Notes on Shares with CES and Frechet

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1 Description of the Model

There are i types of land use, $i \in \{1, ..., I\}$. Take as exogenous for now the count M_i of agents of each type. For simplicity, for now we refer to an agent as a firm.

There are J locations indexed by J. Let S^j be the measure of floor space at location j. (For the purposes here, take this as exogenous)

Let k be used to indicate a particular firm. Each firm k of type i gets a productivity draw for each location j, denoted by z_{ik}^j.

$$F_i^j(z_{ijk}) = e^{-T_{ij}^j \left(z_{ik}^j\right)^{-\theta}},$$

Let M_i^j be the measure of agents of type i locating at j.

All firms employ utilize floor space s and labor n in production.

There are two components to productivity that vary across locations. Let A^j be a Hicks neutral productivity at location j. We take this as exogenous. Let B^j be a labor-specific productivity term that varies across location.

The output of firm k in industry i locating at j is

$$y = z_{ijk} A^j \left[\alpha_i s^{\rho_i} + (1 - \alpha_i)\left(B^j n\right)^{\rho_i}\right]^{\frac{1}{\rho_i}\gamma_i}$$

where the elasticity of substitution between s and n for industry i is given by

$$\sigma_i = \frac{1}{1 - \rho_i}.$$

Let p_n^j be the price of labor in efficiency unit at location j. So if the nominal wage is w^j, then

$$p_n^j = \frac{w^j}{B^j}.$$

2 Input Choice Problem

Take as given a particular agent k of type i locating in j, with idiosyncratic productivity z_{ik}^j and where the productivity is A^j. For simplicity, we drop all subscripts. Since z and A enter multiplicatively the same way, set $z = 1$ for now, and solve the problem for productivity level A. Then later we can substitute in $z_{ik}^j A^j$ for A.

Let p_n, p_ℓ, and p_s be the prices of labor, land, and floor space.

Next the firm solves

$$\max_{(s,n)} A \left[\alpha s^\rho + (1-\alpha)\, n^\rho\right]^{\frac{1}{\rho}\gamma} - p_s s - p_n n,$$

note that are are defining n in terms of efficiency units here, so we leave out B. Define m by

$$p_m = \min p_s s + p_n n$$

subject to

$$\left[\alpha s^\rho + (1-\alpha)\, n^\rho\right]^{\frac{1}{\rho}} = 1$$

Then we can rewrite the problem as

$$\max_m A m^\gamma - p_m m$$

2.1 Calculations for CES

$$\begin{aligned}
\frac{p_s}{p_n} &= \frac{\frac{1}{\rho}\left[\alpha s^\rho + (1-\alpha)\, n^\rho\right]^{\frac{1}{\rho}-1} \alpha\rho s^{\rho-1}}{\frac{1}{\rho}\left[\alpha s^\rho + (1-\alpha)\, n^\rho\right]^{\frac{1}{\rho}-1} (1-\alpha)\,\rho n^{\rho-1}} \\
&= \frac{\alpha s^{\rho-1}}{(1-\alpha)\, n^{\rho-1}} \\
\left(\frac{p_s}{p_n}\right)^{-\frac{1}{1-\rho}} \left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{1-\rho}} &= \frac{s}{n}
\end{aligned}$$

Also

$$\begin{aligned}
\left[\alpha s^\rho + (1-\alpha)\, n^\rho\right]^{\frac{1}{\rho}} &= 1 \\
\left[\alpha \left(\frac{p_s}{p_n}\right)^{-\frac{\rho}{1-\rho}} \left(\frac{\alpha}{1-\alpha}\right)^{\frac{\rho}{1-\rho}} n^\rho + (1-\alpha)\, n^\rho\right]^{\frac{1}{\rho}} &= 1
\end{aligned}$$

So

$$\begin{aligned}
\left[\alpha \left(\frac{p_s}{p_n}\right)^{-\frac{\rho}{1-\rho}} \left(\frac{\alpha}{1-\alpha}\right)^{\frac{\rho}{1-\rho}} + (1-\alpha)\right]^{\frac{1}{\rho}} n &= 1 \\
n &= \left[\alpha \left(\frac{p_s}{p_n}\right)^{-\frac{\rho}{1-\rho}} \left(\frac{\alpha}{1-\alpha}\right)^{\frac{\rho}{1-\rho}} + (1-\alpha)\right]^{-\frac{1}{\rho}}
\end{aligned}$$

By symmetry

$$s = \left[(1-\alpha) \left(\frac{p_s}{p_n}\right)^{\frac{\rho}{1-\rho}} \left(\frac{\alpha}{1-\alpha}\right)^{-\frac{\rho}{1-\rho}} + \alpha\right]^{-\frac{1}{\rho}}$$

Is is straightforward to substitute these in to obtain the price index,

In summary

$$p_m = \left[\alpha^{\frac{1}{1-\rho}} p_s^{\frac{-\rho}{1-\rho}} + (1-\alpha)^{\frac{1}{1-\rho}}\, p_n^{\frac{-\rho}{1-\rho}}\right]^{-\frac{1-\rho}{\rho}}$$

2.2 Choice of m

Consider

$$\max_m A m^\gamma - p_m m$$

The FONC is

$$A\gamma m^{-(1-\gamma)} - p_m = 0$$

Solve for m,

$$\begin{aligned}
m^{-(1-\gamma)} &= \frac{p_m}{A\gamma} \\
m &= \left(\frac{p_m}{A\gamma}\right)^{-\frac{1}{1-\gamma}} \\
&= p_m^{-\frac{1}{1-\gamma}} (A\gamma)^{\frac{1}{1-\gamma}}
\end{aligned}$$

Plug it in to get the value

$$\begin{aligned}
v &= A\left[p_m^{-\frac{1}{1-\gamma}} (A\gamma)^{\frac{1}{1-\gamma}}\right]^{\gamma} - p_m p_m^{-\frac{1}{1-\gamma}} (A\gamma)^{\frac{1}{1-\gamma}} \\
&= A p_m^{-\frac{\gamma}{1-\gamma}} (A\gamma)^{\frac{\gamma}{1-\gamma}} - p_m^{\frac{-\gamma}{1-\gamma}} (A\gamma)^{\frac{1}{1-\gamma}} \\
& A^{\frac{1}{1-\gamma}} \gamma^{\frac{\gamma}{1-\gamma}} p_m^{-\frac{\gamma}{1-\gamma}} - A^{\frac{1}{1-\gamma}} \gamma^{\frac{1}{1-\gamma}} p_m^{-\frac{\gamma}{1-\gamma}} \\
v &= \left[\gamma^{\frac{\gamma}{1-\gamma}} - \gamma^{\frac{1}{1-\gamma}}\right] A^{\frac{1}{1-\gamma}} p_m^{-\frac{\gamma}{1-\gamma}}
\end{aligned}$$

Can we write m as a function of v?

$$m = p_m^{-\frac{1}{1-\gamma}} A^{\frac{1}{1-\gamma}} \gamma^{\frac{1}{1-\gamma}}$$

and

$$\begin{aligned}
v &= \left[\gamma^{\frac{\gamma}{1-\gamma}} - \gamma^{\frac{1}{1-\gamma}}\right] A^{\frac{1}{1-\gamma}} p_m^{-\frac{\gamma}{1-\gamma}} \\
&= p_m^{-\frac{1}{1-\gamma}} A^{\frac{1}{1-\gamma}} \gamma^{\frac{1}{1-\gamma}} \times p_m^{\frac{1}{1-\gamma}} A^{-\frac{1}{1-\gamma}} \gamma^{-\frac{1}{1-\gamma}} \times \left[\gamma^{\frac{\gamma}{1-\gamma}} - \gamma^{\frac{1}{1-\gamma}}\right] A^{\frac{1}{1-\gamma}} p_m^{-\frac{\gamma}{1-\gamma}} \\
&= m\left[\frac{1-\gamma}{\gamma}\right] p_m
\end{aligned}$$

Or

$$m = \frac{v}{p_m}\frac{\gamma}{1-\gamma}$$

Next look at demand for space. Recall space per unit demand is

$$s_1 = \left[(1-\alpha) \left(\frac{p_s}{p_n} \right)^{\frac{\rho}{1-\rho}} \left(\frac{\alpha}{1-\alpha} \right)^{-\frac{\rho}{1-\rho}} + \alpha \right]^{-\frac{1}{\rho}}$$

Therefore

$$s = \left[(1-\alpha) \left(\frac{p_s}{p_n} \right)^{\frac{\rho}{1-\rho}} \left(\frac{\alpha}{1-\alpha} \right)^{-\frac{\rho}{1-\rho}} + \alpha \right]^{-\frac{1}{\rho}} \frac{1}{p_m} \frac{\gamma}{1-\gamma} v$$

Analogously

$$n = \left[\alpha \left(\frac{p_s}{p_n} \right)^{-\frac{\rho}{1-\rho}} \left(\frac{\alpha}{1-\alpha} \right)^{\frac{\rho}{1-\rho}} + (1-\alpha) \right]^{-\frac{1}{\rho}} \frac{1}{p_m} \frac{\gamma}{1-\gamma} v$$

3 Frechet Calculations

Suppose a variable z is distributed Frechet

$$F(z) = e^{-Tz^{-\theta}},$$

For arbitrary parameters $\tilde{\zeta} > 0$ and $\eta > 0$ define a transformation

$$x = \tilde{\zeta} z^{\eta}$$

We can solve out for z.

$$\begin{aligned} z^{\eta} &= \frac{x}{\tilde{\zeta}} \\ z &= \tilde{\zeta}^{-\frac{1}{\eta}} x^{\frac{1}{\eta}} \end{aligned}$$

The c.d.f of the random variable x is then

$$\tilde{F}(x) = F(z(x)) = e^{-T\tilde{\zeta}^{\frac{\theta}{\eta}} x^{-\frac{\theta}{\eta}}}$$

Thus the variable just transformed is also Frechet with parameters

$$\begin{aligned} \tilde{T} &= T\tilde{\zeta}^{\frac{\theta}{\eta}} \\ \tilde{\theta} &= \frac{\theta}{\eta} \end{aligned}$$

Now return to variable z which again is Frechet with T and θ.

$$F(z) = e^{-Tz^{-\theta}},$$

Let

$$z^* = \max\left\{z^1, z^2, ...z^J\right\} \tag{1}$$

Let $F^*(z)$ be the c.d.f of z^*. This equals

$$F^*(z) = \prod_{j=1}^{J} e^{-T^j z^{-\theta}} = e^{-\bar{T}z^{-\theta}}$$

for

$$\bar{T} = T^1 + T^2 + ...T^J$$

What is the density?

$$f^*(z) = e^{-\bar{T}z^{-\theta}}\bar{T}\theta z^{-\theta-1}$$

The expected value is

$$E[z^*] = \bar{T}\Gamma(1 - \frac{1}{\theta})$$

where Γ is the gamma function.

Next note that

$$E\left[z^j | z^j = z^*\right] = E[z^*]$$

Finally, we use the above to calculate $E[z_{ik}^j]$ and expectation of powers of z_{ik}^j Firm k's of type i location choice problem is

$$v^* = \max_j \tilde{\zeta}_i^j \left(z_{ik}^j\right)^\eta$$

$$\eta \equiv \frac{1}{1-\gamma}$$
$$\tilde{\zeta}_i^j \equiv \left[\gamma^{\frac{\gamma}{1-\gamma}} - \gamma^{\frac{1}{1-\gamma}}\right] \left(A^j\right)^{\frac{1}{1-\gamma}} \left(p_{i,m}^j\right)^{-\frac{\gamma}{1-\gamma}}$$

Then define

$$\tilde{T}_i^j = T_i^j \left(\tilde{\zeta}_i^j\right)^{\frac{\theta}{\eta}}$$
$$\tilde{\theta} = \frac{\theta}{\eta}$$

and

$$\tilde{T}_i = \sum_{j=1}^{J} \tilde{T}_i^j$$

So then

$$\tilde{z}_j = \tilde{\zeta}_i^j \left(z^j\right)^{\eta}$$

Then

$$E\left[\tilde{z}_{ik}^j | \tilde{z}_{ik}^j = \tilde{z}_{ik}^*\right] = E[\tilde{z}_{ik}^*]$$
$$= \bar{T}\Gamma(1 - \frac{1}{\tilde{\theta}})$$

4 Floor Space Share

What is the expected quantity of floor space, per firm of type i locating at j?

Note, on account of the Frechet, fixing i,

$$E\left[v|\text{choose } j\right] = \bar{v}$$

Note also that

$$m = \frac{v}{p_m}\frac{\gamma}{1-\gamma}$$

So

$$Em^j = \frac{1}{p_m^j}\frac{\gamma}{1-\gamma}E[v|\text{choose } j] = \frac{1}{p_m^j}\frac{\gamma}{1-\gamma}\bar{v}$$

Now $\bar{v}$ is a constant across j, so let's leave the expression in these terms

Let's leave i and j implicit to start.

$$Em = \frac{1}{p_m}\frac{\gamma}{1-\gamma}\bar{v}$$

This is amount of intermediate, and

$$\begin{aligned} Es &= \left[(1-\alpha)\left(\frac{p_s}{p_n}\right)^{\frac{\rho}{1-\rho}}\left(\frac{\alpha}{1-\alpha}\right)^{-\frac{\rho}{1-\rho}}+\alpha\right]^{-\frac{1}{\rho}} Em \\ &= \left[(1-\alpha)\left(\frac{p_s}{p_n}\right)^{\frac{\rho}{1-\rho}}\left(\frac{\alpha}{1-\alpha}\right)^{-\frac{\rho}{1-\rho}}+\alpha\right]^{-\frac{1}{\rho}} \frac{1}{p_m^j}\frac{\gamma}{1-\gamma}\bar{v} \end{aligned}$$

So the quantity of land equals

Recall that

$$p_m = \left[\alpha^{\frac{1}{1-\rho}} p_s^{\frac{-\rho}{1-\rho}} + (1-\alpha)^{\frac{1}{1-\rho}} p_n^{\frac{-\rho}{1-\rho}}\right]^{-\frac{1-\rho}{\rho}}$$

We can use this to derive

$$\left[(1-\alpha)\left(\frac{p_s}{p_n}\right)^{\frac{\rho}{1-\rho}}\left(\frac{\alpha}{1-\alpha}\right)^{-\frac{\rho}{1-\rho}}+\alpha\right]^{-\frac{1}{\rho}} = \alpha^{\frac{1}{1-\rho}} p_s^{\frac{-1}{1-\rho}} p_m^{\frac{1}{1-\rho}}$$

Therefore quantity of floor space per firm at the location is

$$\begin{aligned} & p_s^{-\frac{1-\lambda}{\lambda}} (1-\lambda)^{-\frac{1-\lambda}{\lambda}} \alpha^{\frac{1}{1-\rho}} p_s^{\frac{-1}{1-\rho}} p_m^{\frac{1}{1-\rho}} \frac{1}{p_m}\frac{\gamma}{1-\gamma}\bar{v} \\ = \; & (1-\lambda)^{-\frac{1-\lambda}{\lambda}} \alpha^{\frac{1}{1-\rho}} \frac{\gamma}{1-\gamma} p_s^{-\frac{1-\lambda}{\lambda}-\frac{1}{1-\rho}} p_m^{\frac{\rho}{1-\rho}} \end{aligned}$$

5 Exercise

Let S_i^j denote the total volume of floor space occupied at location j by firms in industry i. Derive an expression for

$$share_i^j = \frac{S_i^j}{\sum_{j'=1}^{J} S_i^{j'}}$$