Optimal City Hierarchy: A Dynamic Programming Approach to Central Place Theory

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Abstract

Central place theory is a key building block of economic geography and is an empirically plausible description of city systems. As shown recently by Hsu (2008) and Mori et al. (2008), it also provides a route to explain empirical regularities in city size distribution and industrial locations. This paper formalizes central place hierarchy by providing a rationale for it via a social planner's problem in both one-dimensional and two-dimensional spaces. We then use the optimal city hierarchy to study efficiency properties of the equilibrium hierarchy in Hsu (2008).

JEL: R12; R13

Keywords: dynamic programming, central place theory, hexagonal market area, Zipf's law, number-average size rule

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1 Introduction

Central place theory describes how a city hierarchy is formed out of a featureless plain of farmers. It is a key building block of economic geography (King, 1984) and dates back at least to Christaller (1933). Many have argued for its empirical plausibility as a description of city hierarchy (Fujita, Krugman, and Venables, 1999; Mori and Smith, 2008; Berliant, 2008). Although original central place theory is not a rigorous economic theory based on incentives and equilibrium, many economists have found its insights appealing, and a few attempts have been made to formalize it, including those by Eaton and Lipsey (1982), Quinzii and Thisse (1990), Fujita, Krugman, and Mori (1999), Tabuchi and Thisse (2008), and Hsu (2008).

The basic idea of this theory is that goods differ in their degree of scale economies relative to market size. Goods for which this ratio is large, e.g., stock exchanges or symphony orchestras, will be found in only a few places, whereas goods for which it is small, e.g., gas stations or convenience stores, will be found in many places. Moreover, large cities tend to have a wide range of goods, whereas small cities provide only goods with low scale economies. Naturally, small cities are in the market areas of large cities for those goods that they themselves do not provide. In Christaller's scheme, the *hierarchy property*[1] holds if larger cities provide all of the goods that smaller cities also provide and more.

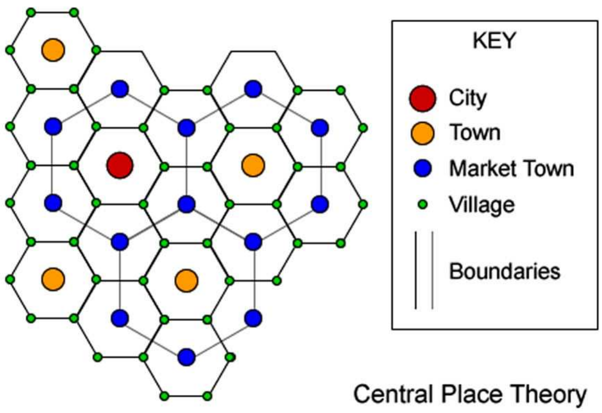

Figure 1: Central Place Hierarchy on the Plane

In this paper, a *city system* has multiple layers of cities, and cities of the same layer have the same functions, i.e., they host the same set of industries. The driving force behind the differentiation of cities is the heterogeneity of scale economies among goods, which is modeled by heterogeneity in the fixed costs of production. In addition to the hierarchy property,

[1]This is often called the *hierarchy principle* in the literature.

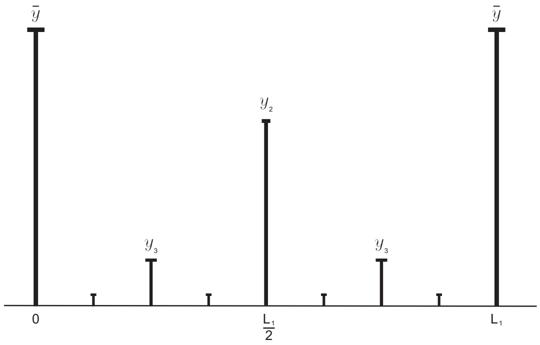

Figure 2: Central Place Hierarchy on the Line

another defining feature of city hierarchy in central place theory, that called the *central place property*, is that there is always *only one* next-layer city in between (theoretically, in the middle of) neighboring larger cities. Christaller (1933) calls this the K = 3 *market principle.*[2] The city hierarchy described by central place theory (hereafter *central place hierarchy*) is a city system in which both the hierarchy and central place properties hold. Figures 1[3] and 2 show illustrations of such city hierarchies on the plane and on the line, respectively. In the case of the plane, the market areas are hexagonal. In the case of the line, the dimension of the range of goods produced in different layers of cities is shown.[4]

This paper takes aim at providing a rationale for central place theory via a social planner's problem. An innovative feature of this paper is that the social planner's problem is formulated as a dynamic programming problem in a geographic space (instead of in time). In this paper, we ask what optimal city hierarchy would arise from a uniformly populated space via the tradeoff between transportation costs and the fixed costs of production. To the best of our knowledge, Quinzii and Thisse (1990) is the only other paper that asks how a central place hierarchy emerges from a socially optimal solution. Although Quinzii and Thisse (1990) provide the conditions under which the hierarchy property emerges in the optimal solution, their optimal solution does not feature the central place property. In contrast, the focus of this paper is on whether and how the central place property emerges conditional on the hierarchy property. To put it in more general terms, this paper does not ask why

[2]On the plane, if there is always only one next-layer city located in the equilateral triangle area in between three neighboring larger cities, then the ratio of the market areas is 3.

[3]Courtesy of Paul Thompson and the Wolf at the Door website. This graph can be downloaded from http://wolf.readinglitho.co.uk/mainpages/sustainability.html.

[4]The total range of the goods indexed by fixed cost of production y is $[0, \bar{y}]$. The hierarchy property implies that each city provides goods in $[0, y]$ for some y. Hence, a layer-i city provides goods in $[0, y_i]$, and, obviously, $y_1 = \bar{y}$.

firms agglomerate or why cities exist; instead, it asks whether, and if yes, why, the spacing in central place theory is optimal.

The two main contributions of this paper are as follows. First, it shows a sufficient condition under which the central place property emerges conditional on the hierarchy property. This condition applies to both spaces of the line and the plane. Second, the paper provides two results for the efficiency properties of the equilibrium hierarchy (on the line) modeled in Hsu (2008). Using a one-good model, Lederer and Hurter (1986) show that equilibrium entry is socially optimal. Here, in an extension to a continuum of goods, we show that the optimal solution can be decentralized if the central place property holds. However, other suboptimal equilibria also exist. With regard to the welfare properties of equilibrium entry in a spatial competition model, Salop (1979), also using a one-good model, shows that there is always greater equilibrium entry than what is optimal. Here, we show that when the distance between the two largest cities is the same between the equilibrium and optimal solution, then the equilibrium and optimal entries for each good coincide. However, in contrast to Salop (1979), we also show that when equilibrium entry deviates from the socially optimal solution, the directions of deviation for different goods are different.

The extension to the plane utilizes Morgan and Bolton's (2002) theorem that the hexagonal market area is most efficient in terms of saving on transportation cost, given the number of cities. Hence, conditional on the hierarchy property, we successfully rebuild the central place hierarchy on the plane as per Christaller (1933) and prove Christaller's conjecture that the market principle (the ratio of the market areas of one layer to the next being 3) is the most efficient way of organizing the hierarchy.

The central place hierarchy in this paper shares the same structure as that in Hsu (2008), who has shown that, under a rather general class of the distribution functions of fixed costs, this hierarchy leads to Zipf's laws for cities[5] and firms, as well as to a newly documented empirical regularity called the Number-Average-Size (NAS) rule.[6] Unlike other theories of urban systems and city size distribution, our explanation of city size distribution is based on what cities do differently and how things occur geographically, rather than on a statistical property that arises from the random growth process of cities with no inter-city spatial relations.[7] Moreover, Mori, Nishikimi, and Smith (2008) have shown that if the hierarchy

[5]Beckmann (1958) has shown that Zipf's law for cities may be the result of a hierarchical structure. However, he does not provide a microfoundation for this structure, and his conditions are different from those of Hsu (2008).

[6]Zipf's law states that the size distribution of cities can be approximated by the Pareto distribution with a tail index close to 1, and the NAS rule states that the number and average size of cities in which an industry is located have a log-linear relationship. For evidence on Zipf's law for firms, see Axtell (2001) and Luttmer (2007). Also see Luttmer (2007) for a theoretical explanation.

[7]For explanations along this line, see Simon (1955), Gabaix (1999), Eeckhout (2004), Duranton (2006,

property holds, then the NAS rule and Zipf's law are essentially equivalent. Hence, central place theory provides a key connection between two empirical regularities, and other urban theories, i.e., those without different industries or without the hierarchy property, have little explanatory power in this regard.[8] In a broad sense, the reason that the central place hierarchy leads to these power-law related regularities is that it is close to a spatial fractal structure. For those readers interested in more details on Zipf's law, the NAS rule, the hierarchy property, and the relationships among them, see Hsu (2008).

The rest of this paper is organized as follows. Section 2 lays out the social planner's problem on the line and derives the central place property. Section 3 retrieves the necessary results from Hsu (2008) and carries out a welfare analysis. Section 4 extends the model to the plane and shows that the central place property still holds. Section 5 concludes.

2 Social Planner's Problem

2.1 Model setup

The geographic space is the real line on which consumers are uniformly distributed.[9] There is a continuum of commodities labeled $x \in [0, z_1]$, and each consumer demands one unit of each $x \in [0, z_1]$. To produce any good x, a fixed cost $\phi(x)$ is required. The marginal cost is a constant c. Rank the goods in terms of their fixed costs, and assume that no two goods have the same fixed cost. With differentiability of ϕ, $\phi'(.) > 0$. To transport any good requires a cost of t per unit of distance. Assume the hierarchy property: at any location, if z is produced, then all $x \in [0, z]$ are also produced.

2.2 The problem

The social planner's problem is to find the optimal allocation of production locations to minimize the per capita cost. We can ignore the variable cost per capita, as it must be cz_1, regardless of the allocation. Imagine now that social planner has to decide how to place the production locations for the goods $[0, z]$, where $z > 0$ is arbitrary. Due to the hierarchy property, there must be locations that produce all $x \in [0, z]$, and they must be evenly spaced to save on transportation costs. Let the distance between these "cities" with $[0, z]$ (z-cities

2007), Rossi-Hansberg and Wright (2007), and Córdoba (2008).

[8]For example, Henderson's (1974) type-of-cities theory and its extension in Rossi-Hansberg and Wright (2007) do not feature the hierarchy property, as they are concerned with the specialization of cities and assume that each city specialize in only one industry.

[9]We can think of these consumers as farmers who would locate themselves uniformly if agricultural productivity were uniform all over the plane and if the farming technology were Leontief in land and labor.

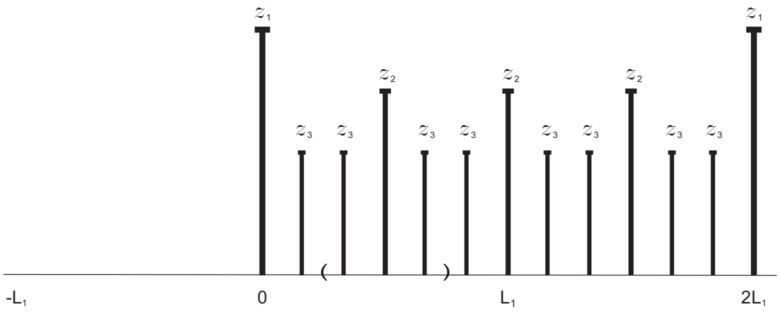

Figure 3: City Planting

hereafter) be denoted as $2L$. Hence, L is the radius of the market area. If the social planner does nothing else, then the total cost (the sum of transportation and fixed costs) per capita must be

$$\frac{1}{2L}[\Phi(z) + ztL^2] \equiv C_f(L, z),$$

where $\Phi(z) = \int_0^z \phi(x)dx$ is the total fixed costs incurred in a z-city.

However, the social planner also contemplates the possibility of having some z'-cities in between two z-cities ($0 < z' \leq z$) so that the cost of transporting these less heavy goods $[0, z']$ can be saved. Having such smaller z'-cities is desirable when the total fixed costs incurred for these goods are not too large. Hence, given $z > 0$ and $L > 0$, the social planner solves the following dynamic programming problem.

$$(FE_f)$$

$$C(L, z) = \min\{C_f(L, z), \min_{n', z'} \quad \frac{1}{2L}[\Phi(z) - \Phi(z') + (z - z')tL^2] + C(L', z')\}$$

$$s.t. \quad L' = \frac{L}{n' + 1}, \; n' \in \mathbb{N}, z' \in [0, z].$$

where n' is the number of z'-cities placed in between two z-cities. This is the core problem that the social planner must solve for any $z, L > 0$. Once this problem has been solved, she can solve the optimal radius for the z_1-cities:

$$L_1 = \arg\min_L \quad C(L, z_1).$$

In sum, the social planner's problem is a *recursive city planting* problem in which the planner first chooses the radius for the z_1-cities, which are the first-layer cities, and then she decides how many second-layer cities (n_2 z_2-cities) to plant in between, and this occurs recursively. An example of recursive city-planting is illustrated in Figure 3, in which $n_2 = 3$ and $n_3 = 2$.

2.3 Characterization

Denote the policy function for the choice of n' solving the sub-problem of (FE_f) as $g(L,z)$. The policy function for z' of the sub-problem is a simple rule given by

$$z'^o = \phi^{-1}\left(\frac{tL^2}{n'+1}\right) = \phi^{-1}\left(\frac{tL^2}{g(L,z)+1}\right). \tag{1}$$

We arrive at this rule by combining the first-order condition (2) and the envelope condition (3) of the sub-problem:[10]

$$\frac{\partial C(\frac{L}{n'+1}, z'^o)}{\partial z} = \frac{1}{2L}[\phi(z'^o) + tL^2], \tag{2}$$

$$\frac{\partial C(L,z)}{\partial z} = \frac{1}{2L}[\phi(z) + tL^2]. \tag{3}$$

Barring a choice of $C_f(L,z)$ in (FE_f), the policy function g determines the sequence of $n_i = g(L_{i-1}, z_{i-1})$, $i \geq 2$, and hence $L_i = L_{i-1}/(n_i+1)$ and

$$z_i^o = \phi^{-1}\left(\frac{tL_{i-1}^2}{n_i+1}\right). \tag{4}$$

Obviously, $\{z_i\}$ and $\{L_i\}$ are decreasing sequences. Whenever the social planner chooses $C_f(L,z)$, given L_I, z_I for some integer $I \geq 1$, then I becomes the number of layers in the city hierarchy. However, it turns out that one simple condition, $\phi(0) = 0$, guarantees infinite layers and hence a cleaner form of the dynamic programming problem:

(FE)

$$C(L,z) = \min_{n',z'} \quad \frac{1}{2L}[\Phi(z) - \Phi(z') + (z-z')tL^2] + C(L', z')$$

$$s.t. \quad L' = \frac{L}{n'+1}, \; n' \in \mathbb{N}, z' \in [0,z].$$

Proposition 1 (Infinite layers)**.** *Suppose $\phi(0) = 0$, the first term in the objective $(C_f(L,z))$ of (FE_f) is never the optimal choice for any pair of L and z. Hence, there are infinitely many layers. If $\phi(0) > 0$, then there are only I layers, and I is the largest integer such that*

$$\frac{tL_{I-1}^2}{n_I+1} \geq \phi(0). \tag{5}$$

Proof. Given $n' = g(L,z)$ and $\phi(0) = 0$, $z'^o(n') = \phi^{-1}\left(\frac{tL^2}{n'+1}\right)$ always exists. The sub-problem in (FE_f) becomes

$$\begin{aligned} &\min_{n'\geq 1} \quad \frac{1}{2L}\int_{z'^o(n')}^{z} [\phi(x) + tL^2]dx + C\left(\frac{L}{n'+1}, z'^o(n')\right) \\ =& C_f(L,z) + \min_{n'\geq 1} \quad C\left(\frac{L}{n'+1}, z'^o(n')\right) - \frac{1}{2L}\int_0^{z'^o(n')} [\phi(x) + tL^2]dx. \end{aligned} \tag{6}$$

[10]The second-order condition is easy to check, and it holds because $\phi'(.) > 0$.

In fact,

$$
\begin{aligned}
&\min_{n'\geq 1} \; C\left(\frac{L}{n'+1}, z'^{o}(n')\right) - \frac{1}{2L}\int_0^{z'^{o}(n')} [\phi(x) + tL^2]dx \\
\leq &\min_{n'\geq 1} \; \frac{n'+1}{2L}\int_0^{z'^{o}(n')} \left[\phi(x) + \frac{tL^2}{(n'+1)^2}\right] dx - \frac{1}{2L}\int_0^{z'^{o}(n')} [\phi(x) + tL^2]dx \\
= &\min_{n'\geq 1} \; \frac{n'}{2L}\int_0^{\phi^{-1}\left(\frac{tL^2}{n'+1}\right)} \left[\phi(x) - \frac{tL^2}{n'+1}\right] dx < 0.
\end{aligned}
$$

Hence, $C_f(L, z)$ is never the optimal choice. If $\phi(0) > 0$, then the decreasing nature of L_i implies that (4) holds only for a finite number of i's. Given any sequence of $\{n_i\}_{i=2}^{\infty}$, consider the largest integer I such that (5) holds. Then, $\phi(0) > tL_I^2/(n_{I+1}+1)$. Given $L = L_I$ and $z = z_I$, the social planner will choose $C_f(L, z)$, and hence the city planting stops. This is because, given L_I and z_I, for any $n_{I+1} = n' \in \mathbb{N}$, the optimal choice in the sub-problem must be $z_{I+1} = z'^{o} = 0$, which is equivalent to choosing $C_f(L, z)$. To see this, simply observe that, for all $z' \in [0, z_I]$, the first-order derivative of the second term in (FE_f) is

$$
\frac{n'}{2L}[\phi(z') - \frac{tL^2}{n'+1}] \geq \frac{n'}{2L}[\phi(0) - \frac{tL^2}{n'+1}] = \frac{n_{I+1}}{2L_I}[\phi(0) - \frac{tL_I^2}{n_{I+1}+1}] > 0.
$$

□

It is also useful to define an equivalent sequence problem as follows.

(SP)

$$
\begin{aligned}
C(L, z) = &\min_{\{n_i, z_i\}_{i=2}^{I}} \sum_{i=2}^{I+1} \frac{1}{2L_{i-1}}[\Phi(z_{i-1}) - \Phi(z_i) + (z_{i-1} - z_i)tL_{i-1}^2] \\
&s.t. \;\; L_i = \frac{L_{i-1}}{n_i+1}, \quad n_i \in \mathbb{N}, z_i \in [0, z_{i-1}] \quad \forall\, i \geq 2, \\
&\qquad L_1 = L, \;\; z_1 = z,
\end{aligned}
$$

where the number of layers I is given by Proposition 1, and if I is finite, then $z_{I+1} = 0$.

2.4 Central place property

It is difficult to solve the sequence of $\{n_i\}_{i=2}^{\infty}$ without assuming a functional form. Therefore, we focus on the two prototypes of the class of fixed cost requirement functions in Hsu (2008) that lead to Zipf's law. These are the power function ($\phi(x) = abx^{b-1}, a > 0, b > 1$) and the exponential function ($\phi(x) = ae^{bx}, a > 0, b > 0$). The exponential function is, in fact, the limit of the power function.[11] The exponential function case is difficult to solve analytically because it allows only finite layers. The solution for this case is thus obtained by solving the problem numerically. The power function, in contrast, has $\phi(0) = 0$ and gives infinite layers. It can be solved analytically using the guess-and-verify technique.

Proposition 2 (Central place property)**.** *Assume that the fixed-cost requirement function is* $\phi(x) = abx^{b-1}, a > 0, b > 1$, *and hence* $\Phi(z) = az^b$. *Then,* $n'^o = 1,\ \forall L, z > 0$.

Proof. The complete proof is given in the Appendix, and a sketch is provided here. With the assumption of $\phi(.)$, (4) becomes

$$z_i^o = \left(\frac{tL_{i-1}^2}{ab(n_i + 1)} \right)^{\frac{1}{b-1}}. \tag{7}$$

By plugging (7) into (SP), the problem is reduced to finding the optimal sequence of $\{n_i\}_{i=2}^{\infty}$. We guess that $n_i^o = 1$ in (SP) to obtain a guess for the functional form of C. By applying the guess-and-verify technique to (FE), we verify that the unique optimal solution is, indeed, $n_i^o = 1$ for all i. □

Next, we show the solution to the exponential function case. Define $\phi(x) = ae^{bx}, a > 0, b > 0$. By applying (4), we have

$$z_i^o = \frac{1}{b} \ln \frac{tL_{i-1}^2}{a(n_i + 1)}.$$

There are only finite layers, as $\phi(0) > 0$.[12] By Proposition 1, given any $\{n_i\}_{i=2}^{K}$, $I < K$ is determined as the largest integer satisfying $z_I \geq 0$. Find the solution using (SP) is quick. For $\{n_i\}_{i=2}^{K}$, we simply try out all of the elements in the set of $\{1, 2, ..., \bar{n}\}^{K-1}$ with (SP). We need K to be large enough such that there exists an $I < K$. We also need $\bar{n}$ to be large to ensure that the solution is correct. The solution is again $n_i^o = 1$ for all $2 \leq i \leq I$, for extensive parameter values.[13]

[11] Hsu (2008) uses the inverse of the fixed cost function, that is, the distribution function of the fixed costs. In Proposition 3 in Hsu (2008), when the density function takes the exponent of $\alpha = 0$, it corresponds to the exponential function in this paper, and when $\alpha > 0$, it corresponds to the power function.

[12] One may wonder whether there perhaps exists a variant of the exponential fixed cost requirement function with $\phi(0) = 0$, and hence we can utilize the stationarity. However, the empirical relevance of such a variant is weak. For example, take $\phi(x) = ae^{bx} - a$; this function gives infinite layers and, according to Hsu (2008), an approximate Pareto distribution with a tail index of 1/3. However, the lowest estimated tail index among the 73 countries examined by Soo (2005) is in Australia, which has a tail index of 0.5855 and only 21 cities.

[13] A Matlab code that computes the solution can be obtained from the authors upon request.

2.5 Zipf's law for cities and the NAS rule

The central place property and (4) imply that $z_i = \phi^{-1}(\frac{tL_1^2}{2^{2i-3}})$. Denote y_i as the fixed cost of z_i; then

$$y_i \equiv \phi(z_i) = \frac{tL_1^2}{2^{2i-3}}, \tag{8}$$

which is exactly the zero-profit condition that pins down the cutoff fixed cost for each layer in Hsu (2008).[14] Therefore, the city hierarchies in both papers share the same structure. The only difference between an optimal solution and an equilibrium may be the radius of the market area of layer-1 cities, L_1. However, as is clear in Hsu (2008), the magnitude of L_1 does not matter in the proof of Zipf's law or the NAS rule. Hence, both empirical regularities also emerge from the optimal solution.

3 Welfare Analysis

The environment of Hsu's (2008) model is the same as that in this paper. Thus, we can compare the equilibrium allocation in his model with the optimal solution presented here. In the firm-entry part, Hsu (2008) has an infinite pool of firms that play the following two-stage game.[15]

1. Entry and location stage:
   Firms simultaneously decide whether to enter, and, upon entering, they must decide their locations. They need to pay a fixed cost to set up at any location. Assume the tie-breaking rule that if a potential firm sees a zero-profit opportunity, then it enters.

2. Price competition stage:
   Firms deliver goods to farmers. Given the locations of firms, each firm sets a (delivered) price schedule over the real line. For each good, each location on the real line is a market in which firms engage in Bertrand competition. For each good, each farmer decides which firm to buy from.

Hsu (2008) focuses on a set of equilibria that are consistent with the hierarchy property, i.e., the hierarchy equilibria. Proposition 1 in Hsu (2008) provides the characterization of this set. Briefly, there is a continuum of hierarchy equilibria, each of which satisfies the central place property. By this property, the market area of cities shrinks by half from one

[14]Note that, in Hsu (2008), L denotes the length of the market area, rather than the radius. With this understanding, the zero-profit condition in Proposition 1 in Hsu (2008) is exactly $y_i = \frac{tL^2}{2^{2i-3}}$.

[15]This game setup first appeared in Lederer and Hurter (1986).

layer to the next, and the top good in each layer has fixed cost $\{y_i\}_{i=2}^{I}$ as given by the zero-profit condition (8). There is a continuum of hierarchy equilibria because the subgame perfect Nash equilibrium of top good z_1 allows a continuum of equilibrium radius. More specifically, $L_1 \in [L[z_1], 2L[z_1])$, where $L[z_1]$ is the zero-profit radius given by

$$L[z_1] = \sqrt{\frac{\phi(z_1)}{2t}}. \tag{9}$$

The key questions in this section are whether the social planner's solution can be decentralized, and, if there is a discrepancy between an equilibrium and the optimal solution, what the pattern of deviation in terms of entry is. We present two major findings in this section.

First, we show that, if the central place property holds, the optimal solution can be decentralized. Including the power and exponential functions, any functional form giving rise to the central place property entails a decentralizable solution. Second, also conditional on the central place property, the entry comparison yields a result that, whenever there is a discrepancy between the equilibrium and the optimal solution, there is a "choppy pattern," which means that the whole range of goods can be partitioned into sets, such that the first set has an equilibrium entry that is less (more) than that in the optimal solution, the second set has one that is more (less), the third set less (more), and the fourth set more (less), etc. This result contrasts Salop's (1979) finding that equilibrium entry is always more than optimal in his one-good model.

3.1 Decentralization or not

In this section, we assume that the central place property holds.

The value function (or indeed the cost function) $C(L, z)$ must be periodic in L. Denote the smallest optimal radius as L^o. The value function is periodic because what can be done at $L = L^o$ should be done at $L = 2L^o$ with $z' = z$. Also, according to (1), z'^o is strictly increasing in L. Denote $\bar{L}(z)$ as the L such that z'^o grows to exactly z. That is,

$$\bar{L}(z) = \sqrt{\frac{2\phi(z)}{t}} = 2L[z], \tag{10}$$

where $L[z]$ denotes the zero-profit radius of z defined similarly to (9). Obviously, $C(\bar{L}(z), z) = C(\bar{L}(z)/2, z)$. Moreover, as L is in the right neighborhood of $\bar{L}(z)$ ($L \geq \bar{L}(z)$), the problem becomes the same as that with $\widetilde{L} = L/2 \geq \bar{L}(z)/2$, where $\widetilde{L}$ denotes the effective market area of z since now $z'^o = z$. Hence, $C(L, z) = C(L/2, z)$ for $L \in [\bar{L}(z), 2\bar{L}(z))$.

Now, let $z = z_1$ and $L_u \equiv \bar{L}(z_1)$. The generalization to the discussion above is that for any $k = 0, 1, 2, ...,$ and for any $L \in [2^{k-1}L_u, 2^k L_u)$, we have $z_{k+1} = z_k = ... = z_2 = z_1$, and the effective radius is given by $\widetilde{L} = L/2^k$. Therefore, the value function $C(L, z_1)$ is periodic in the sense that $C(L, z_1) = C(\widetilde{L}, z_1) = C(L/2^k, z_1)$, for $L \in [2^{k-1}L_u, 2^k L_u)$.

It is now clear that we only need to focus on the optimal radius L_1^o that solves

$$L_1^o = \underset{0<L<L_u}{\arg\min} \;\; C(L, z_1). \tag{11}$$

A solution to (11) must exists because $\lim_{L\to 0} C(L, z_1) = \infty$ and $C(L_u, z_1) = C(L_u/2, z_1)$. To investigate decentralization, first note that (8) denotes both the optimal and equilibrium top good of each layer, given L_1. However, the equilibrium layer-1 radius may be different from that of the optimal one, and we denote

$$z_i^o = \phi^{-1}\left(\frac{tL_1^{o^2}}{2^{2i-3}}\right), \tag{12}$$

$$z_i^* = \phi^{-1}\left(\frac{tL_1^{*^2}}{2^{2i-3}}\right), \tag{13}$$

where superscripts o and $*$ denote the allocation in the optimal solution and in a hierarchy equilibrium, respectively. Thus, an optimal solution can be decentralized if $L_1^o \in [L[z_1], 2L[z_1])$.

Proposition 3 (Decentralization)**.** *Suppose that the central place property holds. Then,* $L_u/2 \leq L_1^o < L_u$. *Hence,* $L_1^o \in [L[z_1], 2L[z_1])$.

Proof. We know from our previous discussion that $L_1^o < L_u$. To see that $L_u/2 \leq L_1^o$, assume the contrary is true, that is, assume $L_u > 2L_1^o$. Note that $C(2L_1^o, z_1) = C(L_1^o, z_1)$, and when $L_1 = 2L_1^o$, $z_2^o = z_1$. However, the strictly increasing nature of ϕ implies that the L_u, the value of L_1 such that z_2^o grows to exactly z_1, must be unique, and hence $2L_1^o = L_u$, which contradicts the assumption that we started with. □

Figure 4 depicts a typical shape of $C(L, z_1)$ for $L < L_u$ when ϕ is a power function.

3.2 Entry comparison

Conditional on the central place property, there are still equilibria that are suboptimal. The following proposition states that, whenever an equilibrium allocation is suboptimal, the entry comparison for all goods exhibits a choppy pattern. In the following proposition, it is convenient to consider that $k = 0$. Figure 5 illustrates such a case where $L_1^* > L_1^o$.

Proposition 4 (Choppy Pattern)**.** *Provided that* $n_i^o = 1$, *the following holds.*

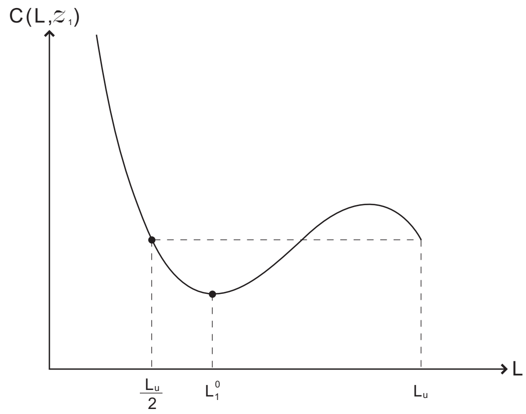

Figure 4: A Typical Value Function on $L < L_u(z)$.

1. *If $L_1^* = L_1^o$, then entry for each good is identical in both the equilibrium and the optimal solution.*

2. *If $L_1^* > L_1^o$, then there exists some $k \in \{0, 1, 2, ...\}$ such that $L_1^* \in (2^k L_1^o, 2^{k+1} L_1^o]$.*

   (a) *$z_{i+1}^o < z_{i+1+k}^* \leq z_i^o$ for all $i \geq 1$. The $[0, z_1]$ continuum can be partitioned into the sets of the form $(z_{i+1+k}^*, z_i^o]$ and $(z_{i+1}^o, z_{i+1+k}^*]$. Running index i from 1 to $I - 1$ completes the partition.*

   (b) *For all $i \geq 1$ and for all $z \in (z_{i+1}^o, z_{i+1+k}^*]$, equilibrium entry is weakly more than the optimal one.*

   (c) *For all $i \geq 1$ and for all $z \in (z_{i+1+k}^*, z_i^o]$, equilibrium entry is less than the optimal one.*

3. *If $L_1^* < L_1^o$, then there exists some $k \in \{0, 1, 2, ...\}$ such that $L_1^o \in (2^k L_1^*, 2^{k+1} L_1^*]$. The result in (b) holds with the superscripts of o and $*$ exchanged.*

Proof. See the Appendix. □

4 Extension to the plane

The only change to the model setup is that the geographic space becomes the infinite plane.

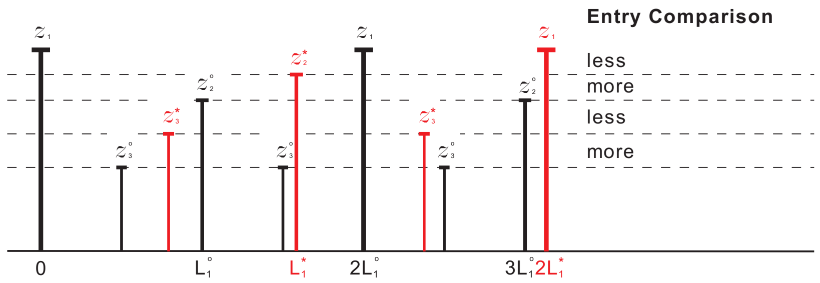

Figure 5: This graph illustrates the entry comparison in the case of $L_1^o < L_1^* < 2L_1^o$. The words "the same/more/less" mean that the equilibrium entry is the same/more/less than the optimal one.

4.1 Hexagonal market areas

Section 2 models the central place hierarchy on the line, but this leaves out the fascinating feature of Christaller and Lösch's central place theory, which is that, on the plane, the market areas are hexagons. Lösch (1940) provided a suggestive, but not rigorous, proof for hexagonal markets being the most efficient for distributing goods to consumers. Many later writers pointed out certain problems in Lösch's proof and tried to improve upon it. However, it was not fully proved until Morgan and Bolton (2002) in the sense that they proved that the result holds on the infinite plane without any additional assumptions with regard to the limit of the edge effect of a finite plane approaching infinity.

The settings in Morgan and Bolton (2002) are exactly the same as those in this paper except that they consider only one good. Thus, for any good, the efficient market areas are equal-sized hexagons, and, when the goods are stratified in this paper, the lattices of the hexagonal market areas are those described by Christaller (1933). Interestingly, it turns out that one can make a heuristic derivation of hexagonal market areas by properly defining "even spacing" on the plane and observing that such even spacing results in hexagonal market areas.

Focus on the market area of any particular good. Given the production locations of this good, the market areas are necessarily convex polygons, as each consumer is assigned to the nearest location. For any location, define its *neighboring locations* as those with which the market areas have parts of their borders in common. Define "even spacing" as the spacing by which any location is an equal distance away from its neighboring locations. It turns out that the only even spacing is constructed in the following way. Without loss of generality, place the production locations evenly on the x-axis of the xy plane with some distance $2L$. From each of these production locations, place more production locations evenly with $2L$ along the 60° and -60° rays. Assign each point (consumer) on the plane to its nearest production location, and the market areas are then regular hexagons.[16]

4.2 Central place property

Let L be the radius of a circle inscribed in a hexagon. Then, the total transportation cost for delivering goods $[0, z]$ to the consumers in the hexagon with L is

$$\frac{z(4 + \ln 27)t}{3} L^3 \equiv z\tau L^3.$$

The market area is $2\sqrt{3}L^2$. Thus, the total cost (excluding the variable costs) per capita for goods $[0, z]$ with a radius of L is

$$\frac{1}{2\sqrt{3}L^2}\left[\Phi(z) + z\tau L^3\right] \equiv C_f^P(L, z).$$

Similar to our discussion in Section 2, given $L > 0$ and $z > 0$, the dynamic programming problem is defined by

$$(FE_f^P)$$

$$C(L, z) = \min\{C_f^P(L, z), \min_{n', z'} \frac{1}{2\sqrt{3}L^2}[\Phi(z) - \Phi(z') + (z - z')\tau L^3] + C(L', z')\}$$

$$s.t. \quad L' = \frac{L}{\sqrt{3}^{n'}}, n' \in \mathbb{N}, z' \in [0, z].$$

Here, the superscript P refers to the "plane," and $L' = L/\sqrt{3}^{n'}$ is derived from the fact that the even spacing of cities necessarily requires that city planting occur on the hexagonal lattices. That is, given any hexagonal lattice of cities, the locations to plant the "next-layer" smaller cities must be at the points of a finer hexagonal lattice.

Similar first-order and envelope conditions give

$$z'^o = \phi^{-1}\left(\frac{\sqrt{3}^{n'} - 1}{(3\sqrt{3})^{n'} - \sqrt{3}^{n'}}\tau L^3\right). \tag{14}$$

---

[16] Another way to check this is based on the fact that the only regular polygons that partition the plane are equal-lateral triangles, squares, and hexagons. However, triangles and squares do not give even spacing. For example, take any production location in the case of squares. For any square and its center, there are eight squares neighboring it, including those that only touch its border at the corners. The centers of the four neighboring squares that touch the corners are actually farther away than those that touch the sides.

The policy function for n' is similarly defined, and that for z' is given by (14). There is also a parallel proposition to Proposition 1. To save space, we do not repeat the proof here. Hence, assuming $\phi(0) = 0$, we have

$$
\begin{aligned}
&(FE^P)\\
&C(L, z) = \min_{n', z'} \frac{1}{2\sqrt{3}L^2}[\Phi(z) - \Phi(z') + (z - z')\tau L^3] + C(L', z')\\
&\quad s.t.\ \ L' = \frac{L}{\sqrt{3}^{n'}}, n' \in \mathbb{N}, z' \in [0, z].
\end{aligned}
$$

Similar to Section 2, we show the result that the central place property holds if ϕ is a power function.

Proposition 5 (Central place property)**.** *Assume that the fixed-cost requirement function is* $\phi(x) = abx^{b-1}, a > 0, b > 1,$ *and hence* $\Phi(z) = az^b$. *Then,* $n'^o = 1,\ \forall L, z > 0.$

Proof. The procedure of the proof is the same as that for Proposition 2. The complete proof is given in the Appendix. □

Any three neighboring cities must form an equilateral triangle. Thus, the city planting process can be thought of as the placing of smaller cities in each such triangular area. As previously mentioned, the only even-spacing method is to place the smaller cities at the lattice points of a finer hexagonal lattice. In this way, the possible number of cities that can be planted in such a triangular area is $m = 1,\ 3,\ 12,\ 36, ...$, and these numbers correspond to $n' = 1,\ 2,\ 3,\ 4, ...$ Suppose the social planner were to plant $m = 2,\ 4,\ 5$, etc., cities; then, the even-spacing principle could not hold. Presumably, the social planner could figure non-even spacing to minimize the transportation cost. A reasonable conjecture is that such minimization could not beat, for some $g(m)$, the solution of

$$
\begin{aligned}
&C(L, z) = \min_{n', z'} \frac{1}{2\sqrt{3}L^2}[\Phi(z) - \Phi(z') + (z - z')\tau L^3] + C(L', z')\\
&\quad s.t.\ \ L' = \frac{L}{\sqrt{3}^{g(m)}}, m \in \mathbb{N}, z' \in [0, z],
\end{aligned}
$$

where $g(m)$ is strictly increasing and satisfies $g(1) = 1,\ g(3) = 2,\ g(12) = 3,\ g(36) = 4, ...$ If the foregoing conjecture is true, then the optimal solution is $m = 1$, which is still the central place property. As shown in the proof of Proposition 5, the objective function in (FE^P) is strictly increasing in $n' \geq 1$ even if n' is treated as a real number. Hence, $n' = g(m) = 1$ defines the optimal m.

5 Conclusion

This paper presents a social planner's problem with regard to the spacing of different layers of cities and the ranges of goods produced in each layer. The model formalizes central place theory via an efficiency rationale. It takes the hierarchy property as given and provides the conditions for the central place property. In this sense, this paper complements Quinzii and Thisse (1990), who model the hierarchy property. It remains to be seen whether the optimality of both properties can be obtained in one concise model.

Our formulation uses a dynamic programming approach, which, to the best of our knowledge, is the first time that such a technique has been applied to economic geography. The central place property is proved by the guess-and-verify technique of a dynamic programming problem, using a power fixed-cost requirement function. As shown in Hsu (2008), this functional form is the prototype of the class of functions leading to Zipf's law and the NAS rule under a central place hierarchy. It is interesting to note that the power function makes the city hierarchy an exact *spatial fractal structure,* which leads to the power-law size distribution.

A comparison of the equilibrium allocation in Hsu (2008) with the optimal solution is made, and a sufficient condition for an optimal solution to be decentralizable is simply the central place property. The central place property is also shown to hold on the infinite plane, and thus, the city hierarchy is fully that of Christaller (1933).

Appendix: Proofs

Proof of Proposition 2

Proof. With the assumption of $\phi(.)$, (4) becomes (7). By plugging (7) into (SP), the problems are reduced to finding the optimal sequence of $\{n_i\}_{i=2}^{\infty}$. The optimal solution of $n_i = 1$ for all i can be proved using (FE) by the guess-and-verify technique. We obtain the guess of the functional form, denoted as $C_0(L, z)$, by plugging the guess of $n_i = 1$ into (SP). This guess is

$$C_0(L,z) = \frac{1}{2L}[az^b + ztL^2] - \frac{2^{\frac{b-2}{1-b}}}{2^{\frac{b+1}{b-1}} - 1} a^{\frac{1}{1-b}} t^{\frac{b}{b-1}} (b^{\frac{1}{1-b}} - b^{\frac{b}{1-b}}) L^{\frac{b+1}{b-1}}.$$

Use (FE) to define C_1 by the following mapping.

$$C_1(L,z) = \min_{n'\in\mathbb{N}} \quad \frac{1}{2L}\left[\Phi(z) - \Phi\left(\left(\frac{tL^2}{ab(n'+1)}\right)^{\frac{1}{b-1}}\right) + \left(z - \left(\frac{tL^2}{ab(n'+1)}\right)^{\frac{1}{b-1}}\right) tL^2\right]$$
$$+ C_0\left(\frac{L}{n'+1}, \left(\frac{tL^2}{ab(n'+1)}\right)^{\frac{1}{b-1}}\right). \tag{15}$$

It is readily verified that $C_1 = C_0$ if $n' = 1$. Thus, what remains is to show that $n'^o = g(L,z) = 1$ for all L, z.

Denoting the objective function in (15) as $\tilde{R}(n')$, a few algebraic manipulations give

$$\frac{d\tilde{R}(n')}{dn'} = \frac{b^{\frac{1}{1-b}} - b^{\frac{b}{1-b}}(n'+1)^{\frac{2b}{1-b}}}{(2^{\frac{b+1}{b-1}} - 1)(b-1)}[(2^{\frac{b+1}{b-1}} - 1)(n'+1)^{\frac{1}{b-1}}(n'+1-b) + 2^{\frac{1}{b-1}}(b+1)]. \tag{16}$$

If $\frac{d\tilde{R}(n')}{dn'} > 0$ for all $n' \in \mathbb{N}$, then the optimal solution of n' is 1. Denote the term in the brackets in (16) as $D(n')$. $\frac{d\tilde{R}(n')}{dn'} > 0$ for all $n' \in \mathbb{N}$ if and only if $D(n') > 0$ for all $n' \in \mathbb{N}$. In fact,

$$\frac{dD(n')}{dn'} = \frac{2^{\frac{b+1}{b-1}} - 1}{b-1} bn'(n'+1)^{\frac{2-b}{b-1}} > 0, \qquad \forall n \in \mathbb{N}.$$

Therefore, if we can show that $D(1) > 0$, we are done.

Define $E(b) = 2^{\frac{1}{1-b}} D(1)$. Thus,

$$E(b) = 2^{\frac{b+1}{b-1}}(2-b) + 2b - 1.$$

Recall that $b > 1$. Thus, $E(b) > 0$ if $b \leq 2$. Consider $b > 2$ and define a new variable $w \equiv \frac{b+1}{b-1}$; then

$$E(b) \equiv H(w) = \frac{1}{w-1}[2^w(w-3) + w + 3].$$

Note that $1 < w < 3$, as $b > 2$. It can be verified that $H(w) > 0$ for $1 < w < 3$. Hence, $E(b) > 0$ for all $b > 1$, and thus $D(1) > 0$ for all $b > 1$. □

Proof of Proposition 4

Proof. The first point becomes trivial by inspecting (12) and (13). The proof for Point 3 is the same as that for Point 2 with the superscripts of o and $*$ exchanged. For (a) of Point 2, first note that $L_1^* \in (2^k L_1^o, 2^{k+1} L_1^o]$ implies

$$\phi^{-1}\left(\frac{tL_1^{o^2}}{2^{2i-1}}\right) < \phi^{-1}\left(\frac{tL_1^{*^2}}{2^{2(i+k)-1}}\right) \leq \phi^{-1}\left(\frac{tL_1^{o^2}}{2^{2i-3}}\right).$$

Hence, $z^o_{i+1} < z^*_{i+1+k} \leq z^o_i$. Running index i from 1 to $I-1$ completes the partition of $[0, z_1]$. Now, consider (b) in Point 2. The distances between any two neighboring locations of z satisfy the following inequality.

$$L^*_z = \frac{L^*_1}{2^{i+k}} \leq \frac{2^{k+1}L^o_1}{2^{i+k}} = \frac{L^o_1}{2^{i-1}} = L^o_z,$$

which implies that the equilibrium entry is more than optimal if $L^*_1 < 2^{k+1}L^o_1$ and is equal to the optimal entry if $L^*_1 = 2^{k+1}L^o_1$. Similarly, for (c) in Point 2, the distance between any two neighboring locations of z satisfies the following inequality.

$$L^*_z = \frac{L^*_1}{2^{i+k-1}} > \frac{2^k L^o_1}{2^{i+k-1}} = \frac{L^o_1}{2^{i-1}} = L^o_z,$$

which implies that the equilibrium entry is less than optimal. □

Proof of Proposition 5

Proof. Similar to Section 2, we can define a sequence problem as follows.

$$(SP^P)$$

$$C(L,z) = \min_{\{n_i,z_i\}_{i=2}^{\infty}} \sum_{i=2}^{\infty} \frac{1}{2\sqrt{3}L^2_{i-1}}[\Phi(z_{i-1}) - \Phi(z_i) + (z_{i-1} - z_i)\tau L^3_{i-1}]$$

$$s.t. \quad L_i = \frac{L_{i-1}}{\sqrt{3}^{n_i}}, \quad n_i \in \mathbb{N}, z_i \in [0, z_{i-1}] \quad \forall\, i \geq 2,$$

$$L_1 = L, \; z_1 = z.$$

With the assumption of $\phi(.)$, (14) becomes

$$z'^o = \left(\frac{\tau L^3}{ab}\frac{\sqrt{3}^{n'} - 1}{(3\sqrt{3})^{n'} - \sqrt{3}^{n'}}\right)^{\frac{1}{b-1}}, \tag{17}$$

or,

$$z^o_i = \left(\frac{\tau L^3_{i-1}}{ab}\frac{\sqrt{3}^{n_i} - 1}{(3\sqrt{3})^{n_i} - \sqrt{3}^{n_i}}\right)^{\frac{1}{b-1}}. \tag{18}$$

By plugging (18) into (SP^P), the problem is reduced to finding the optimal sequence of $\{n_i\}_{i=2}^{\infty}$. The optimal solution of $n_i = 1$ for all i can be proved using (FE^P) by the guess-and-verify technique. We obtain the guess of the functional form, denoted as $C_0(L,z)$, by plugging the guess of $n_i = 1$ into (SP^P). When $n_i = 1$, (18) becomes

$$z_i = \kappa L^{\frac{3}{b-1}}\left(\frac{1}{3\sqrt{3}}\right)^{\frac{i-2}{b-1}}, \tag{19}$$

where $\kappa = \left(\frac{(\sqrt{3}-1)\tau}{2\sqrt{3}ab}\right)^{\frac{1}{b-1}}$. The guess is

$$C_0(L,z) = \frac{1}{2\sqrt{3}L^2}\left[az^b - a\kappa^b L^{\frac{3b}{b-1}} + \left(z - \kappa L^{\frac{3}{b-1}}\right)\tau L^3\right] + \frac{\sqrt{3}\kappa L^{\frac{b+2}{b-1}}}{2(\sqrt{3}^{\frac{3b}{b-1}} - 3)}\left[a\kappa^b\left(\sqrt{3}^{\frac{3b}{b-1}} - 1\right) + \kappa\tau\left(\sqrt{3}^{\frac{3}{b-1}} - 1\right)\right].$$

With $z'^o(n')$ given by (17), use (FE^P) to define C_1 by the following mapping.

$$C_1(L,z) = \min_{n'\in\mathbb{N}} \quad \frac{1}{2\sqrt{3}L^2}\left[\Phi(z) - \Phi(z'^o(n')) + (z - z'^o(n'))\tau L^3\right] + C_0\left(\frac{L}{\sqrt{3}^{n'}}, z'^o(n')\right). \quad (20)$$

It can be verified that $C_1 = C_0$ if $n' = 1$. Thus, what remains is to show that $n'^o = g(L,z) = 1$, for all L, z. Denote the objective function in (20) as $\tilde{R}(n')$. If $\frac{d\tilde{R}(n')}{dn'} > 0$ for all $n' \geq 1$, then the optimal solution of n' is 1.

With some algebraic manipulations,

$$\frac{d\tilde{R}(n')}{dn'} = \frac{d}{dn}\left(\frac{\sqrt{3}^{n'} - 1}{(3\sqrt{3})^{n'} - \sqrt{3}^{n'}}\right)\left(\frac{\sqrt{3}^{n'} - 1}{(3\sqrt{3})^{n'} - \sqrt{3}^{n'}}\right)^{\frac{2-b}{b-1}} \times \frac{n'3^{n'}}{6\sqrt{3}}\frac{\sqrt{3}^{\frac{(2-b)n'}{b-1}}(\sqrt{3}^{n'}+1)^{\frac{b}{b-1}}}{(2+\sqrt{3}^{-n'})\ln 3}\left(\frac{1}{ab}\right)^{\frac{1}{b-1}}\tau^{\frac{b}{b-1}}L^{\frac{b+2}{b-1}}E(n'),$$

where

$$E(n') = \frac{\left(3^{n'} + \sqrt{3}^{n'}\right)^{\frac{1}{1-b}}}{(3\sqrt{3})^{n'}} - \frac{2}{b}\left(\frac{1}{3^{n'} + \sqrt{3}^{n'}}\right)^{\frac{b}{b-1}} + \frac{b+2}{2^{\frac{1}{b-1}}b(\sqrt{3}^{\frac{b+2}{1-b}} - 1)}\left(\frac{3-\sqrt{3}}{3^{n'+1}\sqrt{3}^{n'}}\right)^{\frac{b}{b-1}}.$$

As $\frac{d}{dn}\left(\frac{\sqrt{3}^{n'}-1}{(3\sqrt{3})^{n'}-\sqrt{3}^{n'}}\right) < 0$, $\frac{d\tilde{R}(n')}{dn'} > 0$ if and only if $E(n') < 0$.

$E(n')$ can be further written as $E(n') = \frac{1}{(3\sqrt{3})^{n'}(3^{n'}+\sqrt{3}^{n'})^{\frac{1}{b-1}}}F(n')$. And, we have

$$\frac{dF(n')}{dn'} = -\left(\frac{n'}{2}3^{-\frac{n'}{2}-1} + n'3^{-n'-1}\right)\left[\frac{2\times 3^{2n'}}{b(\sqrt{3}^{n'}+1)^2} - \frac{b+2}{b-1}\left(\frac{3-\sqrt{3}}{3}\right)^{\frac{b}{b-1}}\frac{\left(3^{-\frac{n'}{2}} + 3^{-n'}\right)^{\frac{2-b}{b-1}}}{2^{\frac{1}{b-1}}b(1-\sqrt{3}^{\frac{b+2}{1-b}})}\right].$$

It can be verified that for all $b > 1$,

$$F(1) = 1 - \frac{3\sqrt{3}-3}{b} - \frac{b+2}{b}\frac{3^{\frac{2b-1}{2(b-1)}} - 3^{\frac{b}{2(b-1)}}}{3^{\frac{b+1}{b-1}} - 3^{\frac{b}{2(b-1)}}} < 0.$$

Hence, if the term in the bracket of $\frac{dF(n')}{dn'}$ is positive for all $n' \geq 1$, then $F(n') < 0$ and $E(n') < 0$ for all $n' \in \mathbb{N}$, and we are done. To see that $F(1) < 0$, note that, with $w \equiv \frac{b+2}{b-1}$,

$$F(1) < 0 \Longleftrightarrow \frac{b - 3\sqrt{3} + 3}{b+2} < \frac{\sqrt{3} - 1}{\sqrt{3}^{\frac{b+2}{b-1}} - 1}$$
$$\Longleftrightarrow f(w) \equiv w + 3\sqrt{3} - 1 - [(4 - 3\sqrt{3})w + 3\sqrt{3} - 1]\sqrt{3}^{w} > 0.$$

Also note that $b \in (1, \infty)$ implies that $w \in (1, \infty)$. It is easy to verify that $f(1) = 0$ and $f'(w) > 0$ for all $w > 1$. Hence, $f(w) > 0$ for all $w > 1$.

Because $\frac{3^{n'}}{\sqrt{3}^{n'}+1}$ is strictly increasing in n', the term in the bracket of $\frac{dF(n')}{dn'}$ is positive for all n' if

$$\left(\frac{3}{\sqrt{3}+1}\right)^{\frac{b}{b-1}} > \frac{b+2}{b-1}\left(\frac{3-\sqrt{3}}{6}\right)^{\frac{b}{b-1}} \frac{1}{1 - \sqrt{3}^{\frac{b+2}{1-b}}}.$$

The foregoing inequality holds if and only if

$$\ln 3 > \frac{2(b-1)}{3b} \ln(\frac{4b-1}{b-1}) \equiv G(b).$$

Observe that $\lim_{b \to 1^+} G(b) = 0$, $\lim_{b \to \infty} G(b) = \frac{4 \ln 2}{3}$, and

$$G'(b) = \frac{2}{3b^2(4b-1)}\left[\ln\left(\frac{4b-1}{b-1}\right)(4b-1) - 3b\right] \equiv \frac{2}{3b^2(4b-1)}H(b).$$

In fact, there exists a $b^* > 1$ such that $H'(b^*) = 0$. Since $H''(b) = \frac{9}{(b-1)^2(4b-1)} > 0$, $H(b^*) \doteq 7.01$ is the unique minimum. Together with the limits of $G(b)$ at 1 and ∞, the fact that $G'(b) > 0$ implies that $0 < G(b) < \frac{4 \ln 2}{3} < \ln 3$ for all $b > 1$. □

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