Lecture 2—9/13/04

Evolution of Market Concentration

- Examine concentration in a structure with long-run constant returns to scale
- Static Cournot Duopoly
- Dynamic Duopoly. How calculate
- Dominant Firm Model (use to talk about mergers)

Technology

- K_i capital of firm i
- Q_i output of firm i
- $q = \frac{K}{Q}$ output per unit of capital
- c(q) cost per unit of capital when output intensity is q. c' > 0, c'' > 0.
- C(q) = Kc(q) is total cost

Example:

- Cobb-Douglas $Q = L^{\alpha} K^{1-\alpha}$.
- Suppose L is \$1 per unit.

$$C(Q) = \left[\frac{Q}{K^{1-\alpha}}\right]^{\frac{1}{\alpha}}$$
$$c(q) = q^{\frac{1}{\alpha}}$$

Static Cournot

- K_i fixed
- P(Q) industry demand where P'(Q) < 0.
- Cournot problem. Firm 1 takes q_2 as fixed. Maximize profits per unit of capital

$$\max_{q_1} P(K_1q_1 + K_2q_2)q_1 - c(q_1)$$

• FONC

 $P(K_1q_1 + K_2q_2) + P'(K_1q_1 + K_2q_2)K_1q_1 - c'(q_1) = 0$

• SOC

$$2P'(K_1q_1 + K_2q_2)K_1 + P''(K_1q_1 + K_2q_2)K_1^2q_1 - c''(q_1) < 0$$

- Reaction function $q_1 = R(q_2)$ solves above.
- If $K_1 = K_1$, then weak conditions get existence of symmetric equilibrium (if reaction function continuous. $(P'' \leq 0$ is sufficient)
- Let q^c solve $q^c = R(q^c)$.

Infintely Repeated Game (supergame)

- $K_1 = K_2 = 1$ fixed over time.
- β discount factor
- Can collusion be supported?

$$\max_{q_1,q_2} P(q_1 + q_2) (q_1 + q_2) - c(q_1) - c(q_2)$$

FONC : $P + P' - c'(q_i) = 0$

• Let q^m solve the above

$$\pi^c = P(q^c)q^c - c(q^c)$$

$$\pi^m = P(q^m)q^m - c(q^m)$$

- Can show $\pi^c < \pi^m$. So have standard prisoner's dilemma.
- Can collusive solution be supported?

Trigger Strategies

- $\bullet\,$ If deviate play Cournot forever, otherwise q^m
- Return to cooperation

$$rac{1}{1-eta}\pi^m$$

• Return to deviating

$$\max_{q_1} P(q_1 + q^m)q_1 - c(q_1) + \frac{\beta}{1 - \beta}\pi^c$$
$$= \pi^{dev} + \frac{\beta}{1 - \beta}\pi^c$$

• Won't deviate iff

$$\pi^{dev} - \pi^m \le \frac{\beta}{1-\beta} \left(\pi^m - \pi^c\right)$$

so get cooperation for sufficiently high β .

• More complicated solutions if there is uncertainty, imperfect monitoring, etc. (Abreu, Pearce, and Staccetti).

Markov Perfect Equilibria (Maskin and Tirole)

- Equilibrium policy functions depend only on *payoff relevant* states. Let *s* be a vector of such states.
- π_i(a₁, a₂, s) current period payoff to player i given actions a₁ and a₂ in the current period and state s. π₁
- $s' = f(a_1, a_2, s)$ be transition function
- Let $\tilde{a}_i(s)$ be policy function and suppose $\tilde{v}_i(s)$ satisfies

$$\tilde{v}_1(s) = \max_{a_1} \pi(a_1, \tilde{a}_2(s), s) + \beta \tilde{v}_1(f(a_1, \tilde{a}_2(s), s))$$

and let \tilde{a}_1 be the solution Suppose $\tilde{v}_2(s)$ and $\tilde{a}_2(s)$ satisfy the analogous relationships. Then $(\tilde{a}_1, \tilde{a}_2, \tilde{v}_1, \tilde{v}_2)$ is a Markov-perfect equilibrium.

Cournot Duopoly

• Suppose

$$K_1 = K_2 = 1$$

fixed over time.

—What is the set of Markov-perfect equilibria?

—What is the set of payoff-relevant states?

• Suppose

$$K_{i,t} = Q_{i,t-1}(1-\delta)$$

-Intepretation: use capital to make new capital.

—Adjustment costs (Lucas 1967, Prescott and Visscher (1980))

- Can separate output and investment. Add an output stage after the investment state. Assume Q_i is capital and Y_i is output. Suppose Y_i ≤ Q_i and zero marginal cost up to capacity. Suppose demand is elastic. Then firms always produce up to capacity.
- Define a Markov-perfect equilibrium
- What is a steady state?

Dynamics with $\beta = 0$

- Given (K₁, K₂), solve the (asymmetric) Cournot duopoly problem
- Claim: if $K_1 > K_2$ then $q_1 < q_2$, but $q_1K_1 > q_2K_2$.

-FONC for two firms

$$P + P'q_1K_1 - c'(q_1) = 0$$

$$P + P'q_2K_2 - c'(q_2) = 0$$

Suppose instead that $q_1 \ge q_2$.

 $\Rightarrow c'(q_1) \ge c'(q_2)$

 $\Rightarrow P'q_1K_1 \ge P'q_2K_2$

 $\Rightarrow K_1 \leq K_2$, a contradiction.

• Claim market shares converge to equality.

$$\frac{K'_{1}}{K'_{2}} = \frac{q_{1}K_{1}(1-\delta)}{q_{2}K_{2}(1-\delta)}$$
$$= \frac{q_{1}K_{1}}{q_{2}K_{2}}$$
$$< \frac{K_{1}}{K_{2}}$$

 $1 < \frac{K_1'}{K_2'}$

• So converge to 50-50 monotonically.

-Kydland, Dominant firm literature

- Intuition?
- Suppose $\beta > 0$

-analytic results difficult

-will go to computer and work this out

—Suppose commit to sequence of outputs. Does this matter? Look at T = 2 case.

Benchmark Case of Perfect Competition Steady State

- Suppose agents take as given a constant price p. .
- Let v be the discounted value of owning one unit of capital at the beginning of a period

$$v = \max_{q} pq - c(q) + \beta \sigma q v$$

where

$$\sigma = 1 - \delta$$

• FONC

$$p - c'(q) + \beta \sigma v = 0 \tag{1}$$

• In a stationary equilibrium,

$$egin{array}{rcl} \sigma q &=& 1 \ q^* &=& rac{1}{\sigma} \end{array}$$

• v^* solves

$$v^* = pq^* - c(q^*) + \beta \sigma q^* v^*$$
$$= pq^* - c(q^*) + \beta v^*$$

SO

$$v^* = \frac{pq^* - c(q^*)}{1 - \beta}$$

• From the FONC

$$p = c'(q^*) - \beta \sigma v^*$$

• Plugging in the formula for v^* yields

$$p = c'(q^*) - \beta \sigma \frac{pq^* - c(q^*)}{1 - \beta}$$

Solving for p yields the stationary competitive price

$$p_C^* = (1 - \beta)c'(q^*) + \beta \sigma c(q^*).$$

- Q_C^* be the stationary competitive output
- $x_C^* = \sigma Q_C^*$ be the stationary competitive capital level.

Pure Monopoly.

The state variable is K at the beginning of period capital.
 Let w(K) be discounted maximized monopoly profit. This solves

$$w(K) = \max_{q} P(Kq) Kq - Kc(q) + \beta w(\sigma Kq)$$

• The FONC is

$$PK + P'K^2q - Kc' + \beta\sigma K\frac{dw}{dK} = \mathbf{0}$$

• Dividing by x,

$$P + P'Kq - c' + \beta\sigma\frac{dw}{dK} = \mathbf{0}$$

• Use the envelope theorem to verify that

$$\frac{dw}{dK} = qc'(q) - c(q)$$

(Think of Q as the choice variable....).

• Plugging this into the first-order condition and evaluating at the steady state output level $q^* = \frac{1}{\sigma}$ yields

$$p + P'qK - c' + \beta\sigma \left[qc' - c\right] = \mathbf{0}$$

or

$$p + P'q^*K = (1 - \beta)c' + \beta\sigma c$$

= P_C^* .

- Let K solving the above be denoted $K_M^{\ast}.$.
- Now calculate the equilbrium off the steady state

A Technical Aside

Numerical Solutions of Dynamic Programming Problems

Monopoly Problem

• Statement of problem. w(K) value function and q(K) is policy function. Contraction mapping: Let w_0 be value function beginning next period. Then

$$w_1(K) = \max_q P(Kq) Kq - Kc(q) + \beta w_0(\sigma Kq).$$

A solution is where $w_1(K) = w_0(K)$ for all K.

• Iterate

- How do numerically? Need an approximation for w_0 .
- Discretize? Works well with single agent decision theory. For duopoly problem though continuity is useful.
- Polynomial approximation.

Example with Linear Approximation

1. Start with approximation

$$\hat{w}_{\mathbf{0}}(K) = \alpha_{\mathbf{0}} + \beta_{\mathbf{0}}K$$

- 2. Take a set of m evaluation points $\tilde{K} = { \tilde{K}_1, \tilde{K}_2, ..., \tilde{K}_m }$
- 3. Solve problem at each of this points with $\hat{w}_0(K)$ instead of $w_0(K)$.

$$\tilde{w}_{1,i} = \max_{q} P\left(\tilde{K}_{i}q\right) \tilde{K}_{i}q - \tilde{K}_{i}c(q) + \beta \hat{w}_{0}\left(\sigma \tilde{K}_{i}q\right).$$

4. Yields a vector $\tilde{W}_1 = (\tilde{w}_{1,1}, \tilde{w}_{1,2}, ..., \tilde{w}_{1,m})$

5. Use OLS to determine a new approximation

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = (X'X)^{-1} X' \tilde{W}_1$$
$$X = \mathbf{1}^{\tilde{K}}$$

6. Iterate until obtain convergence in (α_t, β_t)

General Polynomial Approximation

- Chebyshev polynomials (in class of orthogonal polynomials)
- Defined on range $x \in [-1, 1]$

$$T_n(x) = \cos(n\cos^{-1}x)$$



Figure 1:

Recipe in Judd

• Step 1: Evaluation points

$$z_k = -\cos(\frac{2k-1}{2m}\pi), k = 1, ..., m$$

• Step 2: Adjust the notes to the [a,b] interval (here $a = .5K_M^*, b = 1.5K_M^*$)

$$x_k = (z_k + 1)\left(\frac{b-a}{2}\right) + a, \ k = 1, ..., m$$

• Step 3: Evaluate w(x) at the approximation nodes

$$ilde{w}_k = w(x_k)$$
, $k = 1, ..., m$

• Step 4: Compute the Chebyshev coefficients (remember T_i orthogonal)

$$a_i = \frac{\sum_{k=1}^m \tilde{w}_k T_i(z_k)}{\sum_{k=1}^m T_i(z_k)^2}$$

• To arrive at the approximation

$$\hat{w}(x) = \sum_{i=0}^{n} a_i T_i (2\frac{x-a}{b-a} - 1)$$

Hints for Duopoly Problem

- $(a_0, ...a_n)$ coefficient vector for the value function $v_1(K_1, K_2)$ approximation
- $(b_0, ..., b_n)$ coefficient vector for the policy function $q_1(K_1, K_2)$ approximation.
- Use Judd's techniques for approximation in R^2 (page 238)
- You need to iterate on q_1 as well as v_1 since firm 1 takes firm 2's action as given in the problem (and $q_2(x, y) = q_1(y, x)$).