q)
$$

(Think of $Q$ as the choice variable....).

- Plugging this into the first-order condition and evaluating at the steady state output level $q^{*}=\frac{1}{\sigma}$ yields

$$
p+P^{\prime} q K-c^{\prime}+\beta \sigma\left[q c^{\prime}-c\right]=0
$$

or

$$
\begin{aligned}
p+P^{\prime} q^{*} K & =(1-\beta) c^{\prime}+\beta \sigma c \\
& =P_{C}^{*}
\end{aligned}
$$

- Let $K$ solving the above be denoted $K_{M}^{*}$. .
- Now calculate the equilbrium off the steady state


## A Technical Aside

Numerical Solutions of Dynamic Programming Problems

Monopoly Problem

- Statement of problem. $w(K)$ value function and $q(K)$ is policy function. Contraction mapping: Let $w_{0}$ be value function beginning next period. Then

$$
w_{1}(K)=\max _{q} P(K q) K q-K c(q)+\beta w_{0}(\sigma K q)
$$

A solution is where $w_{1}(K)=w_{0}(K)$ for all $K$.

- Iterate
- How do numerically? Need an approximation for $w_{0}$.
- Discretize? Works well with single agent decision theory. For duopoly problem though continuity is useful.
- Polynomial approximation.


## Example with Linear Approximation

1. Start with approximation

$$
\hat{w}_{0}(K)=\alpha_{0}+\beta_{0} K
$$

2. Take a set of $m$ evaluation points $\tilde{K}=\left\{\tilde{K}_{1}, \tilde{K}_{2}, \ldots, \tilde{K}_{m}\right\}$
3. Solve problem at each of this points with $\hat{w}_{0}(K)$ instead of $w_{0}(K)$.

$$
\tilde{w}_{1, i}=\max _{q} P\left(\tilde{K}_{i} q\right) \tilde{K}_{i} q-\tilde{K}_{i} c(q)+\beta \hat{w}_{0}\left(\sigma \tilde{K}_{i} q\right)
$$

4. Yields a vector $\tilde{W}_{1}=\left(\tilde{w}_{1,1}, \tilde{w}_{1,2}, \ldots . \tilde{w}_{1, m}\right)$
5. Use OLS to determine a new approximation

$$
\begin{aligned}
\binom{\alpha_{1}}{\beta_{1}} & =\left(X^{\prime} X\right)^{-1} X^{\prime} \tilde{W}_{1} \\
X & =1^{\sim} \hat{K}
\end{aligned}
$$

6. Iterate until obtain convergence in $\left(\alpha_{t}, \beta_{t}\right)$

## General Polynomial Approximation

- Chebyshev polynomials (in class of orthogonal polynomials)
- Defined on range $x \in[-1,1]$

$$
T_{n}(x)=\cos \left(n \cos ^{-1} x\right)
$$



Figure 1:

## Recipe in Judd

- Step 1: Evaluation points

$$
z_{k}=-\cos \left(\frac{2 k-1}{2 m} \pi\right), k=1, \ldots, m
$$

- Step 2: Adjust the notes to the $[\mathrm{a}, \mathrm{b}]$ interval (here $a=$ $\left..5 K_{M}^{*}, b=1.5 K_{M}^{*}\right)$

$$
x_{k}=\left(z_{k}+1\right)\left(\frac{b-a}{2}\right)+a, k=1, \ldots, m
$$

- Step 3: Evaluate $w(x)$ at the approximation nodes

$$
\tilde{w}_{k}=w\left(x_{k}\right), k=1, \ldots, m
$$

- Step 4: Compute the Chebyshev coefficients (remember $T_{i}$ orthogonal)

$$
a_{i}=\frac{\sum_{k=1}^{m} \tilde{w}_{k} T_{i}\left(z_{k}\right)}{\sum_{k=1}^{m} T_{i}\left(z_{k}\right)^{2}}
$$

- To arrive at the approximation

$$
\hat{w}(x)=\sum_{i=0}^{n} a_{i} T_{i}\left(2 \frac{x-a}{b-a}-1\right)
$$

## Hints for Duopoly Problem

- $\left(a_{0}, \ldots a_{n}\right)$ coefficient vector for the value function $v_{1}\left(K_{1}, K_{2}\right)$ approximation
- $\left(b_{0}, \ldots ., b_{n}\right)$ coefficient vector for the policy function $q_{1}\left(K_{1}, K_{2}\right)$ approximation.
- Use Judd's techniques for approximation in $R^{2}$ (page 238)
- You need to iterate on $q_{1}$ as well as $v_{1}$ since firm 1 takes firm 2's action as given in the problem (and $q_{2}(x, y)=q_{1}(y, x)$ ).

