Econ 8602, Fall 2017

Homework 2

Due Tues Oct 3.

Question 1

Consider the following model of entry. There are two firms. There are two entry scenarios in each period. With probability η only one firm is able to produce, and if that event occurs, it is equally likely for this to be firm 1 or firm 2. With probability $1-\eta$ both firms are able to enter and produce in the period.

Let markets be indexed by m. There is heterogeneity across markets in fixed cost. Let ξ_ℓ be the market specific fixed cost for markets of type ℓ. Assume two ,market types, and $\xi_1 < \xi_2$. Let λ_ℓ be the population fraction of markets that are type ℓ.

There is also a fixed cost that depends upon whether a particular firm produced in the previous period. To describe the state of a particular firm it is therefore necessary to keep track of whether it is able to produce in the particular period, and whether the firm produced the previous period. Let $\omega_{it} \in \{0, 1, 2\}$ summarize firm i's state at time t, with $\omega_{it} = 0$ indicating the firm is unable to enter (implying $\omega_{-it} \neq 0$ for the other firm), and $\omega_{it} = 1$ indicating the firm has the option to produce this period, but did not the previous period, and $\omega_{it} = 2$ indicating the firm has the option to produce this period and did produce the previous period.

Let ϕ_ω be the component of fixed cost that depends upon whether the firm produced in the previous period. Assume $\phi_1 \geq \phi_2$.

Let π_n be operating profits per firm (i.e. excluding fixed cost) which depends upon the number n of firms that produce in the period.

Let $a_{it} \in \{0, 1\}$ indicate the action of firm i at time t, with $a_{it} = 0$ meaning the firm stays out in the period and $a_{it} = 1$ indicating entry in the period.

Finally, there is a stochastic component of profit, $\varepsilon_{it,a}$ for firm i at time t from choice of action a, that is drawn i.i.d. across firms and over time from the standard type 1 extreme value distribution. (Note, the c.d.f of this distribution is the double exponential $F(\varepsilon) = \exp(-\exp(-\varepsilon))$. Let $\boldsymbol{\varepsilon}_{it} = (\varepsilon_{it,0}, \varepsilon_{it,1})$ be the vector of random profits for firm i at time t.

Let $(\omega_{1t}, \omega_{2t}, \ell, \boldsymbol{\varepsilon}_{1t}, \boldsymbol{\varepsilon}_{2t})$ be the state of the industry in market m at time t. Note the market characteristic ℓ is fixed over time and is observed by both firms. The ω_{1t}, and ω_{2t} are publicly observed. The profit shocks are $\boldsymbol{\varepsilon}_{1t}, \boldsymbol{\varepsilon}_{2t}$ are the private information of each firm.

There are $K = 8$ different possible combinations of ω_1 and ω_2 and it is useful to index them by k as follows

k	(ω_1, ω_2)	$u_1 : a_1 = 0, a_2 \in \{0,1\}$	$u_1 : a_1 = 1, a_2 = 0$	$u_1 : a_1 = 1, a_2 = 1$
1	(0,1)	0	$-\infty$	$-\infty$
2	(0,2)	0	$-\infty$	$-\infty$
3	(1,0)	0	$\pi_1 - \phi_1 - \xi_\ell$	$\pi_2 - \phi_1 - \xi_\ell$
4	(1,1)	0	$\pi_1 - \phi_1 - \xi_\ell$	$\pi_2 - \phi_1 - \xi_\ell$
5	(1,2)	0	$\pi_1 - \phi_1 - \xi_\ell$	$\pi_2 - \phi_1 - \xi_\ell$
6	(2,0)	0	$\pi_1 - \phi_2 - \xi_\ell$	$\pi_2 - \phi_2 - \xi_\ell$
7	(2,1)	0	$\pi_1 - \phi_2 - \xi_\ell$	$\pi_2 - \phi_2 - \xi_\ell$
8	(2,2)	0	$\pi_1 - \phi_2 - \xi_\ell$	$\pi_2 - \phi_2 - \xi_\ell$

Let $u_1(k, a_1, a_2)$ gather together the deterministic components of profit for firm i at state k given actions a_1 and a_2. We tabulate this above. Note that to simplify notation, we allow firm 1 to enter if $\omega_{it} = 0$ but then give the firm a payoff of minus infinity so it never happens in equilibrium.

Restrict attention to Markov-perfect equilibria where the firms strategies are symmetric. In a Markov-perfect equilibrium, let q_k^ℓ denote the probability that firm 1 chooses $a_1 = 1$ given state k, and let $\bar{q}_k^\ell$ be the analogous probability for firm 2. Let $\mathbf{q}^\ell$ and $\bar{\mathbf{q}}^\ell$ be the corresponding vectors of entry probabilities. Let $p_{k,k'}^\ell$ be the probability the state is k' next period, given it is k in the current period and the market type is ℓ. Let $\mathbf{P}^\ell$ be the transition matrix.

Let $v_k^\ell(a)$ be the choice-specific value function firm 1 in the Markov-perfect equilibrium, excluding the choice-specific random profit component. This is the expected return to firm 1 from picking action $a \in \{0,1\}$ given state k, excluding ε_a. Let the ex ante value of state k to firm 1 be

$$\tilde{v}_k^{\ell} = E \max \left\{ v_k^{\ell}(0) + \varepsilon_0, v_k^{\ell}(1) + \varepsilon_1 \right\},$$

where this expectation is taken with respect to the random profit draws ε_0 and ε_1.

Finally, denote the discount factor by β.

(a) For the first set of questions, we hold market type ℓ fixed, so for simplicity leave the ℓ index implicit. Use the above notation to define a symmetric Markov perfect equilibrium in this model.

(b) Suppose you are given $\mathbf{q}$ and $\bar{\mathbf{q}}$ and $\mathbf{P}$. Derive an analytical expression for the choice-specific value functions $v_k^{\ell}(a)$. (Hint: If we condition on the state k (8 possibilities) and the action choice of firm 1, $a_1 \in \{0, 1\}$ (two possibilities) there a total of 16 possibilities. Suppose we index these state/firm-1 choice's by $j \in \{1, 2, ..., 16\}$. Let $\mathbf{H}$ be the transition probability matrix from j to j' in the Markov-perfect equilibrium. This can be calculated from $\mathbf{q}$ and $\bar{\mathbf{q}}$ and $\mathbf{P}$. Derive an analytic expression for the discounted amount of time spent at a given j' in future periods, given the current j. Note in class we discussed using simulation to do this, but here I am looking for an expression with matrix algebra.)

(c) Suppose $\pi_1 = 2$, $\pi_2 = 0$, $\phi_1 = 1$, $\phi_2 = 0$, $\eta = 0.5$, and $\beta = 0.5$.

Regarding the market level heterogeneity, assume two market types, $\xi_1 = 0$ and $\xi_2 = 4$. Suppose equal shares for the two types, $\lambda_{\ell} = 0.5$, $\ell \in \{1, 2\}$.

(i) Solve for the equilibrium for the two market types, and calculate $\mathbf{q}^{\ell}$ and $\bar{\mathbf{q}}^{\ell}$ and $\mathbf{P}^{\ell}$. Make a table of $\mathbf{q}^{\ell}$ for $\ell \in \{1, 2\}$.

(ii) Now let's say you have collected data generated by this model and assume you can directly observe the type ℓ of market, as well as the state k and the firm actions. Suppose you use the data to calculate estimates of $\mathbf{q}^{\ell}$ and $\bar{\mathbf{q}}^{\ell}$ and $\mathbf{P}^{\ell}$, and for simplicity, assume your estimates exactly equal what you obtained in part (i) when you solved the model (which is what you estimate would be in the limit when the number of observations get large). Use the partial solution approach to solve for the model parameters, given $\mathbf{q}^{\ell}$ and $\bar{\mathbf{q}}^{\ell}$ and $\mathbf{P}^{\ell}$. You will have to make a normalization here, so set $\phi_2 = 0$. Take the estimate of $\eta = 0.5$ as direct from the data. Take $\beta = 0.5$ as known. It isn't necessary to report your results for π_1, π_2 and ϕ_1, as you should get back the original parameter set.

(iv) Now suppose you do not take into account market heterogeneity. In particular, suppose you mistakenly assume that all the data is being generated by markets where $\xi = 0$. Re-estimate π_1, π_2 and ϕ_1 and in particular calculate $\pi_1 - \pi_2$, and do report these estimates. In what way are the results biased when unobserved heterogeneity is not taken into account?

(v) Briefly outline how you might estimate the model , taking the unobserved heterogeneity into account. Assume that the support of the mixture ξ_1 and ξ_2 is known, and that the distribution parameter λ_ℓ is an unknown parameter to be estimated.

Question 2. Consider the Logit Model of Product Differentiation (For background related to some of the tasks for this question, you can look at Anderson and De Palma, "The Logit as a Model of Product Differentiation," *Oxford Economic Papers* 44 (1992), 51-57.)

Suppose there are n firms plus an "outside good" labeled by 0. Each firm has constant marginal cost equal to c. The is a measure M of consumers. Let i index an individual consumer and suppose the utility of consumer i from purchasing good j is

$$\begin{aligned} U_{i,j} &= \xi - \alpha p_j + \varepsilon_{i,j} \text{ for } j = 1, 2, ...n \\ &= \varepsilon_{i,0} \text{ for good } 0. \end{aligned}$$

Note the parameters ξ and α are constant across the n firms and across consumers, so the firms are symmetric. It is convenient to write the utility has having two parts

$$U_{i,j} = \delta_j + \varepsilon_{ij}$$

(where $\delta_j = \xi - \alpha p_j$ for $j \geq 1$ and $\delta_0 = 0$). The first part δ_j is common to all consumers. The second part is idiosyncratic, capturing random reasons why one consumer i might get value product j. Assume the ε_{ij} are drawn i.i.d. from the type 1 extreme value distribution. It can be shown that the probability of drawing a vector $\varepsilon_i = (\varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3}, ...\varepsilon_{in})$ so that

$$U_{i,j} \geq U_{i,k}, \text{ for } k \neq j \tag{1}$$

is

$$S_j(p_1, p_2, ...p_n) = \frac{\exp(\delta_j)}{1 + \sum_{k=1}^{n} \exp(\delta_k)}, \tag{2}$$

where the δ_j are implicitly functions of the prices. The event (1) is the event that good j provides the consumer the highest utility over of all the choices. Given the continuum of consumers, this is the share of consumers that will select option j. Hence, the quantity of sales of firm j, given the vector of price is

$$q_j(p_1, p_2, ...p_n) = M \times S_j(p_1, p_2, ...p_n)$$

(a) Calculate the slope $\frac{\partial S_j}{\partial p_j}$ and write it in convenient way in terms of S_j.

(b) Suppose the n firms compete in a Bertrand fashion. Set up the problem of firm 1 given the choices of the remaining firms $p_2, p_3, ...p_n$. Derive the first-order necessary condition.

(c) Define a symmetric Bertrand equilibrium.

(d) There exists a symmetric Bertrand price equilibrium $p^e(n)$ that depends upon the number of firms. Derive the equation characterizing this price. Show a price solving this equation exists.

(e) Consider the numerical example where $\alpha = 1$, $c = 1$, $\xi = 1$ and $n = 5$. Plot on the same graph the following functions of price:

$$\begin{aligned} f_1(p) &= p - c \\ f_2(p) &= \frac{1}{\alpha}\frac{1}{1-\tilde{S}(p)} \end{aligned}$$

where $\tilde{S}(p) = S_1(p, p, p, ..p)$ (the representative firm share when all price the same.) What does this graph tell you about existence and uniqueness of the Bertand price equilibrium?

(f) Now make the number of firms n endogenous. Suppose there is a fixed cost ϕ to enter the industry. Suppose there is a two stage game. In stage 1, $n \geq 0$ firms enter the industry. In stage 2 the n firms play a simultaneous move Bertrand price game. We are interested in subgame perfect Nash equilibrium. Suppose n^e is an equilibrium entry level for this game. What condition must it solve?

(g) Set $M = 1$. Determine the interval of fixed costs $[\underline{\phi},\bar{\phi}]$ such that $n^e = 5$ is the equilibrium with free entry in the numerical example of part (f).

(h) Using the δ_j notation above, the formula for consumer surplus for the logit model is (Small and Rosen, *Econometrica,* 1981)

$$CS = M \ln \sum_{j=0}^{n} \exp(\delta_j)$$

Consider the following social planner problem. The social planner picks an integer n in the first stage. Then in stage 2, the firms engage in Bertrand competition to maximize profits. Suppose the social planner chooses n to maximize the sum of CS plus total profit (where profit nets out the fixed cost). Over what range of fixed costs $[\underline{\phi},\bar{\phi}]$ is the social planner's solution equal to $n^* = 5$? How does this compare with the range of fixed cost for $n^e = 5$ in the market allocation that you determined in part (g). (Note: the social planner is picking an integer, so your solution should not include differentiating with respect to n.)