Notes on Proofs for
"Bar Codes Lead to Frequent Deliveries and Superstores"
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1 Proof of Proposition 3

Note the proposition proved here includes the additional result for the $\lambda = \frac{\gamma}{2}$ case mentioned in the footnote in the text.

Proposition 3.

(i) Binary Demand and General λ. Assume $n = 2$. Assume demand is discrete and binary, $z \in \{0, 1\}$. Then $H_{2,3} \leq 0$ and the strict inequality holds for λ close enough to γ.

(ii) Continuous Demand and Small λ. Assume $n = 2$. Assume demand is continuous on $[0, k]$ and that $f(k) > 0$. For positive λ close enough to 0, , $\tilde{\Delta} \approx \sqrt{2}\Delta$ so $H_{2,3} < 0$. But $H_{2,3}$ is negligible compared to H_1,

$$\lim_{\lambda \to 0} \frac{H_{2,3}}{H_1} = 0.$$

(iii) Continuous Demand and Large λ. Assume demand is continuous. For $\lambda < \gamma$ close enough to γ, $\tilde{\Delta} \approx n\Delta(n)$ so $H_{2,3} < 0$. Furthermore, H_1 is negligible compared to $H_{2,3}$,

$$\lim_{\lambda \to \gamma} \frac{H_1}{H_{2,3}} = 0$$

(iv) Continuous Demand and Intermediate λ. Assume that $f(z)$ is symmetric around μ and that $\lambda = \frac{\gamma}{2}$. Then $H_{2,3}(\lambda) < 0$ for $\lambda = \frac{\gamma}{2}$.

1.1 Proof of (i)

(i) Binary Demand and General λ. Assume $n = 2$. Assume demand is discrete and binary, $z \in \{0, 1\}$. Then $H_{2,3} \leq 0$ and the strict inequality holds for λ close enough to γ.

Proof.

I need to show that $\tilde{\Delta}(2) \geq \Delta(2)$. For this proof it is convenient not to normalize the draws of z in the tilde case by dividing through by n. That is, let z_1 and z_2 be the two demand draws in the tilde case and assume that these are i.i.d. and that the probability $z_i = 1$ is f_1. Also normalize $\gamma = 1$.

In the tilde case without bar codes, the optimal inventory is obviously either $r = 0$, $r = 1$, or $r = 2$. The policy of $r = 1$ is better than $r = 0$ if and only if

$$\left[2f_1(1 - f_1) + f_1^2\right] > \lambda$$

The LHS is the expected sale probability of the first unit, the RHS is the holding cost. Analogously the policy of $r = 2$ is better than $r = 1$ if and only if

$$f_1^2 > \lambda.$$

Define the cutoff values of λ by

$$\begin{aligned} \lambda' &= f_1^2 \\ \lambda'' &= 2f_1(1 - f_1) + f_1^2 \\ &= \lambda' + 2f_1(1 - f_1) \end{aligned} \tag{1}$$

So $r^* = 2$ if $\lambda < \lambda'$, $r^* = 1$ if $\lambda \in (\lambda', \lambda'')$, and $r^* = 0$ if $\lambda > \lambda''$. Thus without bar codes, the value in the tilde case is

$$\begin{aligned} \tilde{v}^1 &= 2f_1 - 2\lambda,\ \lambda < \lambda' \\ &= \left[2f_1(1 - f_1) + f_1^2\right] - \lambda,\ \lambda \in (\lambda', \lambda'') \\ &= 0,\ \lambda > \lambda'' \end{aligned}$$

.

If the store has bar codes then the value is $2f_1(1 - \lambda)$. Thus the value of bar codes is (to simplify notation throughout this proof, I do not divide by 2 to convert to a per period basis):

$$\begin{aligned} \tilde{\Delta} &= 2f_1(1 - \lambda),\ \lambda > \lambda'' \\ \tilde{\Delta} &= 2f_1(1 - \lambda) - \left(\left[2f_1(1 - f_1) + f_1^2\right] - \lambda\right), \lambda \in (\lambda', \lambda'') \\ &= 2f_1 - 2f_1\lambda - 2f_1 + 2f_1^2 - f_1^2 + \lambda \\ &= -2f_1\lambda + f_1^2 + \lambda \\ &= (1 - f_1)^2\lambda + f_1^2(1 - \lambda), \\ \tilde{\Delta} &= 2(1 - f_1)\lambda,\ \lambda < \lambda' \end{aligned}$$

Now look at benefit $\Delta(2)$ of bar codes in the 2-period case. Define λ^0 as the case where with bar codes and 2 periods, the store is indifferent between $r' = 1$ and $r' = 0$, where $z_1 + r'$ is the optimal order given z_1. Thus

$$\begin{aligned} f_1 &= 2\lambda^0 \\ \lambda^0 &= \frac{f_1}{2}. \end{aligned}$$

Suppose the store does not have bar codes. If $\lambda > f_1$, the optimal order is obviously 0. For an order of $r = 1$ to be optimal, $f_1 > \lambda$ must hold. For an order of $r = 2$ to be optimal, need

$$\begin{aligned} f_1^2 &> 2\lambda \\ \lambda &< \frac{f_1^2}{2} \end{aligned}$$

If the store orders 1 the expected profit with no bar codes is

$$\begin{aligned} & f_1 - \lambda + (1 - f_1)(f_1 - \lambda) \\ = \; & (2 - f_1)(f_1 - \lambda) \end{aligned}$$

Given the behavior described, the value of bar codes depends on λ in the following manner:

$$\begin{aligned} \Delta_2 &= (1 - f_1)2\lambda, \; \lambda < \frac{f_1^2}{2} \\ \Delta_2 &= f_1(1-\lambda) + f_1 - 2\lambda - (2 - f_1)(f_1 - \lambda) \\ &= f_1 - f_1\lambda + f_1 - 2\lambda - 2f_1 + f_1^2 + 2\lambda - f_1\lambda \\ &= f_1^2 - 2f_1\lambda, \; \lambda \in (\frac{f_1^2}{2}, \frac{f_1}{2}) \\ &= f_1(1-\lambda) - (2 - f_1)(f_1 - \lambda) \\ &= f_1 - f_1\lambda - 2f_1 + f_1^2 + 2\lambda - f_1\lambda \\ &= -2f_1\lambda - f_1 + f_1^2 + 2\lambda, \\ &= 2\lambda(1 - f_1) - f_1(1 - f_1) \\ &= (2\lambda - f_1)(1 - f_1), \; \lambda \in (\frac{f_1}{2}, f_1) \\ &= f_1(1-\lambda), \; \lambda > f_1 \end{aligned}$$

A summary is

$$\begin{aligned} \Delta_2 &= 2(1 - f_1)\lambda, \; \lambda < \frac{f_1^2}{2} \\ &= f_1^2 - 2f_1\lambda, \; \lambda \in (\frac{f_1^2}{2}, \frac{f_1}{2}) \\ &= (2\lambda - f_1)(1 - f_1), \; \lambda \in (\frac{f_1}{2}, f_1) \\ &= f_1(1-\lambda), \; \lambda > f_1 \end{aligned}$$

Now compare

$$
\begin{aligned}
\tilde{\Delta} &= 2(1-f_1)\lambda,\ \lambda < f_1^2 \\
&= (1-f_1)^2\lambda + f_1^2(1-\lambda),\ \lambda \in (f_1^2, \lambda'') \\
&= 2f_1(1-\lambda),\ \lambda > \lambda''
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda'' &= 2f_1(1-f_1) + f_1^2 \\
&= 2f_1 - f_1^2. \qquad (2)
\end{aligned}
$$

Note that $\lambda'' = 2f_1 - f_1^2 > f_1$, so for $\lambda > \lambda''$, $\tilde{\Delta} = 2\Delta_2$, so the result holds.
Suppose $\lambda < \frac{f_1^2}{2}$, then $\tilde{\Delta} = \Delta_2$, and the result holds.
Now break up the remaining analysis into two cases depending upon f_1.

Case 1: $f_1 < \frac{1}{2}$

Suppose $f_1 < \frac{1}{2}$. Then $f_1^2 < \frac{f_1}{2}$. Then when λ is in the range $\frac{f_1^2}{2} < \lambda < f_1^2$, $\lambda < \frac{f_1}{2}$ holds because $f_1^2 < \frac{f_1}{2}$.

$$
\begin{aligned}
\tilde{\Delta} - \Delta &= 2(1-f_1)\lambda - f_1^2 + 2f_1\lambda \\
&= 2\lambda - f_1^2 \geq 0
\end{aligned}
$$

The next breakpoint is at $\lambda = f_1^2$, where there is a kink in the tilde case.

$$
\begin{aligned}
\tilde{\Delta} - \Delta &= (1-f_1)^2\lambda + f_1^2(1-\lambda) - f_1^2 + 2f_1\lambda,\ \lambda \in (f_1^2, \frac{f_1}{2}) \\
&= \lambda - 2f_1\lambda + f_1^2\lambda + f_1^2 - f_1^2\lambda - f_1^2 + 2f_1\lambda \\
&= \lambda > 0
\end{aligned}
$$

The next range to consider is λ from $\frac{f_1}{2}$ to f_1,

$$
\begin{aligned}
\tilde{\Delta} - \Delta &= (1-f_1)^2\lambda + f_1^2(1-\lambda) - (2\lambda - f_1)(1-f_1),\ \lambda \in (\frac{f_1}{2}, f_1) \\
&= \lambda - 2f_1\lambda + f_1^2\lambda + f_1^2 - f_1^2\lambda - 2\lambda + f_1 + 2f_1\lambda - f_1^2 \\
&= \lambda - 2\lambda + f_1 \\
&= f_1 - \lambda \geq 0
\end{aligned}
$$

Next look at λ from f_1 to λ''.

$$
\begin{aligned}
\tilde{\Delta} - \Delta &= (1-f_1)^2\lambda + f_1^2(1-\lambda) - f_1(1-\lambda),\ \lambda \in (f_1, \lambda'') \\
&= \lambda - 2f_1\lambda + f_1^2\lambda + f_1^2 - f_1^2\lambda - f_1 + f_1\lambda \\
&= \lambda - f_1\lambda + f_1^2 - f_1 \\
&= (1-f_1)(\lambda - f_1) \geq 0
\end{aligned}
$$

Since the case of $\lambda > \lambda''$ is already completed, this completes the proof for the case of $f_1 < \frac{1}{2}$.

Case 2: $f_1 > \frac{1}{2}$.

Now assume that $f_1 > \frac{1}{2}$. Then $f_1^2 > \frac{f_1}{2}$. The case of $\lambda < \frac{f_1^2}{2}$ is the same as above. Next consider λ in the range $\frac{f_1^2}{2} < \lambda < \frac{f_1}{2}$, which means $\lambda < f_1^2$ since $\frac{f_1}{2} < f_1^2$. Then

$$\begin{aligned} \tilde{\Delta} - \Delta &= 2(1 - f_1)\lambda - f_1^2 + 2f_1\lambda \\ &= 2\lambda - f_1^2 \geq 0 \end{aligned}$$

Look next between $\frac{f_1}{2}$ and f_1^2

$$\begin{aligned} \tilde{\Delta} - \Delta &= 2(1 - f_1)\lambda - (2\lambda - f_1)(1 - f_1), \ \lambda \in (\frac{f_1}{2}, f_1^2) \\ &= 2\lambda - 2f_1\lambda - 2\lambda + f_1 + 2f_1\lambda - f_1^2 \\ &= f_1 - f_1^2 > 0 \end{aligned}$$

Next look between f_1^2 and f_1,

$$\begin{aligned} \tilde{\Delta} - \Delta &= (1 - f_1)^2\lambda + f_1^2(1 - \lambda) - (2\lambda - f_1)(1 - f_1), \ \lambda \in (f_1^2, f_1) \\ &= \lambda - 2f_1\lambda + f_1^2\lambda + f_1^2 - f_1^2\lambda - 2\lambda + f_1 + 2f_1\lambda - f_1^2 \\ &= \lambda - 2\lambda + f_1 = f_1 - \lambda \geq 0 \end{aligned}$$

For λ above f_1, the same calculation is the same as the calculation for the $f_1 < \frac{1}{2}$ case. Q.E.D.

1.2 Proof of Proposition 3 (ii)

(ii) Continuous Demand and Small λ. Assume $n = 2$. Assume demand is continuous on $[0, k]$ and that $f(k) > 0$. For positive λ close enough to 0, , $\tilde{\Delta} \approx \sqrt{2}\Delta$ so $H_{2,3} < 0$. But $H_{2,3}$ is negligible compared to H_1,

$$\lim_{\lambda \to 0} \frac{H_{2,3}}{H_1} = 0.$$

Proof.

I now return to prove Proposition 3′ (i). It is convenient to make some normalizations. Let $\gamma = 1$. Assume the maximum demand realization $k = 1$, so the demand realization z_t in each period t is distributed on the unit interval.

Let z be the sum of realized demand over two periods $z = z_1 + z_2$ and let $f_2(z)$ be the distribution over the sum of realized demand over two periods, $z_1 + z_2$. For

$z > 0$ (the relevant case below), the distribution function and the derivatives are as follows:

$$\begin{aligned} F_2(z) &= \int_0^{z-1} f(z_1)dz_1 + \int_{z-1}^1 f(z_1)F(z-z_1)dz_1 \\ f_2(z) &= f(z-1) - f(z-1)F(1) + \int_{z-1}^1 f(z_1)f(z-z_1)dz_1 \\ &= \int_{z-1}^1 f(z_1)f(z-z_1)dz_1 \\ f_2'(z) &= -f(r-1)f(1) + \int_{z-1}^1 f(z_1)f'(z-z_1)dz_1 \end{aligned}$$

Thus

$$\begin{aligned} 1 - F_2(z) &= 1 - \int_0^{z-1} f(z_1)dz_1 - \int_{z-1}^1 f(z_1)F(z-z_1)dz_1 \\ &= \int_{z-1}^1 f(z_1)\left[1 - F(z-z_1)\right] dz_1 \end{aligned}$$

The first part of this proof compares $\tilde{\Delta}(2)$ and $\Delta(2)$. Since n is fixed at $n = 2$ throughout this comparison, it is convenient not to normalize by dividing through by 2. Thus $\Delta(2)$ is the benefit of bar codes over 2 periods in the base case where the demand draw is from the unit interval in each period. And $\tilde{\Delta}(2)$ is the benefit in a single period in the tilde case when the store gets 2 draws from the unit interval in each period.

1.2.1 The FONC

Consider the base case with no bar codes (again n is fixed at $n = 2$). Let r_2 be the optimal starting inventory for the cycle. In the limiting case of $\lambda = 0$, $r_2 = 2$ since this is the maximum possible sum of demand over the two periods (with a zero holding cost, the store will hold demand to meet the maximum possible demand). For λ small (the case considered in this proposition), r_2 will be close to 2. Thus inventory will be sufficient to meet all demand in the first period, but there is a probability of a stockout in the second period. It is immediate then that r_2 must solve the FONC

$$1 - F_2(r_2) = 2\lambda. \tag{3}$$

The LHS is the expected probability that the marginal unit is sold. This the probability the marginal unit is sold times the gross margin $\gamma = 1$. The RHS is the two period holding cost (since r_2 is above 1, the marginal unit is necessarily held two periods).

Next consider the tilde case and let $\tilde{r}$ be the optimal starting inventory. This satisfies the FONC

$$1 - F_2(\tilde{r}) = \lambda. \tag{4}$$

The LHS is the same as in (3) since the there are two draws of demand in the tilde case, the same as in the base case, so the probability of sale of the marginal unit is the same. But the marginal holding cost is only λ in the tilde case as opposed to 2λ in the base case.

1.2.2 Change of Variable

This part of the proof explains a change of variable I make for the taking of limits.

For each value of λ there is a unique r_2 solving (3) and a unique $\tilde{r}$ solving (4). By combining these two equations we can eliminate λ,

$$\begin{aligned} 2 - 2F_2(\tilde{r}) &= 1 - F_2(r_2) \\ 1 - 2F_2(\tilde{r}) + F_2(r_2) &= 0. \end{aligned}$$

By solving the above equation we can define a function $r_2(\tilde{r})$. Below it will be convenient to do the analysis in terms of taking the limit of $\tilde{r}$ rather than λ and thinking of r_2 as a function of $\tilde{r}$. Of course for each value of $\tilde{r}$ there is a unique value of λ. In the limit as λ goes to zero, r_2 and $\tilde{r}$ both go to 2. So I when I look at $\tilde{r}$ close to 2, this is equivalent at looking at λ close to 0.

The slope of the relation $r_2(\tilde{r})$ is determined as follows.

$$\begin{aligned} -2f_2(\tilde{r}) + f_2(r_2)\frac{dr_2}{d\tilde{r}} &= 0 \\ \frac{dr_2}{d\tilde{r}} &= 2\frac{f_2(\tilde{r})}{f_2(r_2)} \\ \frac{d^2 r_2}{d\tilde{r}^2} &= 2\frac{f_2'(\tilde{r})}{f_2(r_2)} - 2\frac{f_2(\tilde{r})f_2'(r_2)}{f_2(r_2)^2}\frac{dr_2}{d\tilde{r}} \\ &= 2\frac{f_2'(\tilde{r})}{f_2(r_2)} - 4\frac{f_2(\tilde{r})^2 f_2'(r_2)}{f_2(r_2)^3} \end{aligned}$$

Observe that $f_2(\tilde{r})$ and $f_2(r_2)$ both go to zero as $\tilde{r}$ goes to 2 (see the formula for $f_2(z)$ above). So I need to use l'Hopital's rule.

$$\begin{aligned} \lim \frac{dr_2}{d\tilde{r}} &= 2\lim \frac{f_2'(\tilde{r})}{f_2'(r_2)\frac{dr_2}{d\tilde{r}}} \\ &= 2\frac{\lim f_2'(\tilde{r})}{\lim f_2'(r_2)\lim \frac{dr_2}{d\tilde{r}}} \\ \left[\lim \frac{dr_2}{d\tilde{r}}\right]^2 &= 2\frac{\lim f_2'(\tilde{r})}{\lim f_2'(r_2)} \end{aligned}$$

Now

$$\begin{aligned} f_2'(r) &= -f(r-1)f(1) + \int_{z-1}^{1} f(z_1)f'(z-z_1)dz_1 \\ \lim f_2' &= -f(1)^2 \end{aligned}$$

By assumption, $f(1) > 0$. Thus

$$\lim \frac{dr_2}{d\tilde{r}} = \sqrt{2}.$$

Next, for later use I show

$$\lim f(r_2)\frac{d^2 r_2}{d\tilde{r}^2} = 0. \tag{5}$$

But

$$\begin{aligned} f(r_2)\frac{d^2 r_2}{d\tilde{r}^2} &= f(r_2)\left[2\frac{f_2'(\tilde{r})}{f_2(r_2)} - 4\frac{f_2(\tilde{r})^2 f_2'(r_2)}{f_2(r_2)^3}\right] \\ &= \left[2f_2'(\tilde{r}) - 4\frac{f_2(\tilde{r})^2 f_2'(r_2)}{f_2(r_2)^2}\right] \end{aligned}$$

Note that $\lim f'(r_2) = f'(\tilde{r}) = -f(1)^2 < 0$ since by assumption $f(1) > 0$. In addition,

$$\begin{aligned} & \lim 4\frac{f_2(\tilde{r})^2}{f_2(r_2)^2} \\ = \;& 4\lim \frac{2f_2(\tilde{r})f_2'(\tilde{r})}{2f_2(r_2)f_2'(r_2)\frac{dr_2}{d\tilde{r}}} \\ = \;& 4\lim \frac{f_2(\tilde{r})}{f_2(r_2)} \lim \frac{1}{\frac{dr_2}{d\tilde{r}}} \lim \frac{f_2'(\tilde{r})}{f_2'(r_2)} \\ = \;& 4\left[\lim \frac{1}{\frac{dr_2}{d\tilde{r}}}\right]^2 \\ = \;& 2 \end{aligned}$$

Plugging this in gets the (5) result.

1.2.3 The slope of $\tilde{\Delta}(2)$

This part of the proof looks at the slope of $\tilde{\Delta}$ with respect to change in $\tilde{r}$ near the limit of $\tilde{r} = 2$.

It is useful to determine derivatives of λ as a function of $\tilde{r}$.

$$\begin{aligned}
\lambda &= 1 - F_2(\tilde{r}) \\
\frac{d\lambda}{d\tilde{r}} &= -f_2(\tilde{r}) = -\int_{\tilde{r}-1}^{1} f(z_1) f(\tilde{r} - z_1) dz_1 \\
\frac{d^2\lambda}{d\tilde{r}^2} &= f(\tilde{r} - 1) f(1) - \int_{\tilde{r}-1}^{1} f(z_1) f'(\tilde{r} - z_1) dz_1 \\
\lim \frac{d\lambda}{d\tilde{r}} &= 0 \\
\lim \frac{d^2\lambda}{d\tilde{r}^2} &= f(1)^2 > 0.
\end{aligned}$$

Where the strict inequality holds since $f(1) > 0$ holds by assumption.

Now from equation (8) in the text we can write $\tilde{\Delta}$ as follows:

$$\tilde{\Delta} = \int_{\tilde{r}}^{2} z f_2(z) dz - \lambda 2\mu$$

(To see this note that the tilde case is in the $n = 1$ class, so the formula applies). Differentiating yields,

$$\begin{aligned}
\frac{d\tilde{\Delta}}{d\tilde{r}} &= -\tilde{r} f_2(\tilde{r}) - 2\frac{d\lambda}{d\tilde{r}}\mu \\
&= \frac{d\lambda}{d\tilde{r}} [\tilde{r} - 2\mu] \\
\frac{d^2\tilde{\Delta}}{d\tilde{r}^2} &= \frac{d^2\lambda}{d\tilde{r}^2} [\tilde{r} - 2\mu] + \frac{d\lambda}{d\tilde{r}}
\end{aligned}$$

So in limit

$$\begin{aligned}
\lim \frac{d\tilde{\Delta}}{d\tilde{r}} &= 0 \\
\lim \frac{d^2\tilde{\Delta}}{d\tilde{r}^2} &= 2(1 - \mu) f(1)^2
\end{aligned}$$

1.2.4 The slope of $\Delta(2)$

For the beginning of this subsection it is useful to think of varying r_2 rather than $\tilde{r}$. Then at the end I will have r_2 depend upon $\tilde{r}$. Analogous to the previous section, I begin by determining the derivatives of λ as a function of r_2.

$$
\begin{aligned}
2\lambda &= 1 - F_2(r_2) \\
\lambda &= \frac{1}{2} - \frac{1}{2}F_2(r_2) \\
\frac{d\lambda}{dr_2} &= -\frac{1}{2}f_2(r_2) = -\frac{1}{2}\int_{r_2-1}^{1} f(z_1)f(r_2-z_1)dz_1 \\
\frac{d^2\lambda}{dr_2^2} &= \frac{1}{2}f(r_2-1)f(1) - \frac{1}{2}\int_{r_2-1}^{1} f(z_1)f'(r_2-z_1)dz_1 \\
\lim \frac{d\lambda}{dr_2} &= 0 \\
\lim \frac{d^2\lambda}{dr_2} &= \frac{1}{2}f(1)^2
\end{aligned}
$$

Now consider $\Delta(2)$. Suppose the store has bar codes. When the store makes it order for the two periods, it has already observed the demand z_1 in period 1. It orders inventory to match this and it sets an optimal amount r' to have as the available stock in the second period, so the total order is $z_1 + r'$.

For low λ (or high r_2), sales are the same in the first period with or without bar codes. So the difference $\Delta(2)$ is the difference in expected second period sales minus the difference in holding costs (where Eq_2 denotes expected sales in period 2),

$$
\begin{aligned}
\Delta(n) &= Eq_2^0(r') - Eq_2^1(r_2) - \lambda\left[Ez + 2r' - r_2 - (r_2 - Ez)\right] \\
&= Eq_2^0(r') - Eq_2^1(r_2) + \lambda\left[2r_2 - 2r' - 2\mu\right]
\end{aligned}
$$

Now

$$
\begin{aligned}
Eq_2^0(r') &= \int_0^{r'} f(z)zdz + \int_{r'}^{1} f(z)r' \\
\frac{dEq^0(r')}{dr_2} &= \left[1 - F(r')\right]\frac{dr'}{dr_2} \\
&= 2\lambda\frac{dr'}{dr_2}
\end{aligned}
$$

$$
\begin{aligned}
Eq_2^1(r_2) &= \int_0^{r_2-1} f(z_1)dz_1\mu + \int_{r_2-1}^{1}\left[\int_0^{r_2-z_1} z_2 f(z_2)dz_2 + \int_{r_2-z_1}^{1}(r_2-z_1)f(z_2)dz_2\right] f(z_1)\,dz_1 \\
\frac{dEq^1(r_2)}{dr_2} &= f(r_2-1)\mu - \mu f(r_2-1) + \int_{r_2-1}^{1}\int_{r_2-z_1}^{1} f(z_2)f(z_1)dz_2dz_1 \\
&= \int_{r_2-1}^{1} f(z_1)\int_{r_2-z_1}^{1} f(z_2)dz_2dz_1 \\
&= 2\lambda
\end{aligned}
$$

So the slope of $\Delta(n)$ can be written

$$\begin{aligned}
\frac{d\Delta(n)}{dr_2} &= \frac{dEq^0(r')}{dr_2} - \frac{dEq^1(r_2)}{dr_2} + 2\lambda\left(1 - \frac{dr'}{dr}\right) + \frac{d\lambda}{dr_2}\left[2r_2 - 2r' - 2\mu\right] \\
&= 2\lambda\frac{dr'}{dr_2} - 2\lambda + 2\lambda\left(1 - \frac{dr'}{dr}\right) + \frac{d\lambda}{dr_2}\left[2r_2 - 2r' - 2\mu\right] \\
\frac{d\Delta(n)}{dr_2} &= \frac{d\lambda}{dr_2}2\left[r_2 - r' - \mu\right] \\
\frac{d^2\Delta(n)}{dr_2^2} &= \frac{d^2\lambda}{dr_2^2}2\left[r_2 - r' - \mu\right] + \frac{d\lambda}{dr_2}2\left[1 - \frac{dr'}{dr_2}\right] \\
\lim \frac{d\Delta(n)}{dr_2} &= 0 \\
\lim \frac{d^2\Delta(n)}{dr_2^2} &= \frac{d^2\lambda}{dr_2^2}2\left[1 - \mu\right] \\
&= \frac{1}{2}f(1)^2 2\left[1 - \mu\right] \\
&= f(1)^2\left[1 - \mu\right]
\end{aligned}$$

I lastly can determine the slope of $\Delta(n)$ with respect to changes in $\tilde{r}$. By writing $\Delta(n, r_2(\tilde{r}))$

$$\begin{aligned}
\frac{d\Delta(n)}{d\tilde{r}} &= \frac{d\Delta(n)}{dr_2}\frac{dr_2}{d\tilde{r}} \\
\frac{d^2\Delta(n)}{d\tilde{r}^2} &= \frac{d^2\Delta(n)}{dr_2^2}\frac{dr_2}{d\tilde{r}} + \frac{d\Delta(n)}{dr_2}\frac{dr_2^2}{d\tilde{r}^2}
\end{aligned}$$

Taking limits

$$\begin{aligned}
\lim \frac{d\Delta(n)}{d\tilde{r}} &= 0 \\
\lim \frac{d^2\Delta(n)}{d\tilde{r}^2} &= f(1)^2\left[1 - \mu\right]\sqrt{2}
\end{aligned}$$

where this uses the fact that

$$\begin{aligned}
\lim \frac{d\Delta(n)}{dr_2}\frac{dr_2^2}{d\tilde{r}^2} &= \lim \frac{d\lambda}{dr_2}2\left[r_2 - r' - \mu\right]\frac{dr_2^2}{d\tilde{r}^2} \\
&= \lim \left[-\frac{1}{2}f_2(r_2)\right]2\left[r_2 - r' - \mu\right]\frac{dr_2^2}{d\tilde{r}^2} \\
&= -\left[1 - \mu\right]\lim f_2(r_2)\frac{dr_2^2}{d\tilde{r}^2} \\
&= 0
\end{aligned}$$

Where the equality follows from (5) above.

1.2.5 Gathering Together

From the previous subsection

$$\begin{aligned} \lim \frac{d\Delta(n)}{d\tilde{r}} &= 0 \\ \lim \frac{d^2\Delta(n)}{d\tilde{r}^2} &= f(1)^2 \left[1-\mu\right]\sqrt{2} \end{aligned}$$

From the subsection before that,

$$\begin{aligned} \lim \frac{d\tilde{\Delta}}{d\tilde{r}} &= 0 \\ \lim \frac{d^2\tilde{\Delta}}{d\tilde{r}^2} &= 2(1-\mu)f\left(1\right)^2 \end{aligned}$$

Thus

$$\lim \frac{d^2\tilde{\Delta}}{d\tilde{r}^2} = \sqrt{2}\lim \frac{d^2\Delta(n)}{d\tilde{r}^2}$$

This proves that $\tilde{\Delta} \approx \sqrt{2}\Delta(n)$ for $\tilde{r}$ close to 2 or, equivalently, for λ close to 0.

1.2.6 Proof that $|H_1|$ is large relative to $|H_{2,3}|$ near limit

It remains to show (**??**). Consider the value of bar codes with one period. Note that the optimal inventory r_1 without bar codes solves

$$1 - F(r_1) = \lambda$$

as compared to $\tilde{r}$ solving

$$1 - F_2(\tilde{r}) = \lambda$$

So

$$1 - F(r_1) = 1 - F_2(\tilde{r})$$

As above, think of varying r_1 and let $\tilde{r}(r_1)$ be a function of r_1. Then

$$\begin{aligned} -f(r_1) &= -f_2(\tilde{r})\frac{d\tilde{r}}{dr_1} \\ \frac{d\tilde{r}}{dr_1} &= \frac{f_2(\tilde{r})}{f(r_1)} \end{aligned}$$

As r_1 goes to 1, $\tilde{r}$ goes to 2, so

$$\lim \frac{d\tilde{r}}{dr_1} = 0,$$

since $f_2(2) = 0$. Also,

$$1 - F(r_1) = \lambda$$

implies

$$-f(r_1) = \frac{d\lambda}{dr_1}$$

Now

$$\Delta_1 = \int_{r_1}^{1} zf(z)dz - \lambda\mu$$

And

$$\begin{aligned} \frac{d\Delta_1}{dr_1} &= -r_1 f(r_1) - \frac{d\lambda}{dr_1}\mu \\ \lim \frac{d\Delta_1}{dr_1} &= -(1-\mu)f(1) < 0 \end{aligned}$$

Thus a decrease in r_1 below 1 (and a consequent increase in λ) yields a first-order change in Δ_1. But not there is not first order change in $\tilde{r}$. Moreover, a first-order change in $\tilde{r}$ has no first-order effect on $\tilde{\Delta}$. The result then follows.

1.3 Proof of Proposition 3 (iii)

(iii) Continuous Demand and Large λ. Assume demand is continuous. For $\lambda < \gamma$ close enough to γ, $\tilde{\Delta} \approx n\Delta(n)$ so $H_{2,3} < 0$. Furthermore, H_1 is negligible compared to $H_{2,3}$,

$$\lim_{\lambda\to\gamma} \frac{H_1}{H_{2,3}} = 0$$

Proof.

Consider the n-period case and suppose the store has bar codes. If λ is close to γ, it will not be optimal to sell anything in period 2 and beyond. Profit is obtained only on the first period sales which are observed at the time of order. The average value is $v^0(n) = \frac{1}{n}\mu(\gamma - \lambda)$, so the slope with respect to changes in λ is $-\frac{\mu}{n}$. In the case where the store does not have bar codes, in the limit where λ goes to γ, the optimal available stock goes to zero. By an envelope theorem argument, it follows that

$$\lim_{\lambda\to\gamma} \frac{dv^1(n)}{d\lambda} = 0.$$

Thus

$$\lim_{\lambda\to\gamma} \frac{d\Delta(n)}{d\lambda} = \lim_{\lambda\to\gamma} \frac{dv^0(n)}{d\lambda} - \lim_{\lambda\to\gamma} \frac{dv^1(n)}{d\lambda} = -\frac{\mu}{n}. \tag{6}$$

In the tilde case, the value with bar codes is $\tilde{v}^0(n) = \mu[\gamma - \lambda]$, so the slope is $-\mu$. By an analogous envelope theorem argument, the slope of $\tilde{v}^1(n)$ is zero. Thus

$$\lim_{\lambda\to\gamma} \frac{d\tilde{\Delta}}{d\lambda} = -\mu. \tag{7}$$

Since $\Delta(n) = \tilde{\Delta}(n) = 0$ in the limit, (6) and (7) imply that $\tilde{\Delta}(n) \approx n\Delta(n)$ for λ near γ, as claimed. The same argument for why the slope of $\tilde{\Delta}$ is $-\mu$ in (7) shows that the slope of Δ_1 is $-\mu$, so there is no first-order difference. Thus $H_1 = \tilde{\Delta} - \Delta_1$ is negligible compared to $H_{2,3} = \Delta_n - \tilde{\Delta}$ in the limit.

1.4 Proof of Proposition 3 (iv)

(iv) Continuous Demand and Intermediate λ. Assume that $f(z)$ is symmetric around μ and that $\lambda = \frac{\gamma}{2}$. Then $H_{2,3}(\lambda) < 0$ for $\lambda = \frac{\gamma}{2}$.

Proof.

Given a symmetric distribution and $\lambda = \frac{\gamma}{2}$, it is clear that in the tilde case without bar codes, the optimal holding is $\tilde{r} = \mu$. Thus

$$\tilde{\Delta}(n) = \lambda E|\tilde{z} - \mu| = \frac{\lambda}{n} E\left|\Sigma_{i=1}^{n}(z_i - \mu)\right|. \tag{8}$$

Now consider the n-period case. Note that the cost of holding a good two periods or more is greater or equal to γ. So if the store has bar codes, it will order to match the realization of demand in period 1 and order nothing for future sales. Thus $v^0(n) = \frac{1}{n}\mu\lambda$ noting $\lambda = \gamma - \lambda$. Now suppose the store does not have bar codes. I claim that

$$v^1(n) > \frac{1}{n}\left(\mu\lambda - E|z_1 - \mu|\lambda\right). \tag{9}$$

The right hand side is the average return if the store ordered μ for delivery in the first period of the cycle and then sent back the leftovers $r_2 = r_1 - q_1$ at the beginning of period 2. It is not feasible to send r_2 back. However, since $r_2 \leq \mu$ will hold, the return (beginning with period 2) to starting period 2 with a positive r_2 is strictly larger than what the return would be if all the r_2 were sent back. Inequality (9) and $v^0(n) = \frac{1}{n}\mu\lambda$ imply that

$$\Delta(n) < \frac{\lambda}{n}E|z_1 - \mu|. \tag{10}$$

Using the fact that the z_t are i.i.d., it is straightforward to show that $E|z_1 - \mu| \leq E\left|\Sigma_{i=1}^{n}(z_i - \mu)\right|$. Conditions (8) and (10) then imply $H_{2,3} = \Delta(n) - \tilde{\Delta}(n) < 0$, as claimed. This completes the proof of (ii).

2 Proof of Claim about Scenario 2

Claim in text: Suppose Scenario 2 applies. For large n, $\Delta(n)$ and $\tilde{\Delta}(n)$ go to zero so that

$$\begin{aligned} \lim H_{2,3} &= \lim\left[\tilde{\Delta}(n) - \Delta(n)\right] \\ &= 0 \end{aligned}$$

and

$$\lim H_1 = \lim \left[\tilde{\Delta}(n) - \Delta(1)\right] < 0.$$

Proof.

Consider first the limit of $\tilde{\Delta}(n)$. Recall that in the tilde case the cycle length is one period. The demand realization in a period is

$$\tilde{z} = \frac{\sum_{i=1}^{n} z_i}{n}.$$

Under Scenario 2, bar codes reduce the information lag from $\ell = 2$ to $\ell = 1$.

Observe that when the information lag is $\ell = 0$ then the average long-run value is

$$\tilde{v}^0(n) = \mu(\gamma - \lambda).$$

Suppose the information lag is $\ell = 1$. Consider the policy of starting each order cycle with $\tilde{r} = \mu$. There is some possibility that the realized demand will be different from μ, so that return to this policy will be less than $\tilde{v}^0(n)$, the perfect foresight case $\ell = 0$. However, by the law of large numbers, for any $\varepsilon > 0$, the probability that $|\tilde{z} - \mu| > \varepsilon$ goes to zero for large n. It is immediate then that the return to this policy of starting each period with μ is arbitrarily close to $\mu(\gamma - \lambda)$, for n large enough. Thus

$$\lim \tilde{v}^1(n) = \mu(\gamma - \lambda).$$

Now suppose the information lag is $\ell = 2$. When placing its order for period t, the store observes the available stock r_{t-1} in the previous period. Suppose the store uses the following order policy,

$$x = \mu - \max\{(r_{t-1} - \mu), 0\}$$

It is straightforward to see that for large enough n, the return to using this policy is arbitrarily close to $\mu(\gamma - \lambda)$, so

$$\lim \tilde{v}^2(n) = \mu(\gamma - \lambda).$$

Thus

$$\lim \tilde{\Delta}(n) = \lim \tilde{v}^1(n) - \lim \tilde{v}^2(n) = 0.$$

Now consider $\Delta(n)$. It is obvious that

$$\lim_{n\to\infty} v^1(n) = 0.$$

This follows since for large enough cycle length, the holding cost becomes to high to make it worthwhile to intend to sell in later periods (the holding cost in period n is $n\lambda$). So the total expected return over the cycle is bounded. Since the average return is divided by n, in the limit the value is zero. We also know that $v^1(n) \geq v^2(n)$. Thus

$$\lim_{n\to\infty} \Delta(n) = \lim \left[v^1(n) - v^2(n)\right] = 0.$$

Finally note that the value of bar codes with a one-period cycle $\Delta(1)$ does not depend upon n. With a continuous density that is strictly positive on the support, it is immediate that the value of bar codes is strictly positive for this case, $\Delta(1) > 0$. Thus

$$\begin{aligned} \lim H_1 &= \lim \left[\tilde{\Delta}(n) - \Delta(1)\right] \\ &= \Delta(1) < 0. \end{aligned}$$

3 Proof of Proposition 4

Proposition 4. Suppose Scenario 2 applies and demand is binary. For $n \in \{2, 3\}$, an analytic result shows that $H(n) = \Delta(n) - \Delta(1) \leq 0$. Numerical analysis shows the result holds for $n \geq 4$.

To prove this, I first prove the following claim.

Claim: Suppose $n = 1$ and $\ell = 2$. Define $\hat{\lambda}$ by

$$\hat{\lambda} = \gamma \frac{f_1^2}{1 + f_1 - f_1^2}.$$

If $\lambda \in (\hat{\lambda}, f_1\gamma)$, then the optimal order policy is to order so as to never have more than 1 unit ever in stock. In particular, if the previous available stock is $r_{t-1} = 0$, then the optimal order is $x_t = 1$, but if $r_{t-1} = 1$, then the optimal order is $x_t = 0$. If $\lambda \in [0, \hat{\lambda})$, the optimal order policy is instead to order so that there is always at least one unit in stock. In particular, if $r_{t-1} = 1$, then $x_t = 1$, if $r_{t-1} = 2$, then $x_t = 0$.

Proof. It is clear that the optimal policy is one or the other of the two policies described above. Under the first order policy where there is never more than 1 unit in stock, along the optimal path the available inventory r at the start of each period is either 0 or 1. In periods where the inventory is 1, the current expected return in the period is $f_1\gamma - \lambda$. In periods where the inventory is 0, the current return is 0. Let p^0 denote the fraction of time the store starts with 0. This solves

$$p^0 = f_1(1 - p^0).$$

The store starts with 0 in a period if the store had 1 in inventory the previous period (which happens with probability $1 - p^0$) and the demand realization was positive (which happens with probability f_1). Solving for p^0 yields

$$p^0 = \frac{f_1}{1 + f_1}.$$

The average return to this policy is then

$$v^* = \frac{1}{1 + f_1}\left[f_1\gamma - \lambda\right].$$

Under the second policy, the store starts with either 1 or 2 units each period. The store starts with 2 whenever the sales realization was 0 in the previous period. So the average value is

$$v^{**} = f_1\gamma - \lambda - (1 - f_1)\lambda$$

The second policy is optimal if and only if

$$\begin{aligned}
f_1\gamma - \lambda - (1 - f_1)\lambda &> \frac{1}{1 + f_1}\left[f_1\gamma - \lambda\right] \\
f_1\gamma - \lambda - (1 - f_1)\lambda + f_1^2\gamma - \lambda f_1 - f_1(1 - f_1)\lambda &> f_1\gamma - \lambda \\
-(1 - f_1)\lambda + f_1^2\gamma - \lambda f_1 - f_1(1 - f_1)\lambda &> 0 \\
f_1^2\gamma &> \left[1 - f_1 + f_1 + f_1(1 - f_1)\right]\lambda \\
f_1^2\gamma &> \left[1 + f_1 - f_1^2\right]\lambda
\end{aligned}$$

which proves the claim.

The proof of Proposition 4 has two parts. The first part considers the case where $\lambda < \hat{\lambda}$. Here a general result is obtained that $\Delta(n) \leq \Delta(1)$ for all n. The second part considers the case where $\lambda > \hat{\lambda}$. Here I can only show that the result holds for $n = 2$ and $n = 3$ (but use numerical work to extend the result for large n).

3.1 Case 1: Low λ.

Suppose that $\lambda < \hat{\lambda}$. For the single-period case, average value when $\ell = 2$ is v^{**} from above. With bar codes, ℓ falls to 1, and the store will not optimally begin each period with a single unit. The value of bar codes is the average reduction in inventory holding cost,

$$\Delta(1) = (1 - f_1)\lambda.$$

To prove $\Delta(n) \leq \Delta(1)$, I obtain a bound for $\Delta(n)$ that is no greater than $\Delta(1)$. I consider a particular policy $\overline{X}$ for the no bar code case that has an average value in the

n-period cycle case of $\overline{v}_n$. The return on the optimal policy is of course at least this high, $v_n \geq \overline{v}_n$. I will show that $v_n^1 - \overline{v}_n^2 \leq \Delta(1)$. The result $v_n^1 - v_n^2 \equiv \Delta(n) \leq \Delta(1)$ then follows from the fact that $v_n^2 \geq \overline{v}_n^2$.

The policy $\overline{X}$ is defined as follows. Suppose that in the bar code case in the n period cycle, the optimal policy is to start with r_n^* units in inventory. The $\overline{X}$ order policy is to order so to guarantee that the available stock at the beginning of the cycle is at least r^*. i.e., $\overline{x} = r^* - \max\{r^- - 1, 0\}$.

With this policy, along the optimal path the store will start the cycle with either r_n^* or $r_n^* + 1$ as the available stock. Let $\overline{p}^0$ be the fraction of time it starts with r_n^* with the policy and $\overline{p}^1$ the fraction of time it starts with $r_n^* + 1$.

Under this policy, the event that the store starts with $r_n^* + 1$ only happens when the demand realization was zero in the previous period. Thus it must be the case that

$$\overline{p}^1 \leq 1 - f_1. \tag{11}$$

With bar codes, the store sets the optimal inventory level r^* in every period and gets an average payoff of v_{r^*}. With no bar codes and the $\overline{X}$ policy, the store sets r^* with probability $\overline{p}^0$ and $r^* + 1$ with probability $\overline{p}^1$. The difference in average payoff is

$$\begin{aligned} v_n^1 - \overline{v}_n^2 &= v_{r^*} - \overline{p}^0 v_{r^*} - \overline{p}^1 v_{r^*+1} \\ &= \overline{p}^1 \left[v_{r^*} - v_{r^*+1} \right] \\ &\leq \overline{p}^1 \lambda \\ &\leq (1 - f_1)\, \lambda \\ &= \Delta(1) \end{aligned} \tag{12}$$

To see that the first inequality must hold, note that if the available stock is $r_n^* + 1$ instead of r_n^*, and sales turn out to be less than r^* in the month, then the extra unit held is raises per period holding cost over the cycle by λ, reducing average value by this amount. But if this extra unit is sold before the last period, the average holding cost increase will be less than λ plus there is the benefit of the gross margin from the sale of $r_n^* + 1$ unit.

The second inequality uses (11), completing what needed to be shown to prove the claim.

3.2 Case 2: High λ

Analysis of this case has several steps.

3.2.1 Step 1: Case where $r_n^* = 1$

As above let r_n^* denote the optimal starting inventory with bar codes and a n-period cycle. Suppose that the parameters are such that $r_n^* = 1$.

Suppose that without bar codes the store uses the following policy. If the previous available stock is $r^- = 1$, then the order is zero, if the available stock is $r^- = 0$, then order 1. Analogous to above, call this the $\overline{X}$ order policy and let the average return to this policy be denoted by $\overline{v}_n^2$. With this policy the store never has more than 1 unit in stock. With probability $\overline{p}^0$ it starts with 0 in stock and with probability $\overline{p}^1$ it starts with one unit in stock. This satisfies

$$\overline{p}^0 = (1 - \overline{p}^0)(1 - f_1)^{n-1} f_1.$$

To see this note that the event that the store starts with 0 occurs when the previous cycle starts with 1 unit in stock (which happens with probability $(1 - \overline{p}^0)$) and the unit is not sold in the first $n - 1$ periods of the cycle but is sold in period n of the cycle. Thus

$$\overline{p}^0 = \frac{(1 - f_1)^{n-1} f_1}{1 + (1 - f_1)^{n-1} f_1}$$

The expected return, over the n-period cycle, of starting with 1 unit at the beginning of the n period cycle is

$$\sum_{t=1}^{n} (1 - f_1)^{t-1} \left[-\lambda + f_1 \gamma\right].$$

It is immediate that

$$\frac{\sum_{t=1}^{n} (1 - f_1)^{t-1} \left[-\lambda + f_1 \gamma\right]}{n} < \left[-\lambda + f_1 \gamma\right].$$

Now the difference between the value with bar codes and the value without bar codes under the $\overline{X}$ policy is fraction of time the store starts with 0 instead of 1 times the return to starting with 1,

$$\begin{aligned} v_n^1 - \overline{v}_n^2 &= \overline{p}^0 \frac{\sum_{t=1}^{n} (1 - f_1)^{t-1} \left[-\lambda + f_1 \gamma\right]}{n} \\ &< \overline{p}^0 \left[-\lambda + f_1 \gamma\right] \\ &< \frac{f_1}{1 + f_1} \left[-\lambda + f_1 \gamma\right] \\ &= \Delta(1). \end{aligned}$$

But $v_n^1 - \overline{v}_n^2 \geq \Delta(n)$, since without bar codes the store can do no worse than the return from the $\overline{X}$ policy. Thus $\Delta(1) > \Delta(n)$ as claimed.

3.2.2 Step 2: Case of $n = 2$.

Suppose that $n = 2$ and the store has bar codes. The optimal order size satisfies $r_2^* = 1$ if and only if

$$f_1^2 \gamma < 2\lambda. \tag{13}$$

To see this note that the marginal benefit of holding a second unit of inventory is the probability f_1^2 it is sold times the gross margin. The second unit will surely be held two periods, so the marginal cost is 2λ.

By assumption,

$$\lambda > \hat{\lambda} \equiv \gamma \frac{f_1^2}{1 + f_1 - f_1^2}.$$

But this implies that (13) holds. Thus the optimal order size is $r_2^* = 1$ must hold so the result of step 1 applies. Thus for $n = 2$, the result that $\Delta(2) \leq \Delta(1)$ is proven.

3.2.3 Step 3: A Sufficient Condition for General n

This step derives a sufficient condition under which $\Delta(n) \leq \Delta(1)$. This sufficient condition is used for the case of $n = 3$ below and it is used in the numerical analysis for $n \geq 4$.

Take $n > 1$ and suppose the optimal policy with bar codes is r_n^*. As in step 1, consider the $\overline{X}$ policy where the store orders to ensure that there is no more than r_n^* ever in stock. Let $\overline{p}^0$ denote the probability the store starts with $r_n^* - 1$ and let $\overline{p}^1$ be the probability it starts with r_n^*.

Consider the following additional notation. At the risk of causing confusion, let r_n denote the available stock as of period n in the order cycle (Don't confuse this with r_n^* which is the optimal starting inventory in the n-period cycle). Let a_0 be the probability that $r_n = 0$ given that the cycle started with inventory r_n^* and let a_1 be the probability $r_n = 0$ given the starting inventory was $r_n^* + 1$. Let b_0 and b_1 be the analogous conditional probabilities that $r_n \geq 1$, so $a_i + b_i = 1$. It is straightforward to see that

$$\begin{aligned} p_i^0 &= f_1 b_i \\ p_i^1 &= a_i + f_0 b_i \end{aligned}$$

$$\begin{aligned} p^0 &= f_1 b_0 p^0 + f_1 b_1 \left(1 - p^0\right) \\ p^0 &= \frac{f_1 b_1}{1 + f_1 b_1 - f_1 b_0} \end{aligned}$$

Let $v_n(r)$ be the average value starting when the cycle starts with r. The change in value from an increase in the initial inventory by one unit can be written as,

$$v_n(r') - v_n(r' - 1) = \frac{1}{n} \left[\gamma pr(sold) - \lambda E[\text{time}]\right],$$

where $pr(sold)$ is the probability that the marginal unit is sold and $E[\text{time}]$ is the expected holding time of the marginal unit. The difference in value between the bar code case and the return from no bar codes and the $\overline{X}$ policy is the probability that no bar code case starts at $r_n^* - 1$ times the change in value when that happens,

$$v_n^1 - \overline{v}_n^2 = p^0 \frac{1}{n} \left[\gamma pr(sold) - \lambda E[\text{time}] \right].$$

The value of bar codes with a one-period cycle is

$$\Delta(1) = \frac{f_1}{1+f_1} \left[f_1 \gamma - \lambda \right]$$

Thus a sufficient condition for $\Delta(1) \geq \Delta(n)$ is

$$\frac{f_1}{1+f_1} \left(f_1 \gamma - \lambda \right) \geq p^0 \frac{1}{n} \left[\gamma pr(sold) - \lambda E[\text{time}] \right] \tag{14}$$

I examine this condition by varying λ in this condition holding $pr(sold)$ fixed (Of course as λ varies it varies r_n^* which affects $pr(sold)$. But by showing the condition holds for any λ it will of course hold for the λ that is consistent with the given $pr(sold)$.

Suppose first that

$$\frac{f_1}{1+f_1} \geq p^0 \frac{1}{n} E[\text{time}]$$

Then varying λ holding $pr(sold)$ fixed, to show the condition holds it is sufficient to show that it holds at the maximum λ of $\lambda = \gamma f_1$, since the slope of the LHS is less than the slope of the RHS. But since

$$pr(sold) \leq f_1 E[\text{time}]$$

must clearly hold, the RHS is nonpositive at $\lambda = \gamma f_1$, while the LHS is zero, so the condition holds there.

Next consider the alternative case where

$$\frac{f_1}{1+f_1} < p^0 \frac{1}{n} E[\text{time}]$$

Since the slope of the LHS is greater than the slope of the RHS, it is sufficient to show (14) is true at $\lambda = 0$. Evaluating at $\lambda = 0$ and dividing through by γ, the condition is

$$\frac{f_1}{1+f_1} f_1 > p^0 \frac{1}{n} pr(sold)$$

Let $g(x,t)$ be the probability that exactly x show up over t periods.

$$\begin{aligned} b_1 &= g(0, n-1) + g(1, n-1) + ...g(r_n^* - 1, n-1) \\ b_0 &= b_1 - g(r_n^* - 1, n-1) \end{aligned}$$

The sufficient condition can be rewritten as

$$\frac{f_1}{1+f_1} f_1 > \frac{f_1 \left[g(0,n-1) + g(1,n-1) + ...g(r_n^*-1,n-1)\right]}{1+f_1 g(r_n^*-1,n-1)} \frac{1}{n} \left[\sum_{r=r_n^*}^{n} g(r,n)\right]$$

or

$$f_1 > \frac{1+f_1}{1+f_1 g(r_n^*-1,n-1)} \frac{1}{n} \left[\sum_{r=0}^{r_n^*-1} g(r,n-1)\right] \left[\sum_{r=r_n^*}^{n} g(r,n)\right] \tag{15}$$

3.2.4 Step 4: Case of $n = 3$

For $n = 3$, the optimal initial inventory r_3^* with bar codes equals 3 if and only if

$$f_1^3 \gamma < 3\lambda.$$

But as above, $\lambda > \hat{\lambda}$ implies this does not hold, so $r_3^* = 3$ cannot hold in the case under consideration. If $r_3^* = 1$, the result of step 1 shows that $\Delta(3) \leq \Delta(1)$. So assume the remaining case where $r_3^* = 2$ applies. The sufficient condition (15) reduces to

$$f_1 > \frac{1+f_1}{1+f_1 g(1,2)} \frac{1}{n} \left[g(0,2) + g(1,2)\right] \left[g(2,3) + g(3,3)\right] \tag{16}$$

$$= \frac{1+f_1}{1+f_1 2 f_0 f_1} \frac{1}{3} \left(1-f_1^2\right) \left[3 f_0 f_1^2 + f_1^3\right] \tag{17}$$

This holds iff

$$\begin{aligned} 1 &> \frac{1+f_1}{1+f_1 2 f_0 f_1} \frac{1}{3} \left(1-f_1^2\right) \left[3 f_0 f_1 + f_1^2\right] \\ &= \frac{1+f_1}{1+f_1 2 f_0 f_1} \frac{1}{3} \left(1-f_1^2\right) f_1 \left[2 f_0 + f_0 + f_1\right] \\ & \quad \frac{1+f_1}{1+f_1 2 f_0 f_1} \frac{1}{3} \left(1-f_1^2\right) f_1 \left[1 + 2 f_0\right] \end{aligned}$$

A sufficient condition for this to hold is

$$1 > \frac{1+f_1}{1+f_1 2 f_0 f_1} \left(1-f_1^2\right) f_1 \tag{18}$$

This holds iff

$$\begin{aligned} 1 + f_1 2 f_0 f_1 &> (1+f_1)\left(1-f_1^2\right) f_1 \\ 1 + 2 f_1^2 (1-f_1) &> f_1 + f_1^2 - f_1^3 - f_1^4 \end{aligned}$$

or

$$1 + 2 f_1^2 - 2 f_1^3 > f_1 + f_1^2 - f_1^3 - f_1^4.$$

This holds iff

$$1 - f_1 + f_1^2 - f_1^3 + f_1^4 > 0$$

which holds. This completes the analysis of the $n = 3$ case.

4 Proof of Proposition 5

Proposition 5. Under either formulation of the objective function, the advent of bar codes (weakly) decreases the order cycle length and (weakly) increases the optimal number of products m.

Proof.

Case 1: Profit Maximization

Consider first the case where the objective is to maximize profit. Following Milgrom and Roberts, it is sufficient to show that the objective function

$$\pi(m, n, \ell) = mv^{\ell}(n) - \frac{\phi}{n} - mc_S(m). \tag{19}$$

is supermodular in m, $-n$, and $-\ell$. Let $x = (n, \ell)$ and $x' = (n', \ell')$. Take m and m' and assume without loss of generality that $m \leq m'$. Following Milgrom and Roberts, I need to show that

$$\pi(m, x) + \pi(m', x') \leq \pi(m, \max(x, x')) + \pi(m', \min(x, x')). \tag{20}$$

There are four cases.

(i) $n \geq n'$ and $\ell \geq \ell'$.

In this case $x = \max(x, x')$ and $x' = \min(x, x')$ so obviously (20) holds.

(ii) $n < n'$ and $\ell \geq \ell'$.

I need to show

$$\pi(m, n, \ell) + \pi(m', n', \ell') \leq \pi(m, n', \ell) + \pi(m', n, \ell').$$

or

$$\begin{aligned} & mv^{\ell}(n) - \frac{\phi}{n} - mc_S(m) + m'v^{\ell'}(n') - \frac{\phi}{n'} - m'c_S(m') \\ \leq \quad & mv^{\ell}(n') - \frac{\phi}{n'} - mc_S(m) + m'v^{\ell'}(n) - \frac{\phi}{n} - m'c_S(m') \end{aligned}$$

or

$$mv^{\ell}(n) + m'v^{\ell'}(n') \leq mv^{\ell}(n') + m'v^{\ell'}(n)$$

or

$$m\left[v^{\ell}(n) - v^{\ell}(n')\right] \leq m'\left[v^{\ell'}(n) - v^{\ell'}(n')\right].$$

Since $n < n'$, both bracketed terms are positive. Since $m \leq m'$, to show the inequality holds it is sufficient to show that

$$v^{\ell}(n) - v^{\ell}(n') \leq v^{\ell'}(n) - v^{\ell'}(n')$$

or equivalently

$$v^{\ell'}(n') - v^{\ell}(n') \leq v^{\ell'}(n) - v^{\ell}(n).$$

If $\ell = \ell'$ then this obviously holds. So assume $\ell > \ell'$. Then the above equality is the same as

$$\Delta(n') \leq \Delta(n)$$

which holds since $n < n'$.

(iii) $n \geq n'$ and $\ell < \ell'$. I need to show that

$$\pi(m, n, \ell) + \pi(m', n', \ell') \leq \pi(m, n, \ell') + \pi(m', n', \ell)$$

or

$$\begin{aligned} & mv^{\ell}(n) - \frac{\phi}{n} - mc_S(m) + m'v^{\ell'}(n') - \frac{\phi}{n'} - m'c_S(m') \\ \leq \;\; & mv^{\ell'}(n) - \frac{\phi}{n} - mc_S(m) + m'v^{\ell}(n') - \frac{\phi}{n'} - m'c_S(m') \end{aligned}$$

or

$$mv^{\ell}(n) + m'v^{\ell'}(n') \leq mv^{\ell'}(n) + m'v^{\ell}(n')$$

or

$$m\left[v^{\ell}(n) - v^{\ell'}(n)\right] \leq m'\left[v^{\ell}(n') - v^{\ell'}(n')\right]$$

or

$$m\Delta(n) \leq m'\Delta(n')$$

which holds since $m \leq m'$ and $\Delta(n) \leq \Delta(n')$ (since $n \geq n'$).

(iv) $n < n'$ and $\ell < \ell'$. I need to show

$$\pi(m, n, \ell) + \pi(m', n', \ell') \leq \pi(m, n', \ell') + \pi(m', n, \ell)$$

or

$$\begin{aligned} & mv^{\ell}(n) - \frac{\phi}{n} - mc_S(m) + m'v^{\ell'}(n') - \frac{\phi}{n'} - m'c_S(m') \\ \leq \;\; & mv^{\ell'}(n') - \frac{\phi}{n'} - mc_S(m) + m'v^{\ell}(n) - \frac{\phi}{n} - m'c_S(m') \end{aligned}$$

or

$$mv^{\ell}(n) + m'v^{\ell'}(n') \leq mv^{\ell'}(n') + m'v^{\ell}(n)$$

or

$$m\left[v^{\ell}(n) - v^{\ell'}(n')\right] \leq m'\left[v^{\ell}(n) - v^{\ell'}(n')\right]$$

which holds since $m \leq m'$ and $v^{\ell}(n) \geq v^{\ell'}(n')$ (the latter follows from $n < n'$ and $\ell < \ell'$). This completes the proof of the case where the objective function is total profit.

Case 2: Total Surplus Maximization

As above, I show that the objective function

$$w(m,n,\ell) = v^{\ell}(n) - \frac{\phi}{nm} - c_S(m). \tag{21}$$

is supermodular in m, $-n$, and $-\ell$. I need to show that

$$w(m,x) + w(m',x') \leq w(m, \max(x,x')) + w(m', \min(x,x')). \tag{22}$$

Assume without loss of generality that $m \leq m'$. There are four cases.

(i) $n \geq n'$ and $\ell \geq \ell'$.

In this case $x = \max(x,x')$ and $x' = \min(x,x')$ so obviously (21) holds.

(ii) $n < n'$ and $\ell \geq \ell'$.

I need to show

$$w(m,n,\ell) + w(m',n',\ell') \leq w(m,n',\ell) + w(m',n,\ell').$$

or

$$\begin{aligned} & v^{\ell}(n) - \frac{\phi}{mn} - c_S(m) + v^{\ell'}(n') - \frac{\phi}{m'n'} - c_S(m') \\ \leq \;\; & v^{\ell}(n') - \frac{\phi}{mn'} - c_S(m) + v^{\ell'}(n) - \frac{\phi}{m'n} - c_S(m') \end{aligned}$$

or

$$\begin{aligned} & v^{\ell}(n) - \frac{\phi}{mn} + v^{\ell'}(n') - \frac{\phi}{m'n'} \\ \leq \;\; & v^{\ell}(n') - \frac{\phi}{mn'} + v^{\ell'}(n) - \frac{\phi}{m'n}. \end{aligned}$$

I will show this holds in two steps. The first step is to show that

$$v^{\ell}(n) + v^{\ell'}(n') \leq v^{\ell}(n') + v^{\ell'}(n),$$

but this follows from the fact that $v^{\ell}(n) - v^{\ell'}(n)$ is decreasing in n for $\ell < \ell'$. The second step is to show that

$$-\frac{\phi}{mn} - \frac{\phi}{m'n'} \leq -\frac{\phi}{mn'} - \frac{\phi}{m'n}$$

or

$$\frac{1}{m'}\left[\frac{\phi}{n} - \frac{\phi}{n'}\right] \leq \frac{1}{m}\left[\frac{\phi}{n} - \frac{\phi}{n'}\right]$$

which follows because $n < n'$ (so the bracketed term is positive) and $m \leq m'$.

(iii) $n \geq n'$ and $\ell < \ell'$. I need to show that

$$w(m,n,\ell) + w(m',n',\ell') \leq w(m,n,\ell') + w(m',n',\ell)$$

or

$$\begin{aligned} & v^{\ell}(n) - \frac{\phi}{mn} - c_S(m) + m'v^{\ell'}(n') - \frac{\phi}{m'n'} - c_S(m') \\ \leq \;\; & v^{\ell'}(n) - \frac{\phi}{mn} - c_S(m) + m'v^{\ell}(n') - \frac{\phi}{m'n'} - c_S(m') \end{aligned}$$

or

$$v^{\ell}(n) + v^{\ell'}(n') \leq v^{\ell'}(n) + v^{\ell}(n')$$

or

$$\left[v^{\ell}(n) - v^{\ell'}(n)\right] \leq \left[v^{\ell}(n') - v^{\ell'}(n')\right]$$

or

$$\Delta(n) \leq \Delta(n')$$

which holds since $n \geq n'$.

(iv) $n < n'$ and $\ell < \ell'$. I need to show

$$w(m,n,\ell) + w(m',n',\ell') \leq w(m,n',\ell') + w(m',n,\ell)$$

or

$$\begin{aligned} & v^{\ell}(n) - \frac{\phi}{mn} - c_S(m) + v^{\ell'}(n') - \frac{\phi}{m'n'} - c_S(m') \\ \leq \;\; & v^{\ell'}(n') - \frac{\phi}{mn'} - c_S(m) + v^{\ell}(n) - \frac{\phi}{m'n} - c_S(m') \end{aligned}$$

or, canceling terms,

$$-\frac{\phi}{mn} - \frac{\phi}{m'n'} \leq -\frac{\phi}{mn'} - \frac{\phi}{m'n}.$$

But since $n < n'$, this holds by the same calculation as in case (ii) above. *Q.E.D.*