Arcidiacono, Bayer, Blevins, Ellickson

- Discussion of Arcidiacono and Miller (2011) of a way to do Rust bus problem

- Use $d_{1t} = 1$ replace at or $d_{2t} = 1$ if don't replace bus engine

- State x_t is age, s other characteristics

- Choice specific value function (net of ε_{jt})

$$\begin{aligned} v_j(x, s) &= \beta V(0, s), \text{ if } j = 1 \\ &= \theta_1 x + \theta_2 s + \beta V(x+1, s), \text{ if } j = 2 \end{aligned}$$

- Probability of replacement

$$p_1(x,s) = \frac{1}{1 + \exp\left[v_2(x,s) - v_1(x,s)\right]}$$

We know the choice-specific value function can be rewritten as

$$v_2(x,s) = \theta_1 x + \theta_2 s + \beta \ln\left[\exp(v_1(x+1,s) + \exp(v_2(x+1,s)\right] + \beta\gamma$$

which we can rewrite as

$$\begin{aligned} v_2(x,s) &= \theta_1 x + \theta_2 s + \beta \ln\left[\exp(v_1(x+1,s)\left[1 + \frac{\exp(v_2(x+1,s))}{\exp(v_1(x+1,s))}\right]\right] + \beta\gamma \\ &= \theta_1 x + \theta_2 s + \beta \ln\left[\exp(v_1(x+1,s)\left[1 + \exp(v_2(x+1,s) - v_1(x+1,s\right.\right.\right. \\ &= \theta_1 x + \theta_2 s + \beta \ln\left[\frac{\exp(v_1(x+1,s)}{p_1(x+1,s)}\right] + \beta\gamma \\ &= \theta_1 x + \theta_2 s + \beta v_1(x+1,s) - \ln p_1(x+1,s) + \beta\gamma \end{aligned}$$

Analogously

$$v_1(x,s) = \beta v(0,s) - \ln p_1(0,s) + \beta\gamma$$

Also

$$v_1(x+1, s) = v_1(0, s) = \beta V(0, s)$$

So difference expressionf for $v_2(x, s)$ and $v_1(x, s)$ to get

$$v_2(x, s) - v_1(x, s) = \theta_1 x + \theta_2 s + \ln p_1(0, s) - \ln p_1(x+1, s)$$

Now the likelihood of the data

$$\frac{d_{1t} + d_{2t} \exp(\theta_1 x + \theta_2 s + \ln p_1(0, s) - \ln p_1(x+1, s))}{1 + \exp(\theta_1 x + \theta_2 s + \ln p_1(0, s) - \ln p_1(x+1, s))}$$

- Notice a key step. Can get to same place, from any state (x big, can choose to get $x = 0$ tomorrow)

- Can generalize if do something that gets you some place where the next time you can get there.

Model with continuous time

- Time $t \in [0, \infty)$

- State is an element $k \in X = \{1, 2, ...K\}$

- finite state Markov jump process on X with $K * K$ intensity matrix Q_0 governs moves by nature

- X_t current, transition to X_{t+1} at a random time τ, exponentially distributed

$$Q = \begin{pmatrix} q_{11} & q_{12} & \\ & & \\ & & q_{KK} \end{pmatrix}$$

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$$q_{kl} = \lim_{h \to 0} \frac{\Pr\left(X_{t+h} = l | X_t = k\right)}{h}$$

hazard of k to l and

$$q_{kk} = -\sum_{l \neq k} q_{kl}$$

- Poisson arrival process λ governs when agent can move
    - agent chooses among J alternatives from $A = \{0, ..., J-1\}$

- transitions out of state k follow exponential distribution with rate parameter $-q_{kk}$, and conditional on leaving k transitions to $l \neq k$ with probability $p_{kl}/\sum_{l' \neq k} q_{kl'}$.

- Agent's problem
  - discounts future payoffs at rate ρ
  - While in state k, gets u_k
  - At rate λ agent makes a decision, choosing action $j \in A$, receiving instantaneous payoff $\psi_{jk} + \varepsilon_j$,
  - σ_{jk} probability agent chooses j in state k.
  - Can lead to deterministic state change, $l(j,k)$ denote state that results upon choice j in state k

Instantaneous Bellman equation

- For time increment h, probability of event with rate λ is λh

- Given ρ, discount factor is $1/(1+\rho h)$

$$V_k = \frac{1}{1+\rho h}\left[\begin{array}{c} u_k h + \sum_{l\neq k} q_{kl} h V_l + \lambda h E \max_j \left\{\psi_{jk} + \varepsilon_j + V_{l(j,k)}\right\} \\ + \left(1 - \lambda h - \sum_{l\neq k} q_{kl}\right) V_k + o(h) \end{array}\right]$$

- Rearranging and setting $h \to 0$, we get

$$V_k = \frac{u_k + \sum_{l\neq k} q_{kl} V_l + \lambda E \max_j \left\{\psi_{jk} + \varepsilon_j + V_{l(j,k)}\right\}}{\rho + \lambda + \sum_{l\neq k} q_{kl}}$$

Or

$$\rho V_k = u_k + \sum_{l\neq k} q_{kl}\left(V_l - V_k\right) + \lambda E \max_j \left\{\psi_{jk} + \varepsilon_j + V_{l(j,k)} - V_k\right\}$$

- Policy rule assigns to each state k and $\varepsilon = (\varepsilon_0, \varepsilon_1, ..\varepsilon_{J-1})$ the action which maximizes payoff

- CCP

$$\sigma_{jk} = \Pr\left(\delta(k, \varepsilon) = j|k\right)$$

- λ and σ_{jk} imply a jump process on X with intensity matrix Q_1

- Summing $Q = Q_0 + Q_1$

Single Agent Renewal

- Single state: miles on bus. q_{k1} and q_{k2} be rates at which one unit and two-unit increments occur

- with arrival of move, binary choice $j = 1$ or $j = 0$

- If set to $k = 1$, pay replacement cost

$$V_k = \frac{u_k + q_{k1}V_{k+1} + q_{k2}V_{k+2} + \lambda E \max\{V_k + \varepsilon_0, V_0 + c + \varepsilon_1\}}{\rho + q_{k1} + q_{k2} + \lambda}$$

where in earlier notation

$$\begin{aligned} \psi_{jk} &= 0,\ j = 0 \\ &= c,\ \text{if } j = 1. \end{aligned}$$

CCP Representation

- Primary difference: rather than state changes and choices made simultaneously at predetermined intervals, only one event occurs at any given instant almost surely.

- Show insights of Hotz and Hiller, etc on expressing value functions as CCP apply

- Assumptions

  - 1. $\rho > 0$

  - 2. choice specific shocks ε are iid over time

- Prop 1: The value function can be written

$$V(\sigma) = \frac{1}{[(\rho+\lambda)\,I - \lambda\boldsymbol{\Sigma}(\sigma) - Q_0]}\,[u + \lambda E(\sigma)]$$

where $E(\sigma)$ is is the $K \times 1$ $vector$ containing

$$\sum_j \sigma_{jk}[\psi_{jk} + e_{jk}(\sigma)]$$

where $e_{jk}(\sigma)$ is the expected value of the ε given choice j is optimal.

- Proof: write the value function in matrix form ($\tilde{Q}$ replace diagonal with zeros).

$$\left[(\rho+\lambda)\,I - \left(Q_0 - \tilde{Q}_0\right)\right] V(\sigma) = u + \tilde{Q}_0 V(\sigma) + \lambda\,[\boldsymbol{\Sigma}(\sigma)V(\sigma) + E(\sigma)]$$

solve this linear equation

- Prop 2: There exists a function $\Gamma^1(j, j', \sigma_k)$ such that

$$V_{l(j,k)} = V_{l(j',k)} + \psi_{j'k} - \psi_{jk} + \Gamma^1(j, j', \sigma_k)$$

- For ε standard type 1 extreme value we have seen this already works, as Γ^1 is

$$\Gamma^1(j, j', \sigma_k) = \ln\left(\sigma_{jk}\right) - \ln\left(\sigma_{j'k}\right)$$

- Prop 3

$$E \max_j \left\{\psi_{jk} + \varepsilon_j + V_{l(j,k)}\right\} = V_{l(j',k)} + \psi_{l(j',k)} + \Gamma^2(j', \sigma_k)$$

- ε type I extreme value (intuition go to board for static case)

- Prop 2 allows links of value functions across states
  - let action 0 be a continuation action that does not change the state $l(0,k) = k$ and $\psi_{0k} = 0$.
  - If in state k can move to k' by taking action j' , and to k' to k'' by taking j'' then

$$\begin{aligned} V_k &= V_{k'} + \psi_{j',k} + \Gamma^1(0, j', \sigma_k) \\ &= V_{k''} + \psi_{j'',k'} + \psi_{j',k} + \Gamma^1(0, j'', \sigma_{k'}) + \Gamma^1(0, j', \sigma_k) \end{aligned}$$

  - keep doing this, collecting all terms involving V_k yields an expression for V_k in terms of the flow payoff of state k and the conditional choice probabilities.

- Def: a state k^* is attainable from state k if there exists a sequence of actions from k that result in k^*

- Prop 4: Suppose further for a given k, $j = 0$ is a continuation action with $l(0, k) = k$, and all states $l \neq k$ with $q_{kl} > 0$ there exists a state k^* that is attainable from both k and l. Then there exists a function $\Gamma_k(\psi, Q_0, \lambda, \sigma)$ such that

$$\rho V_k = u_k + \Gamma_k(\psi, Q_0, \lambda, \sigma)$$

Example Single Agent Renewal

- Recall

$$V_k = \frac{u_k + q_{k1}V_{k+1} + q_{k2}V_{k+2} + \lambda E \max\{V_k + \varepsilon_0, V_0 + c + \varepsilon_1\}}{\rho + q_{k1} + q_{k2} + \lambda}$$

Apply Prop 3

$$E \max_j \left\{\psi_{jk} + \varepsilon_j + V_{l(j,k)}\right\} = V_{l(j',k)} + \psi_{l(j',k)} + \Gamma^2(j', \sigma_k)$$

$$\begin{aligned} V_k &= \frac{u_k + q_{k1}V_{k+1} + q_{k2}V_{k+2} + \lambda V_k + \lambda\Gamma^2(0, \sigma_k)}{\rho + q_{k1} + q_{k2} + \lambda} \\ &= \frac{u_k + q_{k1}V_{k+1} + q_{k2}V_{k+2} + \lambda\Gamma^2(0, \sigma_k)}{\rho + q_{k1} + q_{k2}} \end{aligned}$$

- No direct link between value function at k and $k+1$. But can link through the replacement decision

$$\begin{aligned} V_k &= V_0 + c + \Gamma^1(0, 1, \sigma_k) \\ V_{k+1} &= V_0 + c + \Gamma^1(0, 1, \sigma_{k+1}) \end{aligned}$$

so

$$V_{k+1} = V_k + \Gamma^1(0, 1, \sigma_{k+1}) - \Gamma^1(0, 1, \sigma_k)$$

Game

- Key step: "Estimating the other value functions, however, is problematic as each play may only be able to move the process to some subset of the state space via a unilateral action, sine they only have direct control over their own state."

- Important: in models with a terminal choice, such as a firm permanently existing a market, that state the value of the terminal choice does not include other values functions.

Literature, progression of literature

- Static, deterministic, Bresnahan and Reiss (1991), Berry (1992), (solve for equilibrium N entrants)

- Static, but ε shocks. (Brock and Durlauf (2001) general social interactions), Seim (Rand 2006)

- Dynamic and Stochastic, Aguirregabira and Mira, Bajari, Benkard, and Levin, Pakes, Ovstrovsky, and Berry, Pesondorfer and Schmitd-Dengler

Wal-Mart

- Jia (2008), static and deterministic. But allow complementarities in cost. State space blowing up, in terms of calculation solution to firm's problem. But had a nice result about supermodularity

- Holmes (2011) adds complementaries in costs.

- ABBE
  - no complementarities in cost
  - dynamic and stochastic
  - asymmetric

* Wal-Mart
* Chains (can have up to 7 different ones)
* Independent Grocers (just have one)

Overview

- Estimate CCP

- Turn it into structural parameters?

  - assumption that have terminal state assumed for chains and independents, so use CCP approach to estimate structural parameters

  - don't get structural parameters of Wal-Mart

- Counterfactual: No Wal-Mart! So don't need structural parameters