

Class Notes on Chaney (2008)

(With Krugman and Melitz along the Way)

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1 Model of Chaney AER (2008)

As a first step, let's write down the elements of the Chaney model.

- N asymmetric countries produce goods only with labor.
- L_n population of country n .
- $H + 1$ sectors,
 - Sector 0 is boring constant returns, no product differentiation; 1 unit of labor makes w_n units of output. Assume the parameters are such that in equilibrium, each country makes a positive amount sector 0 good.
 - Sectors $h \geq 1$ are "interesting" in that they have CES composite utilities, and fixed costs lurking around.
- Utility function of representative consumer is Cobb-Douglas with shares μ_h ,

$$U = q_0^{\mu_0} \prod_{h=1}^H \left(\int_{\Omega_h} q_h(\omega)^{(\sigma_h-1)/\sigma_h} d\omega \right)^{(\sigma_h/(\sigma_h-1))\mu_h}$$

for $\sigma_h > 1$, (we need $\sigma_h > 1$ so that the demand curves faced by the differentiated products monopolists will be elastic. If inelastic, such firms will want to set infinitely high prices.)

- Firms have random productivity ϕ . The cost of producing q units in i and selling in j equals

$$c_{ij}^h(q) = \frac{w_i \tau_{ij}^h}{\phi} q + f_{ij}^h$$

where τ_{ij}^h is the “iceberg” cost specific to sector h good going from i to j . That is, to deliver one unit, τ_{ij}^h units must be shipped.

- The productivity distribution is Pareto

$$\Pr(\tilde{\phi}_h < \phi) = G(\phi) = 1 - \phi^{-\gamma_h}$$

for $\gamma_h \geq \sigma_h - 1$. (Interpret γ_h as an inverse measure of heterogeneity of firms.)

- Let π be the value of aggregate portfolio. (Unlike standard Dixit-Stiglitz setups, like in Melitz, there won't be entry and exit of firms at a given county. Rather the measure of firms will be fixed, and no fixed cost is incurred at initial entry. Thus firms will get positive profit. In Dixit-Stiglitz setups, entry drive revenues minus variable costs down to the fixed cost of entry so there is zero profit. Note that Chaney still gets the Melitz channel going regarding exports since there is a fixed cost to entering an export market.)

2 Krugman Version

Before crunching through the above, it is a useful background exercise to work through a Krugman variation of the above. And even before we do that, it is useful to just crunch through a version of Dixit-Stiglitz in a one-country world.

2.1 Autarky Special Case

So let's do the following:

- Take the spending $Y_{j,h}$ as given in each country j for sector h . Also, let's simplify for now and drop the h notation for sector.

- Suppose initially that international trade is impossible, $\tau_{ij} = \infty$, $i \neq j$. Hence for now we can drop subscript j for country, since each country is in autarky.
- Assume firms are homogenous in productivity, $\phi = 1$.
- Let's make the measure of differentiated goods endogenous by allowing free entry at the a fixed entry cost of f , where f is denominated in units of labor. Let M be the measure of firms who enter. Of course, in equilibrium firms will all choose a different differentiated product ω . Given symmetry, if M firms enter, we can without loss of generality that the products in the interval $[0, M]$ are selected. We can write the composite as

$$Q = \left(\int_0^M q(\omega)^{(\sigma-1)/\sigma} d\omega \right)^{(\sigma/(\sigma-1))}. \quad (1)$$

Now, having explained the model, we define the equilibrium as a

$\{M, p(\omega), q(\omega), \omega \in [0, M]\}$ such that

1. Consumer demand $q(\omega)$ for each good ω maximizes utility given the budget constraint.
2. $p(\omega)$ is the profit maximizing price of firm ω , taking as given the prices of all other firms.
3. Firms that enter make nonnegative profit.
4. There is no incentive for further entry.

Note that (3) and (4) together imply firms make zero profit in equilibrium.

Let's look at the problem of a consumer in this environment. The consumer solves

$$\max_{q(\cdot)} \left[\int_0^M q(\omega)^{(\sigma-1)/\sigma} d\omega \right]^{(\sigma/(\sigma-1))} \quad (2)$$

subject to the budget constraint

$$\int_0^M p(\omega)q(\omega)d\omega = \mu wL = Y.$$

From this problem we can derive the marginal rate of substitution condition; i.e., that the ratio of the marginal utility of any goods ω_1 and ω_0 must equal the ratio of the prices,

$$\begin{aligned} \frac{(\sigma/(\sigma-1)) \prod^{(\sigma/(\sigma-1))^{-1}} \left(\frac{\sigma-1}{\sigma}\right) q_1^{-\frac{1}{\sigma}}}{(\sigma/(\sigma-1)) \prod^{(\sigma/(\sigma-1))^{-1}} \left(\frac{\sigma-1}{\sigma}\right) q_0^{-\frac{1}{\sigma}}} &= \frac{p_1}{p_0} \\ \left(\frac{q_1}{q_0}\right)^{-\frac{1}{\sigma}} &= \frac{p_1}{p_0} \\ q_1 &= p_1^{-\sigma} (p_0^\sigma q_0) \\ &= p_1^{-\sigma} k \end{aligned}$$

for

$$k \equiv p_0^\sigma q_0.$$

Now let's turn to the firm producing product q_1 (call this firm 1). Given the continuum of firms, its actions have a negligible impact on aggregate variables. In particular, if it takes as given the prices of all other firms as in Bertrand (or alternatively the quantities of all other firms in Cournot, with a continuum it doesn't make a difference), a firm can regard the variable k in its demand as an exogenous parameter, so with k fixed, demand faced by the firm is constant elasticity equal to the elasticity of substitution parameter σ ,

$$q_1(p_1) = p_1^{-\sigma} k.$$

(But note that if we were to work with a finite number of firms rather than a continuum, a firm would have to take into account that k would vary with its choice of p_1 .)

We can now solve firm 1's problem. Let w denote the wage rate. The marginal cost in labor units is one. Hence marginal cost in terms of the numeraire equals w . The fixed cost in terms of numeraire is wf . The problem of firm 1 is

$$\max_{p_1} (p_1 - w) q_1(p_1) - wf$$

The FONC is

$$\begin{aligned} p_1^{-\sigma} k - \sigma (p_1 - w) p_1^{-\sigma-1} k &= 0 \\ p_1 &= \sigma (p_1 - w) \\ \frac{p_1 - w}{p_1} &= \frac{1}{\sigma}. \end{aligned}$$

This is the standard result that the price cost margin equals the inverse of the elasticity of demand. Alternatively, we can solve for the price as a constant markup $\sigma/(\sigma - 1)$ over cost,

$$p^e = \frac{\sigma}{\sigma - 1} w. \quad (3)$$

Next we can use the zero-profit condition to pin down firm size,

$$\frac{\sigma}{\sigma - 1} wq - wq - wf = 0, \quad (4)$$

or

$$\frac{1}{\sigma - 1} q = f,$$

So the equilibrium firm output size is

$$q^e = (\sigma - 1) f. \quad (5)$$

Note the comparative statics here. The larger is σ , the more substitutable are competing firms products and hence the lower the markup $\sigma/(\sigma - 1)$. The zero-profit constraint then implies firm size must be bigger (with a lower markup, the firm needs a greater sales volume

to cover the fixed cost.) Analogously, if the fixed cost f increases, quantity increases.

You can see from the simple equations (3) and (5) for the price p^e and quantity q^e of each good why the Dixit-Stiglitz model has been so popular over the years!

Our next step is to calculate the price index. All firms have the same marginal cost, so all firms set the same price. Hence, in any utility maximizing bundle, a consumer will consume equal amounts of each of the M differentiated goods. We can calculate the price index (the price of one unit of composite good) as follows:

$$\begin{aligned}
 P &= \frac{M^{\frac{\sigma}{\sigma-1}} w}{[M]^{(\sigma/(\sigma-1))}} & (6) \\
 &= M^{\frac{\sigma-1}{\sigma-1} - \frac{\sigma}{\sigma-1}} \left(\frac{\sigma}{\sigma-1} w \right) \\
 &= \frac{\sigma}{\sigma-1} w M^{-\frac{1}{\sigma-1}}
 \end{aligned}$$

To understand this formula, suppose a consumer purchased one unit of each differentiated good, i.e. $q(\omega) = 1$, $\omega \in [0, M]$. The denominator in the first equation above equals the amount of composite good that is produced. (Just plug $q(\omega) = 1$, $\omega \in [0, M]$, into equation (1)). The numerator is what such a bundle would cost. By taking the ratio, we calculate price per composite unit.

For now take spending Y as fixed. If we let Q be the total quantity of composite, then

$$\begin{aligned}
 QP &= Y \\
 \left[M (q^e)^{\frac{\sigma-1}{\sigma}} \right]^{(\sigma/(\sigma-1))} P &= Y \\
 M^{\frac{\sigma}{\sigma-1}} q^e \frac{\sigma}{\sigma-1} w M^{-\frac{1}{\sigma-1}} &= Y \\
 M q^e \frac{\sigma}{\sigma-1} w &= Y,
 \end{aligned}$$

so we can back out the equilibrium variety given Y .

$$M = \frac{\sigma-1}{\sigma} \frac{Y}{w} \frac{1}{(\sigma-1)f} = \frac{Y}{\sigma w f}$$

Observe that all expansion in this economy is on the *extensive margin* of a greater variety of products rather than the *intensive margin* of greater quantity per product. If we double spending Y , variety M doubles, and the output of each firm stays constant at q^e given above by (5).

In the above, we held spending Y and the wage fixed. Now we can endogenize these by letting L be the population and assume each individual is endowed with one unit of labor. Let labor be the numeraire, so $w = 1$. Spending equals total income, which is just labor income since equilibrium profits are zero, i.e., $Y = L$. Thus equilibrium variety is

$$M = \frac{L}{\sigma f},$$

so aggregate output equals

$$\begin{aligned} Q &= \left[M (q^e)^{\frac{\sigma-1}{\sigma}} \right]^{(\sigma/(\sigma-1))} \\ &= M^{\frac{\sigma}{\sigma-1}} q^e \\ &= \left(\frac{L}{\sigma f} \right)^{\frac{\sigma}{\sigma-1}} (\sigma - 1) f \\ &= L^{\frac{\sigma}{\sigma-1}} \sigma^{-\frac{\sigma}{\sigma-1}} f^{-1/(\sigma-1)} \end{aligned}$$

Output per person equals

$$\begin{aligned} \text{Composite Output per Person} &= \frac{Q}{L} \\ &= L^{\frac{1}{\sigma-1}} \sigma^{-\frac{\sigma}{\sigma-1}} f^{-1/(\sigma-1)} \end{aligned}$$

Observe how average consumption (or utility) increases in population through the love of variety. As population doubles, each individual consumer purchases twice as many different goods at half the volume. Given curvature in the preferences, this increases per person utility. Observe that as σ gets large, and the goods become better substitutes for each other, the impact of doubling population on average utility declines in percentage terms.

2.2 Trade

Let's add trade to the above model. But let's do it in a particular way to keep things simple and focused as a classroom exercise.

We bring in sector 0 that is boring constant returns to scale and make that the numeraire. Sector 1 is the differentiated products sector.

There are two countries (that I sometimes call two locations). Assume it is costless to ship the sector 0 good in either direction. Let there be an asymmetry between the two countries. The iceberg cost to ship segment 1 good from country 1 to 2 is $\tau_{12} = \tau$. Shipping segment 1 good the other direction from 2 to 1 is impossible, i.e. $\tau_{21} = \infty$. (Perhaps location 1 is upriver from location 2...) With these assumptions, the only kind of trade that will occur is country 1 shipping sector 1 goods in exchange for the sector 0 homogenous good from country 2. Note that while it is possible to ship sector 0 goods from 1 to 2, it will never happen since country 2 can only pay for them with sector 0 goods. There is no point trading sector 0 goods for sector 0 goods because it is a homogenous product.

Let sector 0 be the numeraire and suppose one unit of labor produces w_j units of sector 0 good at location j . Let $Y_{j,h}$ be the spending in country j on sector h . Let $P_{j,h}$ be the sector price index. Now as the numeraire good that is costlessly transported across the two countries, $P_{1,0} = P_{2,0} = 1$. The variables $P_{1,1}$ and $P_{2,1}$ are the costs of sector 1 composite at each location.

Like Chaney, assume utility is Cobb-Douglas

$$U = Q_0^{\mu_0} Q_1^{\mu_1}$$

for $\mu_0 + \mu_1 = 1$. Let L_j be the labor endowment in each country.

Since country 1 is unable to important segment 1 goods, we can use formula (6) for calculating the composite price in the autarky case,

$$P_{1,1} = \frac{\sigma}{\sigma - 1} w_1 M_1^{-\frac{1}{\sigma-1}},$$

where M_1 is the measure of differentiated producers at location 1. Note that each firm in location 1 will be setting the local price equal to $\frac{\sigma}{\sigma-1} w_1$, using the markup rule derived earlier.

Next consider the segment 1 composite price in country 2. This is more complicated because consumers in country 2 in general can obtain segment 1 goods from both countries. Let p_{12} and p_{22} be the prices of differentiated goods at location 2 sourced from country 1 and country 2. (In equilibrium, all prices from a given source will be the same and we impose that directly.) Let M_1 and M_2 be the variety available at each country. Finally, let \bar{q}_{12} and \bar{q}_{22} be cost minimizing quantities used at location 2 of goods from each source to construct one unit of final good composite. We must have the MRS condition for cost minimization satisfied:

$$\frac{\bar{q}_{12}}{\bar{q}_{22}} = \left(\frac{p_{12}}{p_{22}} \right)^{-\sigma}$$

Also

$$\begin{aligned} 1 &= \left[\int_0^{M_1} \bar{q}_{12}^{(\sigma-1)/\sigma} d\omega + \int_{M_1}^{M_1+M_2} \bar{q}_{22}^{(\sigma-1)/\sigma} d\omega \right]^{(\sigma/(\sigma-1))} \\ &= \left[M_1 \bar{q}_{12}^{\frac{\sigma-1}{\sigma}} + M_2 \bar{q}_{22}^{\frac{\sigma-1}{\sigma}} \right]^{(\sigma/(\sigma-1))} \\ &= \left[M_1 \left(\left(\frac{p_{12}}{p_{22}} \right)^{-\sigma} \bar{q}_{22} \right)^{\frac{\sigma-1}{\sigma}} + M_2 \bar{q}_{22}^{\frac{\sigma-1}{\sigma}} \right]^{(\sigma/(\sigma-1))} \\ &= \bar{q}_{22} \left[M_1 \left(\frac{p_{12}}{p_{22}} \right)^{-(\sigma-1)} + M_2 \right]^{(\sigma/(\sigma-1))} \\ &= \bar{q}_{22} \left[p_{22}^{(\sigma-1)} \left(M_1 p_{12}^{-(\sigma-1)} + M_2 p_{22}^{-(\sigma-1)} \right) \right]^{(\sigma/(\sigma-1))} \\ &= \bar{q}_{22} p_{22}^{\sigma} \left(M_1 p_{12}^{-(\sigma-1)} + M_2 p_{22}^{-(\sigma-1)} \right)^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

So

$$\begin{aligned}\bar{q}_{22} &= p_{22}^{-\sigma} \left(M_1 p_{12}^{-(\sigma-1)} + M_2 p_{22}^{-(\sigma-1)} \right)^{-\frac{\sigma}{\sigma-1}} \\ \bar{q}_{12} &= p_{12}^{-\sigma} \left(M_1 p_{12}^{-(\sigma-1)} + M_2 p_{22}^{-(\sigma-1)} \right)^{-\frac{\sigma}{\sigma-1}}\end{aligned}$$

And the price index for segment 1 composite at country 2 is

$$\begin{aligned}P_{2,1} &= M_1 \bar{q}_{12} p_{12} + M_2 \bar{q}_{22} p_{22} \tag{7} \\ &= M_1 p_{12}^{1-\sigma} \left(M_1 p_{12}^{-(\sigma-1)} + M_2 p_{22}^{-(\sigma-1)} \right)^{-\frac{\sigma}{\sigma-1}} \\ &\quad + M_2 p_{22}^{1-\sigma} \left(M_1 p_{12}^{-(\sigma-1)} + M_2 p_{22}^{-(\sigma-1)} \right)^{-\frac{\sigma}{\sigma-1}} \\ &= \left(M_1 p_{12}^{1-\sigma} + M_2 p_{22}^{1-\sigma} \right) \left(M_1 p_{12}^{-(\sigma-1)} + M_2 p_{22}^{-(\sigma-1)} \right)^{-\frac{\sigma}{\sigma-1}} \\ &= \left(M_1 p_{12}^{-(\sigma-1)} + M_2 p_{22}^{-(\sigma-1)} \right)^{-\frac{1}{\sigma-1}}\end{aligned}$$

Note the form of this equation can be extended when location 2 purchases from N countries rather than 2 to

$$P_{2,1} = \left(\sum_{j=1}^N M_j p_{j2}^{-(\sigma-1)} \right)^{-\frac{1}{\sigma-1}}.$$

Moreover, we can reinterpret this formula to include cases where different firms within the same country charge different prices (e.g., because their costs differ). In this case, let M_j be the measure of firms that are charging price p_{j2} and sum over all the different price groups j . The formula also generalizes if there are a continuum of types. Letting $G(j)$ be the c.d.f. of type j setting price $p(j)$, then for a continuum the price index is

$$\text{price index} = \left[\int p(j)^{-(\sigma-1)} dG(j) \right]^{1/(1-\sigma)}.$$

Returning back to the two-country case at hand, the price p_{j2} of a differentiated good

from source j to location 2 follows the constant markup rules,

$$\begin{aligned} p_{12} &= \frac{\sigma}{\sigma - 1} w_1 \tau, \\ p_{22} &= \frac{\sigma}{\sigma - 1} w_2. \end{aligned}$$

Note that the marginal cost for a firm at location 1 to sell one unit at location 2 is $w_1 \tau$ on account of the iceberg transportation cost. There is an interesting point to be made about these prices related to price discrimination and arbitrage. Suppose we think of a firm at location 1 as being able to price discriminate between consumers at location 1 and consumers at location 2. In this case, it sets a separate price at each location depending on the elasticity of demand at each location and the marginal cost, i.e. $p_{11} = \frac{\sigma}{\sigma - 1} w_1$, $p_{12} = \frac{\sigma}{\sigma - 1} w_1 \tau$. But note that $p_{12} = p_{11} \tau$. So consumers at location 2 have no incentive to try to pretend they live at location 1 to get the good there and then ship it to location 2 themselves at an iceberg cost of τ . So the prices here are equivalent to a firm a location 1 charging $\frac{\sigma}{\sigma - 1} w_1$ for delivery at location 1, letting consumers pay the transportation cost to any other destination. Outside the constant elasticity demand case, this won't ordinarily be true. When firms try to price discriminate across locations, in general there may be an incentive for consumers to arbitrage.

One last thing regarding the demands per unit of composite: Below it is convenient to rewrite the \bar{q}_{12} and \bar{q}_{22} as

$$\begin{aligned} \bar{q}_{12} &= p_{12}^{-\sigma} \left(M_1 p_{12}^{-(\sigma-1)} + M_2 p_{22}^{-(\sigma-1)} \right)^{-\frac{\sigma}{\sigma-1}} \\ &= p_{12}^{-\sigma} P_{2,1}^\sigma \\ \bar{q}_{22} &= p_{22}^{-\sigma} P_{2,1}^\sigma. \end{aligned}$$

Next, we need to solve for equilibrium variety at each location. Recall that w_j is the productivity of labor in sector 0 at j . Like Chaney, let's assume the parameters are such

that both locations produce sector 0 good. (This will be satisfied if the utility weight μ_0 on the sector 0 good is high enough.) This pins down the wage at each location at w_j . Given the zero-profit entry condition, total income at each location is labor income $w_j L_j$. Spending at each sector is

$$\begin{aligned} Y_{j,0} &= \mu_0 w_j L_j, \\ Y_{j,1} &= \mu_1 w_j L_j. \end{aligned}$$

Note that the w_j are parameters, so these spending formulas are functions of exogenous model parameters.

Next let's gather together the price indices at the two locations for sector 1 composite,

$$\begin{aligned} P_{1,1}(M_1) &= \left(M_1 \left(\frac{\sigma w_1}{\sigma - 1} \right)^{-(\sigma-1)} \right)^{-\frac{1}{\sigma-1}} = M_1^{-\frac{1}{\sigma-1}} \frac{\sigma w_1}{\sigma - 1} \\ P_{2,1}(M_1, M_2) &= \left(M_1 \left(\frac{\sigma w_1 \tau}{\sigma - 1} \right)^{-(\sigma-1)} + M_2 \left(\frac{\sigma w_2}{\sigma - 1} \right)^{-(\sigma-1)} \right)^{-\frac{1}{\sigma-1}}, \end{aligned} \tag{8}$$

where we note the dependence of these variables on the endogenous variables M_1 and M_2 .

We can solve out for sector 1 consumption composite at each location,

$$\begin{aligned} Q_{1,1}(M_1) &= \frac{Y_{1,1}}{P_{1,1}(M_1)}, \\ Q_{2,1}(M_1, M_2) &= \frac{Y_{2,1}}{P_{2,1}(M_1, M_2)}. \end{aligned}$$

From these we can determine firm level demands. Sales to a firm located in country 1 equal

$$\begin{aligned}
q_1(M_1, M_2) &= M_1^{-\frac{\sigma}{\sigma-1}} Q_{1,1}(M_1) + \tau \bar{q}_{12} Q_{2,1}(M_1, M_2) \\
&= M_1^{-\frac{\sigma}{\sigma-1}} Q_{1,1}(M_1) + \tau p_{12}^{-\sigma} P_{2,1}(M_1, M_2)^\sigma Q_{2,1}(M_1, M_2) \\
&= M_1^{-\frac{\sigma}{\sigma-1}} \frac{Y_{1,1}}{P_{1,1}(M_1)} + \tau \left(\frac{\sigma w_1 \tau}{\sigma - 1} \right)^{-\sigma} P_{2,1}(M_1, M_2)^{\sigma-1} Y_{2,1} \\
&= M_1^{-1} \frac{Y_{1,1}}{\frac{\sigma}{\sigma-1} w_1} + \tau \left(\frac{\sigma w_1 \tau}{\sigma - 1} \right)^{-\sigma} P_{2,1}(M_1, M_2)^{\sigma-1} Y_{2,1},
\end{aligned}$$

where the second equation substitutes in for \bar{q}_{12} from above. To understand the first line, observe that $Q_{1,1}$, the composite for country 1, is sourced entirely at 1, so if sales are q_{11} from each firm then total composite there is $Q_{1,1} = M^{\frac{\sigma}{\sigma-1}} q_{11}$. The first term in the first line above just solves out for q_{11} . To derive the second term, recall that \bar{q}_{12} is the demand at location 2 per unit of composite for a firm at location 1. We multiple through by the quantity of composite $Q_{2,1}$ at location 2 and multiply through by τ to determine the amount that has to be sent in order that the amount demanded be received.

Analogously, sales of a firm at location 2 equal

$$\begin{aligned}
q_2(M_1, M_2) &= \bar{q}_{22} Q_{2,1}(M_1, M_2) \\
&= p_{22}^{-\sigma} P_{2,1}(M_1, M_2)^\sigma \frac{Y_{2,1}}{P_{2,1}(M_1, M_2)} \\
&= \left(\frac{\sigma w_2}{\sigma - 1} \right)^{-\sigma} P_{2,1}(M_1, M_2)^{\sigma-1} Y_{2,1}.
\end{aligned} \tag{9}$$

Analogous to the autarky case, the zero-profit condition pins down the equilibrium sales volume for firms at each location. That is, if we have positive entry into the differentiated products sector at both locations, $M_1 > 0$ and $M_2 > 0$, then M_1 and M_2 must solve

$$(\sigma - 1) f = q_1(M_1, M_2)$$

$$(\sigma - 1) f = q_2(M_1, M_2).$$

Note that it is immediate that $q_1(M_1, M_2)$ and $q_2(M_1, M_2)$ both strictly decrease in M_1 and M_2 .

2.2.1 Case Where Country 2 Specializes in the Homogenous Goods

It is not necessarily the case that both locations produce homogenous goods. In fact, if τ is close to 1 (so transportation costs are small), then there will be no entry in the differentiated products sector at country 2.

We prove this claim by considering the extreme case where $\tau = 1$. Also assume $w_2 \geq w_1$. We can write the sales of a firm at location 1 as

$$\begin{aligned} q_1(M_1, M_2) &= M_1^{-1} \frac{Y_{1,1}}{\frac{\sigma}{\sigma-1} w_1} + \left(\frac{\sigma w_1}{\sigma - 1} \right)^{-\sigma} P_{2,1}(M_1, M_2)^{\sigma-1} Y_{2,1} \\ &= M_1^{-1} \frac{Y_{1,1}}{\frac{\sigma}{\sigma-1} w_1} + \left(\frac{w_1}{w_2} \right)^{-\sigma} q_2(M_1, M_2) \\ &> q_2(M_1, M_2), \end{aligned}$$

where the last inequality uses $w_2 \geq w_1$. Suppose suppose the marginal entrant is breaking even at location 1 (i.e. the sale volume just covers the fixed cost. Then since sales volume is strictly lower at location 2, any differentiated products firm must have less then the break-even level of output.

$$(\sigma - 1) f = q_1(M_1, M_2) > q_2(M_1, M_2).$$

It is immediate that $M_2 = 0$ must hold at $\tau = 1$. By continuity, this must be true for τ close to 1.

Suppose that $w_1 = w_2$. If $\tau = 1$, the price index is the same at both places. The two countries are then equally desirable places to live as income and prices are identical.

But now consider the case where $w_1 = w_2$ and $\tau > 1$, but τ is close enough to one so that $M_2 = 0$. In this case

$$P_{2,1}(M_1, 0) = \tau P_{1,1}(M_1).$$

Country 2 must import all its differentiated goods, so the price index for segment 1 is just a factor τ times what the index is in at location 1. Individuals at location 1 have the same nominal income as in location 2. However, since prices are lower, individuals at location 1 are strictly better off than their counterparts at location 2. All differentiated goods producers are located at 1 because it is possible to generate high sales volume by selling at both location 1 and location 2. Firms at location 2 are precluded from selling in both markets. Consumers at location 1 enjoy local access to these goods while consumers at location 2 have to pay transportation cost to import them.

2.2.2 Symmetric Transportation Cost

We have imposed asymmetry ex ante in the transportation cost, $\tau_{12} = \tau < \infty$, and $\tau_{21} = \infty$. Now consider the case of symmetric transportation costs, $\tau_{21} = \tau_{12} = \tau$. It is straightforward to tweak the analysis above to derive the equilibrium conditions.

If we also assume that $w_1 = w_2$, it is straightforward to derive a symmetric equilibrium where $M_1 = M_2$ and there is trade only in differentiated goods. That is, this model generates two-way trade in the same sector. The original purpose of the Krugman model was to generate such trade; i.e., to understand why the U.S. sells automobiles and medical equipment and aircraft to Europe, while Europe sells these kinds of goods (but different varieties) to the U.S.

3 Melitz

Let's take the model from above, where country 1 has production in a differentiated products sector and country 2 completely specializes in the sector 0 good.

Now tweak the model in two ways.

First, assume that instead of every sector 1 firm having marginal cost of $\phi = 1$, assume there is variation in ϕ . Suppose ϕ is distributed on the support $[\underline{\phi}, \bar{\phi}]$ with c.d.f. $G(\phi)$.

Second, assume that addition to the entry cost f^e , there is a fixed cost f^+ that must be paid if any positive output is produced and a fixed cost f^x to set up an export distribution network.

Suppose firms pay the entry cost f^e and draw ϕ . After seeing ϕ , they decide whether to pay the fixed cost f^+ and whether to set up an export channel.

Let's characterize what an equilibrium must look like (or at least piece of it). Suppose we have an equilibrium. Using the arguments from above, if we take a firm at country 1, we can write the demand it faces in country 1 as a function of the price p_1 it sets as follows,

$$q_{11}(p_{11}) = p_{11}^{-\sigma} k_{11}$$

Analogously, we can write the sales to location 2 as a function of the delivered price as

$$q_{12}(p_{12}) = p_{12}^{-\sigma} k_{12}$$

For a firm of productivity ϕ , the optimal prices are

$$\begin{aligned} p_{11} &= \frac{\sigma}{\sigma - 1} \frac{w_1}{\phi}, \\ p_{12} &= \frac{\sigma}{\sigma - 1} \frac{\tau w_2}{\phi}. \end{aligned}$$

The maximized profit at each location, not including any fixed cost, is

$$\begin{aligned} \left(p_{11} - \frac{w_1}{\phi} \right) q_{11} &= \left(\frac{\sigma}{\sigma - 1} \frac{w_1}{\phi} - \frac{w_1}{\phi} \right) q_{11} \\ &= \frac{1}{\sigma - 1} \frac{w_1}{\phi} \left(\frac{\sigma}{\sigma - 1} \frac{w_1}{\phi} \right)^{-\sigma} k_{11} \\ &= \frac{(\sigma - 1)^{(\sigma-1)}}{\sigma^\sigma} \left(\frac{\phi}{w_1} \right)^{\sigma-1} k_{11} \end{aligned}$$

and

$$\left(p_{12} - \frac{w_1 \tau}{\phi}\right) q_{12} = \frac{(\sigma - 1)^{(\sigma-1)}}{\sigma^\sigma} \left(\frac{\phi}{\tau w_1}\right)^{\sigma-1} k_{12}$$

It is convenient to write these variable profit terms as functions of ϕ ,

$$\begin{aligned}\pi_{11}(\phi) &= \frac{(\sigma - 1)^{(\sigma-1)}}{\sigma^\sigma} \left(\frac{\phi}{w_1}\right)^{\sigma-1} k_{11} \\ \pi_{12}(\phi) &= \frac{(\sigma - 1)^{(\sigma-1)}}{\sigma^\sigma} \left(\frac{\phi}{\tau w_1}\right)^{\sigma-1} k_{12}\end{aligned}$$

Given ϕ and k_{11} and k_{12} , for a firm located at country 1, its decision problem is to decide whether to produce a positive amount of output or shut down to avoid f^+ . And if positive, the firm decides whether or not to set up the export channel. The return to the various actions is

$$\begin{aligned}v^{shutdown}(\phi) &= -f^e \\ v^{domestic_only}(\phi) &= -f^e - f^+ + \pi_{11}(\phi) \\ v^{exporter}(\phi) &= -f^e - f^+ - f^x + \pi_{11}(\phi) + \pi_{12}(\phi)\end{aligned}$$

Observe that

$$\begin{aligned}0 &> \\ v^{shutdown}(0) &= -f^e > \\ v^{domestic_only}(0) &= -f^e - f^+ > \\ v^{exporter}(0) &= -f^e - f^+ - f^x.\end{aligned}$$

Moreover

$$0 = \frac{dv^{shutdown}(\phi)}{d\phi} < \frac{dv^{domestic_only}(\phi)}{d\phi} < \frac{dv^{exporter}(\phi)}{d\phi}.$$

This ranking follows because $\pi_{11}(\phi)$ and $\pi_{12}(\phi)$ both strictly increase in ϕ . Also $\pi_{11}(\phi)$ and

$\pi_{12}(\phi)$ get arbitrarily large as ϕ gets large. It is immediate then that there are two cutoffs ϕ' and ϕ'' , $0 < \phi' < \phi''$, such that if $\phi < \phi'$, then shutdown is optimal, while if $\phi \in (\phi', \phi'')$ the domestic only is optimal and if $\phi > \phi''$, then exporter status is optimal.

This ranking by productivity is the key result in the Melitz paper. Melitz cites empirical findings plants with higher productivity tend to be more likely to be exporters. This fact is consistent with the result just obtained.

4 Back to Chaney

Note key elements:

- Like Melitz, productivity is heterogenous. More specifically, impose Pareto $F(\tilde{\phi}_h < \phi) = 1 - \phi^{-\gamma_h}$ (this gives a very tractable structure)
- Shut down free-entry of firms into domestic markets. Domestic entry assumed proportion to $w_n L_n$ (where again both w_n and L_n are parameters.)
- Still have Dixit-Stiglitz-like fixed cost to entering foreign markets. (Now called Melitz-like)

Let's gather together the key equations. Suppose a firm with productivity ϕ is located in i and sells in j . Its price will use the standard markup,

$$p_{ij}^h(\phi) = \frac{\sigma_h}{\sigma_h - 1} \frac{w_i \tau_{ij}^h}{\phi}$$

Let $q_{ij}^h(\phi)$ be sales in segment h from i to j for a firm with productivity ϕ . Analogous to equation (9), this equals

$$q_{ij}^h(\phi) = p_{ij}^h(\phi)^{-\sigma} (P_j^h)^{\sigma-1} Y_j^h.$$

The value of these sales equals

$$x_{ij}^h(\phi) = p_{ij}^h(\phi)q_{ij}^h(\phi) = \mu_h Y_j \left(\frac{p_{ij}^h(\phi)}{P_j^h} \right)^{1-\sigma_h},$$

where we substitute $Y_j^h = \mu_h Y_j$

Next, let's determine the marginal firm at location i who is indifferent to exporting to $j \neq i$, (all firms will sell to their home location). The profit of firm ϕ from doing this is

$$\pi_{ij}^h(\phi) = \frac{x_{ij}^h(\phi)}{\sigma_h} - f_{ij}^h,$$

since with constant elasticity, a fraction $1/\sigma$ of price is profit net of variable cost, so analogously a fraction $1/\sigma$ of revenues equals revenues minus variable cost. Substituting in above yields

$$\begin{aligned} \pi_{ij}^h(\phi) &= \frac{1}{\sigma_h} \mu_h Y_j \left(\frac{p_{ij}^h(\phi)}{P_j^h} \right)^{1-\sigma_h} - f_{ij}^h \\ &= \frac{1}{\sigma_h} \mu_h Y_j \left(\frac{\frac{\sigma_h}{\sigma_h-1} \frac{w_i \tau_{ij}^h}{\phi}}{P_j^h} \right)^{1-\sigma_h} - f_{ij}^h \end{aligned}$$

Set equal to zero to get cutoff and have

$$\bar{\phi}_{ij}^h = \lambda_1^h \left(\frac{f_{ij}^h}{Y_j} \right)^{\frac{1}{\sigma_h-1}} \frac{w_i \tau_{ij}^h}{P_j^h}$$

where λ_1^h is a constant.

Next the price index equals

$$P_j^h = \left(\sum_{k=1}^N w_k L_k \int_{\bar{\phi}_{kj}^h}^{\infty} \left(\frac{\sigma_h}{\sigma_h-1} \frac{w_k \tau_{kj}^h}{\phi} \right)^{1-\sigma_h} dG_h(\phi) \right)^{\frac{1}{1-\sigma_h}}$$

(Note that $w_k L_k$ is the analog of M_k in the above formula, it is the variety variable (not conditioning upon which ϕ are above the cutoff).

Next step: plug in the cutoffs as well as the functional form for the density of G_h . Carrying around the h is getting annoying, so let's drop it. So $G(\phi) = 1 - \phi^{-\gamma}$, $G'(\phi) = g(\phi) = \gamma\phi^{-1-\gamma}$. Plugging in this and $\bar{\phi}_{kj}^h$ yields

$$\begin{aligned} P_j &= \left(\sum_{k=1}^N w_k L_k \int_{\bar{\phi}_{kj}}^{\infty} \left(\frac{\sigma}{\sigma-1} \frac{w_k \tau_{kj}}{\phi} \right)^{1-\sigma} \gamma \phi^{-1-\gamma} d\phi \right)^{\frac{1}{1-\sigma}} \\ &= \left(\sum_{k=1}^N w_k L_k \left(\frac{\sigma w_k \tau_{kj}}{\sigma-1} \right)^{1-\sigma} \gamma \int_{\bar{\phi}_{kj}}^{\infty} \phi^{-2+\sigma-\gamma} d\phi \right)^{\frac{1}{1-\sigma}} \end{aligned}$$

Now

$$\int_{\bar{\phi}_{kj}}^{\infty} \phi^{-2+\sigma-\gamma} d\phi = \left[\frac{1}{-2+\sigma-\gamma} \phi^{-1+\sigma-\gamma} \right]_{\bar{\phi}_{kj}}^{\infty}$$

Next recall that we assumed $\gamma \geq \sigma - 1$. At this point better put in the strict inequality.

Then

$$\begin{aligned} \int_{\bar{\phi}_{kj}}^{\infty} \phi^{-2+\sigma-\gamma} d\phi &= 0 - \frac{1}{-2+\sigma-\gamma} [\bar{\phi}_{kj}]^{-1+\sigma-\gamma} \\ &= \frac{1}{1+(\gamma-(\sigma-1))} \left[\lambda_1 \left(\frac{f_{kj}}{Y_j} \right)^{\frac{1}{\sigma-1}} \frac{w_k \tau_{kj}}{P_j} \right]^{-(\gamma-(\sigma-1))} \end{aligned}$$

Putting this into P_j yields

$$\begin{aligned} P_j &= \left(\sum_{k=1}^N w_k L_k \left(\frac{\sigma w_k \tau_{kj}}{\sigma-1} \right)^{1-\sigma} \gamma \int_{\bar{\phi}_{kj}}^{\infty} \phi^{-2+\sigma-\gamma} d\phi \right)^{\frac{1}{1-\sigma}} \\ &= \left(\sum_{k=1}^N \frac{w_k L_k \left(\frac{\sigma w_k \tau_{kj}}{\sigma-1} \right)^{1-\sigma} \gamma}{1+(\gamma-(\sigma-1))} \left[\lambda_1 \left(\frac{f_{kj}}{Y_j} \right)^{\frac{1}{\sigma-1}} \frac{w_k \tau_{kj}}{P_j} \right]^{-(\gamma-(\sigma-1))} \right)^{\frac{1}{1-\sigma}} \end{aligned}$$

or

$$P_j^{1-\sigma} = \sum_{k=1}^N \frac{w_k L_k \left(\frac{\sigma w_k \tau_{kj}}{\sigma-1} \right)^{1-\sigma} \gamma}{1+(\gamma-(\sigma-1))} \left[\lambda_1 \left(\frac{f_{kj}}{Y_j} \right)^{\frac{1}{\sigma-1}} w_k \tau_{kj} \right]^{-(\gamma-(\sigma-1))} P_j^{\gamma-(\sigma-1)}$$

or

$$\begin{aligned}
P_j^{-\gamma} &= \sum_{k=1}^N \frac{w_k L_k \left(\frac{\sigma w_k \tau_{kj}}{\sigma-1}\right)^{1-\sigma} \gamma}{1 + (\gamma - (\sigma - 1))} \left[\lambda_1 (f_{kj})^{\frac{1}{\sigma-1}} w_k \tau_{kj} \right]^{-(\gamma - (\sigma - 1))} Y_j^{\frac{\gamma - (\sigma - 1)}{\sigma - 1}} \\
P_j &= \tilde{\lambda} \left[\sum_{k=1}^N w_k L_k (w_k \tau_{kj})^{1-\sigma} \left[(f_{kj})^{\frac{1}{\sigma-1}} w_k \tau_{kj} \right]^{-(\gamma - (\sigma - 1))} \right]^{-\frac{1}{\gamma}} Y_j^{-\frac{1}{\gamma} \frac{\gamma - (\sigma - 1)}{\sigma - 1}} \\
&= \tilde{\lambda} \left[\sum_{k=1}^N w_k L_k (w_k \tau_{kj})^{-\gamma} (f_{kj})^{\frac{-(\gamma - (\sigma - 1))}{\sigma - 1}} \right]^{-\frac{1}{\gamma}} Y_j^{-\frac{1}{\gamma} \frac{\gamma - (\sigma - 1)}{\sigma - 1}}
\end{aligned}$$

Note that Y_k/Y will be proportionate to $w_k L_k$. So can stick this in here,

$$P_j = \tilde{\lambda} \left[\sum_{k=1}^N \frac{Y_k}{Y} (w_k \tau_{kj})^{-\gamma} (f_{kj})^{\frac{-(\gamma - (\sigma - 1))}{\sigma - 1}} \right]^{-\frac{1}{\gamma}} Y_j^{-\frac{1}{\gamma} \frac{\gamma - (\sigma - 1)}{\sigma - 1}}$$

Bottom line is that Chaney claims it is of the form ...

$$P_j = \lambda_2 \times Y_j^{1/\gamma - 1/(\sigma - 1)} \times \theta_j$$

for

$$\theta_j^{-\gamma} = \sum_{k=1}^N \left(\frac{Y_k}{Y} \right) \times (w_k \tau_{kj})^{-\gamma} \times f_{kj}^{-[\gamma/(\sigma - 1) - 1]}$$

where Y is world output and λ_2 is a constant. (I think we are close, but I am running out of time....)

The variable θ_j is “index of j ’s remoteness.” It is bigger the bigger is any iceberg transportation cost and the bigger is any f_{kj} .

The purpose of the rest of the paper is to get everything of interest in terms of θ_j and the Y_j and the exogenous parameters.

Let's look at the individual sales of a firm of type ϕ from i to j , assuming the firm is above the cutoff. From above,

$$\begin{aligned}
x_{ij}(\phi) &= \mu_h Y_j \left(\frac{p_{ij}(\phi)}{P_j} \right)^{1-\sigma} \\
&= \mu_h Y_j \left(\frac{\frac{\sigma}{\sigma-1} \frac{w_i \tau_{ij}}{\phi}}{P_j} \right)^{1-\sigma} \\
&= \mu_h Y_j \left(\frac{\frac{\sigma}{\sigma-1} \frac{w_i \tau_{ij}}{\phi}}{\lambda_2 \times Y_j^{1/\gamma-1/(\sigma-1)} \times \theta_j} \right)^{1-\sigma} \\
&= \mu_h Y_j Y_j^{(1-\sigma)/\gamma-1} \left(\frac{\frac{\sigma}{\sigma-1} \frac{w_i \tau_{ij}}{\phi}}{\lambda_2 \theta_j} \right)^{1-\sigma} \\
&= \lambda_3 Y_j^{(1-\sigma)/\gamma} \left(\frac{\theta_j}{w_i \tau_{ij}} \right)^{\sigma-1} \phi^{\sigma-1}
\end{aligned}$$

Next report:

$$\begin{aligned}
\bar{\phi}_{ij} &= \lambda_4 \times \left(\frac{Y}{Y_j} \right)^{\frac{1}{\gamma}} \times \left(\frac{w_i \tau_{ij}}{\theta_j} \right) \times f_{ij}^{1/(\sigma-1)} \\
Y_i &= (1 + \lambda_5) \times w_i L_i, \text{ for } \pi = \lambda_5 \text{ (solve for this as equilibrium variable?)}
\end{aligned}$$

Note that the elasticity of dollar value of trade flows on the intensive margin of a change in τ_{ij} for a given firm (holding everything else fixed, if just its own tariff was lower) is $\sigma - 1$. (This is the "traditional view".)

Instead look at aggregate flows, taking into account the extensive margin of who is selling.

Proposition 1 (Aggregate Trade)

Total exports f.o.b. (Free on board, means buyer pays the transportation cost. As this is an iceberg transportation cost model it means value at port of departure before ice melts.)

$$X_{ij}^h = \mu_h \times \frac{Y_i \times Y_j}{Y} \times \left(\frac{w_i \tau_{ij}^h}{\theta_i^h} \right)^{-\gamma h} \times (f_{ij}^h)^{-[\gamma h / (\sigma_h - 1) - 1]}$$

Note $\gamma_h > (1 - \sigma_h)$. So holding fixed the θ_i^h , see that elasticity is higher than firm level elasticity given firm is in. But as change τ_{ij}^h , also increase θ_i^h , so adds even more responsiveness. ■

(Proof just takes integral like before and plugs in $\bar{\phi}_{ij}^h$ and the Pareto density.)

Discussion

Anderson and Wincoop. Take a straight Krugman-style model with no heterogeneity. Then estimate the implied barriers at the U.S. and Canadian border. If $\sigma = 8$, the get an implied barrier of $\tau = 1.46$. Here given greater sensitivity, fixing $\sigma = 8$, and using γ from firm level data equal to $(\frac{\gamma}{\sigma-1} \approx 2)$, the same trade volume implies a trade barrier of $\tau = 1.21$. Chaney argues this is more plausible.