Lecture—Doraszelski and Judd

Dynamic *and* Stochastic Model of Industry

Discrete Time Model

- Time discrete, horizon infinite, N players
- States $\omega_t \in \Omega$, finite stet
- Action of player i at t $x_t^i \in$ a set that may depend upon ω_t.
- $x_t^{-i} = (x_t^1, x_t^2, ..x_t^{i-1}, x_t^{i+1}, ..., x_i^N)$
- $\Pr(\omega'|\omega_t, x_t) = \Pi_{i=1}^N((\omega')^i|\omega_t^i, x_t^i)$, case where evolution of indidual i's position independent of what others do.
- Payoffs
  - $\pi^i(x_t, \omega_t)$ current profit
  - $\Phi^i(x_t, \omega_t, \omega_{t+1})$ return in next period dollars when move to ω_t from ω_{t+1}.
- Markov Perfect Equilibrium value function $V^i(\omega)$, $X^i(\omega)$ a policy function such that

$$X^i(\omega) = \arg\max_{x^i} \pi^i(x^i, X^{-i}(\omega), \omega) + \beta E_{\omega'} \left\{ \Phi^i(x^i, X^{-i}(\omega), \omega, \omega') + V^i(\omega')|\omega, x^i, X^{-i}(\omega) \right\}$$

Continuous Time Model

- Path peicewise-constant, right continuous function of time.
  - At time t, the hazard of a jump is $\phi(x_t, \omega_t)$
  - Probability moves to ω' $f(\omega'|\omega_{t-}, x_{t-})$ where ω_{t-} and x_{t-} is the state and action right before the jump.
  - $\pi^i(x_i, \omega_t)$ flow of dollars per unit of time
  - $\Phi^i(x_{t-}, \omega_{t-}, \omega_t)$ shance in stock of wealth in dollars.
  - $\rho > 0$. discount rate
- Bellman equation

$$\begin{aligned}\rho V^i(\omega) &= \max_i \pi^i(x^i, X^{-i}(\omega), \omega) + \\ &\quad \phi(x^i, X^{-i}(\omega), \omega)\left(-V^i(\omega) + E_{\omega'}\left\{\Phi^i(x^i, X^{-i}(\omega), \omega, \omega') + V^i\left(\omega'|\omega, x^i, X^{-i}(\omega)\right)\right\}\right)\end{aligned}$$

Computational Strategies—Discrete Time

Start with $X^i(\omega)$, $V^i(\omega)$. From these define

$$\hat{X}^i(\omega) = \arg\max_{x^i} \pi^i(x^i, X^{-i}(\omega), \omega) + \beta E_{\omega'} \left\{ \Phi^i(x^i, X^{-i}(\omega), \omega, \omega') + V^i(\omega') | \omega, x^i, X^{-i}(\omega) \right\}$$

and $\hat{V}^i(\omega)$ is the value of this. Then

$$\begin{aligned} X^i(\omega) &\leftarrow \hat{X}^i(\omega) \\ V^i(\omega) &\leftarrow \hat{V}^i(\omega) \end{aligned}$$

How update, PM1 pre-Gauss-Jacobi method. Go through each ω, then update.

Block Gauss-Seidel. Go through each ω. But then update.

Computational Strategies—Continuous Time

$$\begin{aligned}\hat{X}^i(\omega) &= \arg\max_{x^i} \pi^i(x^i, X^{-i}(\omega), \omega) - \phi(x^i, X^{-i}(\omega), \omega), V^i(x^i, X^{-i}(\omega), \omega) \\ &\quad +\phi(x^i, X^{-i}(\omega), \omega) E_{\omega'}\left\{\Phi^i(x^i, X^{-i}(\omega), \omega, \omega') + V^i(\omega')|\omega, x^i, X^{-i}(\omega)\right\}\end{aligned}$$

$$\begin{aligned}\hat{V}^i(\omega) &= \frac{1}{\rho + \phi(\hat{X}(\omega), X^{-i}(\omega), \omega)} \pi^i\left(\hat{X}(\omega), X^{-i}(\omega), \omega\right) \\ &\quad + \frac{1}{\rho + \phi(\hat{X}(\omega), X^{-i}(\omega), \omega)} \times \\ &\quad E_{\omega'}\left\{\Phi^i(\hat{X}(\omega), X^{-i}(\omega), \omega, \omega') + V^i(\omega')|\omega, \hat{X}(\omega), X^{-i}(\omega)\right\}\end{aligned}$$

Point about contractions. Individual problems are contractions. But entire systems not contractions.

Curse of dimensionality

Look at special case where can stay the same, go up one, or go down one.

For a given ω, (suppose no-one at the bound).

Then suppose N guys. Look at expectation. There are 3^N different possibilities. So have to sum over a mess of things.

Look at continuous case. $2N$. Key point, measure zero event that two change states the same time.

Storage issue.

Remembers still a curse of dimensionality.

Suppose there are M states $\omega_i \in \{1, 2, 3..., M\}$.

Then whether continuous or discreate, still have Ω with N^M states.

Can pare these down. Impose symmetry

$$V^i(\omega) = V^1(\omega^i, \omega^2, ...\omega^{i-1}, \omega^1, \omega^{i+1}, ...)$$

Then can look at representative firm.

Anonymity, exchangeability. Only care about distribution of the other ω, not the identifies of which have it.

So for player 1, $(1, 1, 3)$ same as $(1, 3, 1)$.

Business about the address matching. (Key point do as much work as possible before hand and store it).

Example, Pakes and McGuire 1

Demand Caplin and Nalebuff

$$\begin{aligned} U_{ik} &= g(\omega^i) - p^i + \varepsilon^{ik} \\ &= \delta^i + \varepsilon^{ik} \end{aligned}$$

outside

$$\begin{aligned} g(\omega^i) &= 3\omega^i - 4,\, \omega^i \leq 5 \\ &= 12 + \ln(2 - \exp(16 - 3\omega^i) \end{aligned}$$

and for the outside good

$$U_{0k} = \varepsilon^{0k}$$

Suppose ε^{ik} is i.i.d. extreme value then Consumers pick

$$\max_i \{U_{0k}, ...U_{Nk}\}$$

$$q^i(p^1, ..., p^N, \omega) = m \frac{\exp(\delta^i)}{1 + \sum_{j=1}^N \exp(\delta^j)}$$

Show a picture easy case of monopoly (with outside good.

Price competition Bertrand competition

$$\max_{p^i \geq 0} q^i(p^1, ...p^N; \omega) \left(p^i - c\right)$$

The FONC is

$$0 = \frac{\partial}{\partial p^i} q^i(p^1, ...p^N, \omega) \left(p^i - c\right) + q^i$$

Law of motion

Random depreciation of δ

invest x^i then advance with probability $\frac{\alpha x^i}{1+\alpha x^i}$

$\Pr^i((\omega')^i \,|\omega^i, x^i) = \frac{\alpha x^i}{1+\alpha x^i}(1-\delta)$, $(\omega')^i = \omega^i + 1$.

$=\frac{\delta}{1+\alpha x^i}(1-\delta)$, $(\omega')^i = \omega^i - 1$.

completement if $\omega^i = \omega^i$.

But bang up at bound of 1 or M.

Now for continous time

$$\phi^i(x^i, \omega^i) = \frac{\alpha x^i}{1+\alpha x^i} + \delta$$

$$f^i = \frac{\frac{\alpha x^i}{1+\alpha x^i}}{\frac{\alpha x^i}{1+\alpha x^i} + \delta}, \ (\omega')^i = \omega_i + 1$$