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Firms, Frictions, and Barriers to Growth*

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ABSTRACT

We construct a model in which aggregate growth is driven by the continual entry of new firms that face barriers to entry that are exacerbated by financial frictions. We show that economies with more severe financial frictions have lower levels of output and consumption along the balanced growth path compared to economies with lower levels of financial frictions, even though all economies grow at the same, constant, rate. Improvements in financial markets generate faster-than-trend growth as the economy transitions to the new balanced growth path. The model generates sharp predictions regarding the rate of firm creation and aggregate output levels, as well as aggregate growth rates; these predictions are borne out in the cross country data.

* The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1. Introduction

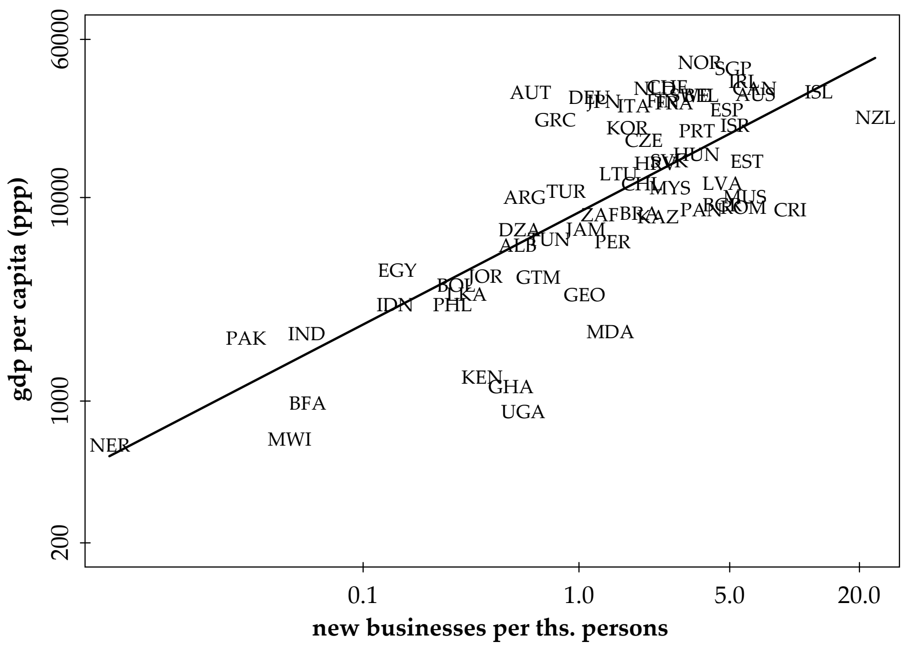
Cross country income differences, as measured by purchasing power parity gross domestic product (GDP) per capita, are large and persistent. GDP per capita in Liberia is less than one-hundredth of that in the United States; in Lebanon, the median country in the Penn World Table, income per capita is roughly one-fifth that of the United States. Even in the developed world—the OECD, for example—GDP per capita in the poorest countries are less than one-third that of the United States.

We provide a model in which income differences across countries arise from barriers to new technology adoption. Barriers to technology adoption have been widely studied, but we provide a link, in both theory and data, between the creation of new production units and the levels of aggregate output in a country. In our theory, new firms draw their productivity from a distribution that is common to all countries and whose mean improves over time at a constant rate. As older, less efficient cohorts of firms exit, they are replaced by more productive ones. This process leads to a balanced growth path in which output per capita grows at the same rate as the distribution of new firms. Barriers to new firm entry—in our model imperfect capital markets—differ across countries for institutional reasons and lead to different levels of income across countries, even though countries will grow at identical rates.

This mechanism yields the empirical prediction that countries with higher levels of income per capita should also have higher firm entry rates. In figure 1 we plot log GDP per capita against firm entry rates from the World Bank. The positive relationship between output and business entry is a salient feature of the data. We discuss these data in more detail in section 2.

Our model addresses cross country differences in income levels; in our model, countries on balanced growth paths will all grow at the same rate. Parente and Prescott (2002) and Kehoe and Prescott (2007) argue that persistent differences in income levels—not growth rates—are the bigger puzzle, particularly in middle income and high income countries. Building up aggregate differences from firm level decisions, as we do, has been taken up by several authors. One line of research has focused on monopoly power in factor or goods markets, such as Parente and Prescott (1999) and Herrendorf and Teixeira (2009). In these models a coalition secures monopoly rights that allow the coalition to block the adoption of better technologies. In our theory better technology is embodied in new firms and inefficiencies in the capital market drive suboptimal levels of new firm entry. Our work is closely related to Amaral and Quintin (2010), who also use lack of commitment in financial contracts to drive income differences, but an important channel in their model is the that firms operate on too small of a scale, distorting the allocation of capital across plants. In our model, conditional on operating, firms operate at the appropriate scale. In contrast to all of these studies, our model generates a balanced growth path in an extremely tractable framework, and the underlying rate of growth is a factor in the extent to which financial imperfections decrease aggregate output; the faster an economy grows, the more important are financial frictions.

Figure 1: Business entry and GDP per capita



Empirically, the link between firm turnover and aggregate variables—particularly aggregate productivity—has been documented for several countries using firm level data that allows for the calculation of firm specific productivity. Many of these studies find that a significant fraction of the growth in aggregate productivity is driven by reallocation of resources across firms. Petrin, White, and Reiter (2009), using U.S. data, show that, of the average productivity growth of 2.2 percent per year, almost all of it is accounted for by reallocation across plants. Brandt, Van Biesbroeck, and Zhang (2009) document that roughly half of the increase in Chinese manufacturing productivity is accounted for by the net entry of plants.

Our model of heterogeneous firms is based on Hopenhayn (1992), but we have abstracted from firm level uncertainty. This abstraction implies that firms exit in a deterministic way, which greatly simplifies the characterization of the equilibrium. In return for this simplification, we are able to develop a framework that generates an extremely tractable balanced growth path, rather than the stationary distributions considered in other models of heterogeneous firms.

2. Firm Entry, Barriers to Entry and Aggregate Output

In this section we document two patterns in a cross section of countries: (i) the firm entry rate is decreasing in costs of entry and increasing in the availability of finance, and (ii) the availability of finance is more important for firm entry, the larger are firm entry costs.

Our data on the entry rate of businesses is constructed using the World Bank's World Development Indicators business entry statistic. This variable measures the number of new firms officially registered per year. We construct the business entry rate as the number of newly registered firms per thousand working age people. The average entry rate in our sample is about 2.8 new businesses per thousand working age people. The minimum entry rate belongs to Niger at less than 0.01, and the maximum to New Zealand with an entry rate of 23.7 businesses per thousand working age people.

We plot GDP per capita against firm entry rates in figure 1. GDP per capita is measured at purchasing power parity and is from the Penn World Tables. The data are for the year 2004. Higher levels of firm entry are clearly correlated with higher levels of income per capita: the correlation coefficient is 0.40. A simple regression of log GDP per capita on the log business entry rate yields a coefficient of 0.55. A one percent increase in the business entry rate is associated with a 0.55 percent increase in GDP per capita.

What drives the rate of firm entry? We consider two broad factors that have been discussed in the previous literature, policy induced barriers to entry and access to financing. In figure 2a, we plot the firm entry rate against a measure of the costs of opening a new business. The cost variable is from the World Bank's *Doing Business* surveys, which calculates the costs of starting up a representative industrial or commercial business. The cost includes both official fees and costs associated with professional services, such as lawyers or accountants, if required by law. The costs are expressed as a percentage of the country's gross domestic income per capita.

Figure 2a: Business entry and entry costs

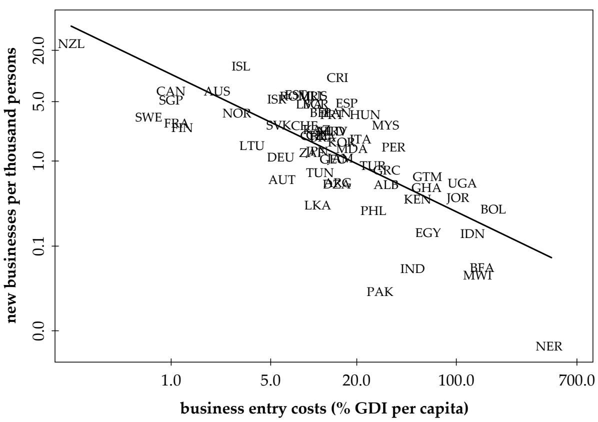
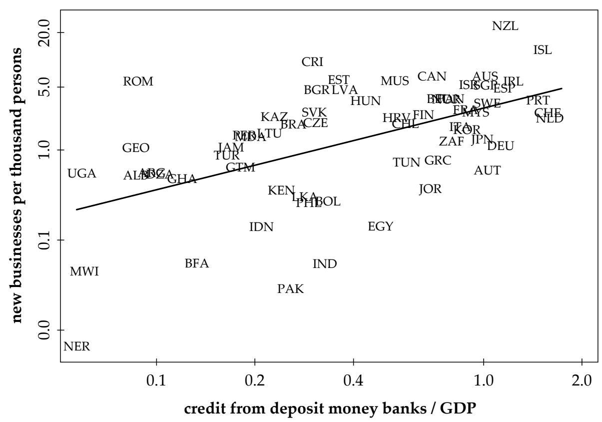


Figure 2b: Business entry and financing



The negative relationship between business entry and costs of entry is striking. This pattern has been studied by several researchers, including Barseghyan and DiCecio (2009) who construct a model in which cross country differences in entry costs lead to differences in output and productivity. The second column of table 1 reports estimates from regressing the log entry rate on a constant and the log entry cost. A one percent increase in the costs of starting a new firm lead to a 0.8 percent decrease in the firm entry rate.

If starting a new firm requires significant upfront costs—either from policy induced costs or from the costs of building production capacity—the availability of finance can be an important determinant in the business entry rate. Following the literature, we take as evidence of the availability of financing in a country the amount of financing outstanding. We use the amount of private credit by deposit money banks as a measure of financing. We take this variable as a proxy for the financial frictions in the economy: when financial frictions are significant, we expect to observe lower volumes of credit. We plot these data in figure 2b: economies with more finance outstanding have higher rates of firm entry.

Column 3 in table 2 reports the results from regressing the log entry rate on the log *inverse* of the finance-GDP ratio. We have chosen the inverse of the credit-GDP ratio so that the entry rate is decreasing in the variable. This is merely a convention on signs: since the variable is expressed in logarithms, the absolute value of the coefficient is unchanged. The coefficient on financing is negative and significant; larger financial frictions are associated with lower entry rates.

Table 1

	log entry rate	log entry rate	log entry rate	log entry rate
log entry cost	-0.809***		-0.505**	-0.465*
	(0.0004)		(0.026)	(0.050)
log GDP-credit ratio		-0.908**	0.421**	0.089
		(0.029)	(0.09)	(0.067)
interaction			-0.249**	-0.060***
			(0.012)	(0.001)
log gdp per capita				0.764*
				(0.101)
log informal				0.973**
				(0.0245)
offshore				0.877**
				(0.019)
Observations	132	132	132	132
Time FE	Yes	Yes	Yes	Yes
R^2 (adj.)	0.56	0.60	0.65	0.74

Notes: interaction = log(entry cost)×log(GDP-credit ratio). OLS estimates of for the years 2004 and 2005. Robust standard errors displayed in parenthesis, clustered by year. ***, **, * denote statistical significance at 1%, 5%, and 10%, respectively

Our focus in this paper is on the interaction between the costs of entry a firm faces and the financial frictions that make it difficult for the entrepreneur to finance these costs. The estimates reported in column 4 of table 1 lend support to this idea. Here we regress the entry rate on our measure of entry costs, financial development, and the interaction of the two.

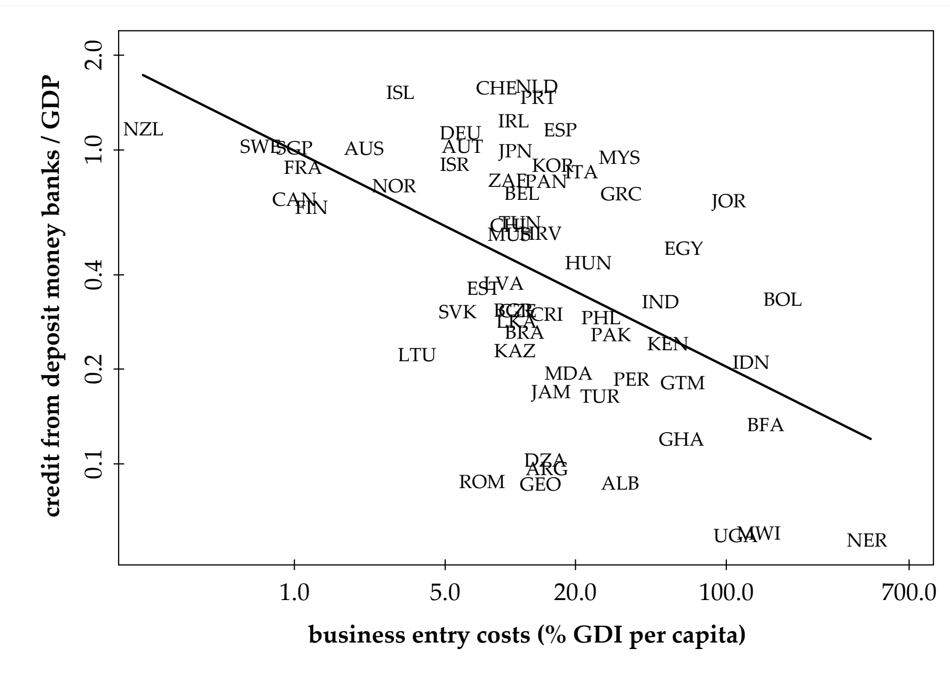
$$\log(entry_{it}) = \beta_0 + \beta_1 \log(finance_{it}) + \beta_2 \log(costs_{it}) + \beta_3 \log(finance_{it}) \times \log(costs_{it}) + \varepsilon_{it} \quad (1)$$

The interaction is significant and negative: financial frictions have a larger impact on firm entry rates when entry costs are large. While the coefficient on financial barriers is now positive, the total impact ($\beta_3 \log(costs) + \beta_1$), evaluated at the median entry cost is -0.22.

Lastly, we estimate a version of (1) controlling for several other factors that might influence firm entry rates. One potential source of concern with our measure of firm entry is that it covers only officially registered firms: entry into the informal sector is not included in our measure of entry. Since poorer countries tend to have larger informal sectors, this could bias our findings. To control for this, we introduce a measure of the size of the informal economy into our previous regression. The data on the informal economy are from Buehn and Schneider (2007), which estimates the size of the informal economy as a percent of the country's official GDP. A second concern is that some firms may be created for tax advantages in offshore tax havens. We control for this by including a dummy variable for countries associated with being offshore locations.

The impacts of financing and entry costs on firm entry have been studied separately by several researchers. Our focus is on the interaction between the two: without access to finance, potential entrants cannot spread the costs of entry across future, profitable, periods, increasing the impact of entry costs on entry. The interaction between these frictions is especially interesting considering that countries with high levels of entry costs also tend to have little financing. In figure 4 we plot the finance-GDP ratio and the entry cost variables for our sample of countries; the correlation coefficient is –0.64.

Figure 3: Availability of finance and entry costs



3. Model

The production side of the economy is comprised of a representative final good producer and a continuum of monopolistically competitive intermediate goods producers. The economy is closed to foreign trade and capital flows.

3.1. Households

The representative household is endowed with labor, L, which is inelastically supplied to firms. The household chooses consumption of composite good, C_t, and bond holdings, B_{t+1}, to solve

$$\max_{c_t, B_{t+1}} \sum_{t=0}^{\infty} \beta^t \log C_t$$

$$P_t C_t + q_{t+1} B_{t+1} = w_t L + D_t + B_t \tag{2}$$

$$C_t \geq 0,\ B_t \geq -g^t B,\ B_0 \text{ given,}$$

Where β is the time discount factor and D_t is the aggregate dividends paid by firms in the economy. We normalize the wage so that $w_t = 1$ in each period. The constraint $B_t \geq -g^t B$ rules out Ponzi schemes but we choose the constant B large enough so that the constraint does not otherwise bind in equilibrium; $g \geq 1$ is the growth factor for the economy.

3.2. Producers

The representative final good firm, which operates in perfect competition, purchases intermediate goods from intermediate firm z, $y(z)$, to solve

$$\min_{y_t(z)} \int_0^{\eta_t} p_t(z) y_t(z) di$$

$$\text{s.t.} \left(\int_0^{\eta_t} y_t(z)^\rho di \right)^{\frac{1}{\rho}} = Y_t$$

The measure of goods available, η_t, is an important variable in this model. We will show that entry costs and financial frictions decrease the measure of intermediate goods available. The elasticity of substitution between intermediate good varieties is $1/(1-\rho)$; the varieties are close, but imperfect, substitutes, $\rho < 1$. Solving the final good producer's problem, we obtain the demand function for good z

,

$$y_t(z) = \frac{Y_t}{p_t(z)^{\frac{1}{1-\rho}} P_t^{\frac{-\rho}{1-\rho}}}, \tag{3}$$

with

$$P_t = \left(\int_0^{\eta_t} p_t(z)^{\frac{-\rho}{1-\rho}} \, dz \right)^{\frac{1-\rho}{-\rho}}. \tag{4}$$

There is a continuum of intermediate good firms which live for a maximum of N periods, unless they choose to exit the market earlier. The firm producing good z that is of age j produces according to

$$y_t(z) = x(z)\ell_t(z), \tag{5}$$

where $x(z)$ is the marginal productivity of firm z. Production is subject to a fixed cost of operating, κ_j, which is denominated in units of labor, and may be conditional on the age of the firm, j. If a firm chooses not to pay κ_j the firm exits the economy forever. Conditional on choosing to produce, the firm chooses $p_t(j)$ to maximize profit, taking as given the final good price, P_t, and aggregate output Y_t,

$$\pi_{jt}(x(z), Y_t, P_t) = \max_{p_{jt}(z)} p_{jt}(z) \frac{Y_t}{p_{jt}(z)^{\frac{1}{1-\rho}} P_t^{\frac{-\rho}{1-\rho}}} - \frac{Y_t}{x(z) p_{jt}(z)^{\frac{1}{1-\rho}} P_t^{\frac{-\rho}{1-\rho}}} \tag{6}$$

The solution to this problem yields the standard markup over marginal cost pricing, $p_{jt}(z) = (\rho x(z))^{-1}$. We assume that firms can costlessly differentiate their products, so no two firms will choose to produce the same good z. Every firm with productivity x, however, chooses the same price and sells the same identical quantity. In what follows, we no longer characterize a good by its label, z, but by the productivity, x, of the firm that produces it.

The firm's choice over producing or exiting is dynamic. Let $I_t = 0$ if the firm in period t chooses to exit and $I_t = 1$ if the firm produces. Firms maximize the expected discounted sum of profits. The problem of an existing firm with productivity x and age $j > 2$ can be written recursively as

$$\begin{aligned} V_{jt}(b_t, x) &= \max_{b_{t+1}, I_t} I_t \left\{ \pi_{jt}(x) - q_{t+1} b_{t+1} + b_t + q_{t+1} V_{j+1,t+1}(b_{t+1}, x) \right\} \\ &\text{s.t. } V_{jt}(b_t, x) \geq \theta \pi_{jt} \\ d_{jt}(x) &\equiv \pi_{jt}(x) - q_{t+1} b_{t+1} + b_t \geq 0, \end{aligned} \tag{7}$$

where once a firm has exited, $I_t = 0$, it cannot return to produce in a later period. The operator of the firm may leave with fraction θ of the period's profits. The first constraint in (7) ensures that this will not happen: the value of continuing to operate the firm must be at least as profitable as absconding with a fraction of the current period's profits. The second constraint in (7) implies that dividends, $d_{jt}(x)$, must be nonnegative and that, if a firm generates positive profits net of fixed costs, it uses them to pay down debt. Any remaining profits are returned to the households as dividend payments.

Every period, a mass μ of potential entrants draw their productivities from a Pareto distribution that improves each period before deciding to produce—an *entry*—or to exit immediately. A potential entrant in period t draws its productivity from the distribution

$$F_t(x) = 1 - \left(\frac{x}{g^t}\right)^{-\gamma},$$

which is characterized by a mean that is improving at rate $g-1$. Additionally, we require that $\gamma(1-\rho)-\rho>0$. The problem of a potential entrant in period t is to choose to operate or not, I_t, and how much to borrow, $-b_{t+1}$, to solve

$$\begin{aligned} V_{1t}(x) = \max_{b_{t+1}, I_t} \; & I_t \left\{ \pi_{1t}(x) - q_{t+1}b_{t+1} + q_{t+1}V_{2,t+1}(b_{t+1}, x) \right\} \\ \text{s.t. } & V_{1t}(x) \geq \theta \pi_{1t} \\ & d_{1t}(x) \equiv \pi_{1t}(x) - q_{t+1}b_{t+1} \geq 0. \end{aligned} \tag{8}$$

We assume that new entrants pay a higher fixed cost than existing firms. In what follows, we consider a simple structure for the fixed costs of operating. A new entrant pays κ and, regardless of a firm's age, all continuing firms pay $\kappa_c < \kappa$.

In equilibrium, the behavior of firm entry and exit can be summarized by simple cutoff rules. Let $\hat{x}_{jt}$ be the productivity of the firm in period t, of age $j>2$, that is indifferent between producing and exiting,

$$\pi_{jt}(\hat{x}_{jt}) - q_{t+1}b_{t+1}(\hat{x}_{jt}) + b_t(\hat{x}_{j-1,t-1}) + q_{t+1}V_{j+1,t+1}(b_{t+1}(\hat{x}_{jt}), \hat{x}_{jt}) = 0. \tag{9}$$

Profits, $\pi(x)$, increase monotonically with productivity, so every firm of age j with $x \geq \hat{x}_{jt}$ produces in period t. Notice that firms of the same age and productivity have identical bond holdings, so it is not necessary to make the exit threshold a function of bond holdings. A similar condition characterizes the entry threshold,

$$\pi_{1t}(\hat{x}_{1t}) - q_{t+1}b_{t+1}(\hat{x}_{1t}) + V_{2,t+1}(b_{t+1}(\hat{x}_{1t}), \hat{x}_{1t}) = 0. \tag{10}$$

Since $\kappa_c < \kappa$, profits for a new entrant will be lower than that of an existing firm with the same productivity and debt level. In the parameterizations that we consider, $\pi_{1t}(\hat{x}_{1t}) < 0$, which implies that the marginal firm must borrow to enter the market. The amount the intermediary is willing to lend to the entrant will be influenced by the degree of enforcement in the economy, $1-\theta$, which creates a link between enforcement, firm entry, and aggregate output.

Given this notation, the mass of firms that are producing at time t is

$$\eta_t = \mu \sum_{i=1}^{N} \left(1 - F_{t-i+1}(\hat{x}_{i,t-i+1})\right). \tag{11}$$

3.3. *Equilibrium*

In this paper, we focus on balanced growth paths. To define a balanced growth path, we first define an equilibrium. The definition is also useful for potential future research that studies transition paths of the model.

To specifying the equilibrium, we need to provide initial conditions on the number of firms entering period 0 that have ages $j > 1$, given by the cutoff levels $\hat{x}_{j0}$, $j = 2,..., N$, the bond holdings by households, B_0, and those of firms $b_{j0}(x)$, $x \geq \hat{x}_{j0}$, $j = 2,..., N$.

Definition. Given these initial conditions, an **equilibrium** is sequences of entry-exit threshold values, $\left\{\hat{x}_{jt}\right\}_{t=0}^{\infty}$, $j = 1,\ldots, N$, prices and allocations for intermediate firms, $\left\{p_t(x), y_t(x), \ell_t(x), b_{j,t+1}(x)\right\}_{t=0}^{\infty}$, for all $x \geq \hat{x}_{jt}$, $j = 1,\ldots, N$, bond prices and bond holdings, $\left\{q_{t+1}, B_{t+1}\right\}_{t=0}^{\infty}$, aggregate dividends and final output, $\left\{D_t, Y_t\right\}_{t=0}^{\infty}$, and household consumption, $\left\{C_t\right\}_{t=0}^{\infty}$, such that:

1. Given $\left\{P_t, D_t, q_{t+1}\right\}_{t=0}^{\infty}$, $\left\{C_t\right\}_{t=0}^{\infty}$ and $\left\{B_{t+1}\right\}_{t=0}^{\infty}$ solve the household's problem in (2).
2. Given $\left\{P_t, Y_t, q_{t+1}\right\}_{t=0}^{\infty}$ and the firm's period profit function $\pi_{jt}(x, Y_t, P_t)$ in (6), $\left\{p_t(x), \ell_t(x), b_{j,t+1}(x)\right\}_{t=0}^{\infty}$ solves the problem of the firm with productivity x and age j in (7) and (8) for all $x \geq \hat{x}_{jt}$.
3. The labor market clears for all $t \geq 0$,

$$L = \sum_{i=1}^{N} \int_{\hat{x}_{jt}}^{\infty} (\ell_t(x) + \kappa_i) dF_{t-i+1}(x) dx . \quad (12)$$

4. Entry and exit cutoffs satisfy the zero-profit conditions (9) and (10) for all $j = 1,\ldots,N$, and all $t \geq 0$.
5. The bond market clears for all $t \geq 0$,

$$B_{t+1} + \sum_{i=1}^{N-1} \int_{\hat{x}_{it}}^{\infty} b_{i,t+1}(x) dF_{t-i+1}(x) dx = 0 . \quad (13)$$

6. Dividend payments satisfy

$$D_t = \sum_{i=1}^{N} \int_{\hat{x}_{it}}^{\infty} d_{it}(x) dF_{t-i+1}(x) dx . \quad (14)$$

4. Balanced Growth Path

In this section, we prove that the model has a balanced growth path, and we characterize the behavior of the key variables along the balanced growth path.

Let aggregate profit of firms operating in the economy be

$$\Pi_t = \sum_{i=1}^{N} \int_{\hat{x}_{it}}^{\infty} \pi_{it}(x) dF_{t-i+1}(x) dx . \quad (15)$$

Lemma 0. In equilibrium, aggregate profits of the firms is equal to sum of aggregate dividends and debt payments, that is

$$\Pi_t = D_t + B_t - q_{t+1} B_{t+1} , \text{ all } t , \quad (16)$$

Definition. A **balanced growth path** is a path of entry-exit threshold values, $\left\{\hat{x}_{jt}\right\}_{t=0}^{\infty}$, $j = 1,\ldots,N$, real aggregate profits, $\left\{\tilde{\Pi}_t\right\}_{t=0}^{\infty}$ where $\tilde{\Pi}_t = \frac{\Pi_t}{P_t}$, household consumption, $\left\{C_t\right\}_{t=0}^{\infty}$, and final good output, $\left\{Y_t\right\}_{t=0}^{\infty}$, such that $\hat{x}_{jt}$, $j = 1,\ldots,N$, and, $\tilde{\Pi}_t$, C_t, Y_t, grow at the same rate

$$\frac{\hat{x}_{jt+1}}{\hat{x}_{jt}} = \frac{\tilde{\Pi}_{t+1}}{\tilde{\Pi}_{t}} = \frac{C_{t+1}}{C_{t}} = \frac{Y_{t+1}}{Y_{t}} = g\,,\ j = 1,\ldots,N\,.$$

We will denote $\hat{x}_{1t}$ as a potential entrant's minimum productivity necessary to operate and $\hat{x}_{jt}$ as the minimum productivity necessary for an incumbent with age $j \geq 2$ to remain in the market. Furthermore, let $n_t(x)$ be the last age at which a firm of productivity x born at time t remains in the market.

Lemma 1. In any balanced growth path, the enforcement constraint of the cutoff firm of time t only holds with equality only when $j = 2$, that is

$$V_{2,t+1}(b_{2,t+1}(\hat{x}_{1t}), \hat{x}_{1t}) = \theta \pi_{2,t+1}(\hat{x}_{1t})\text{, all } t, \tag{17}$$

$$V_{jt}(b_{jt}(x), x) > \theta \pi_{jt}(x)\text{, all } j > 2\text{, all } t\text{, all } x \geq \hat{x}_{jt} \tag{18}$$

Lemma 2. In any balanced growth path, the cutoffs $\hat{x}_{jt}$ are characterized by

$$\hat{x}_{1t} = \tilde{\kappa}(\kappa,\theta)^{\frac{1-\rho}{\rho}} \left[(1-\rho)(L+\Pi_t)\right]^{\frac{1-\rho}{-\rho}} P_t^{-1}\rho^{-1} \tag{19}$$

where

$$\tilde{\kappa}(\kappa,\theta) = \frac{\kappa + \beta(1-\theta)\kappa_c + \sum_{i=3}^{n_t(\hat{x}_{1t})} \beta^{(i-1)}\kappa_c}{1 + \beta(1-\theta) g^{-\frac{\rho}{1-\rho}} + \sum_{i=3}^{n_t(\hat{x}_{1t})} \left[\beta^{(i-1)} g^{\frac{-(i-1)\rho}{1-\rho}}\right]} \tag{20}$$

Proposition 1. The economy has a balanced growth.

The proof of proposition 1 involves guessing and verifying the existence of an equilibrium with a balanced growth path.

We can now characterize $n_t(\hat{x}_{1t})$. An existing firm of age $2 \leq j \leq N$ will remain in the market as long as it remains profitable. This implies that

$$\hat{x}_{j,t+j-1} = \max\left\{ \tilde{\kappa}(\kappa,\theta)^{\frac{1-\rho}{\rho}} \left[(1-\rho)(L+\Pi_t)\right]^{\frac{1-\rho}{-\rho}} P_t^{-1}\rho^{-1}, \kappa_c^{\frac{1-\rho}{\rho}} \left[(1-\rho)(L+\Pi_{t+j-1})\right]^{\frac{1-\rho}{-\rho}} P_{t+j-1}^{-1}\rho^{-1} \right\} \tag{21}$$

The first term is the cutoff productivity for firm age j and the second term is the minimum productivity to be profitable that period. Using the balanced growth path conditions that $\Pi_{t+j-1} = \Pi_t$ and $P_{t+j-1} = g^{1-j} P_t$, we get that if $n_t(\hat{x}_{1t}) < N$ then the following condition must be satisfied

$$g^{(n_t(\hat{x}_{1t})-1)\frac{\rho}{1-\rho}} \leq \frac{\kappa + \beta(1-\theta)\kappa_c + \sum_{i=3}^{n_t(\hat{x}_{1t})} \beta^{(i-1)} \kappa_c}{1 + \beta(1-\theta) g^{-\frac{\rho}{1-\rho}} + \sum_{i=3}^{n_t(\hat{x}_{1t})} \left[\beta^{(i-1)} g^{\frac{-(i-1)\rho}{1-\rho}} \right]} \frac{1}{\kappa_c} < g^{n_t(\hat{x}_{1t})\frac{\rho}{1-\rho}} \tag{22}$$

and if $n_t(\hat{x}_{1t}) = N$ then

$$\frac{\kappa + \beta(1-\theta)\kappa_c + \sum_{i=3}^{N} \beta^{(i-1)} \kappa_c}{1 + \beta(1-\theta) g^{-\frac{\rho}{1-\rho}} + \sum_{i=3}^{N} \left[\beta^{(i-1)} g^{\frac{-(i-1)\rho}{1-\rho}} \right]} \frac{1}{\kappa_c} > g^{(N-1)\frac{\rho}{1-\rho}} \tag{23}$$

Note that $n_t(\hat{x}_{1t})$ does not depend on t in the balanced growth path. Thus, we will denote $\hat{n}(\kappa,\theta) = n_t(\hat{x}_{1t};\kappa,\theta)$.

There are three possible cases for firm exit. First, firms only exit exogenously at age N if

$$\frac{\tilde{\kappa}(\kappa,\theta)}{\kappa_c} \geq g^{\frac{(N-1)\rho}{1-\rho}}.$$

Second, marginal entrant firms exit endogenously if

$$g^{\frac{\rho}{1-\rho}} < \frac{\tilde{\kappa}(\kappa,\theta)}{\kappa_c} < g^{\frac{(N-1)\rho}{1-\rho}}.$$

Third, marginal entrant firms operate only one period if

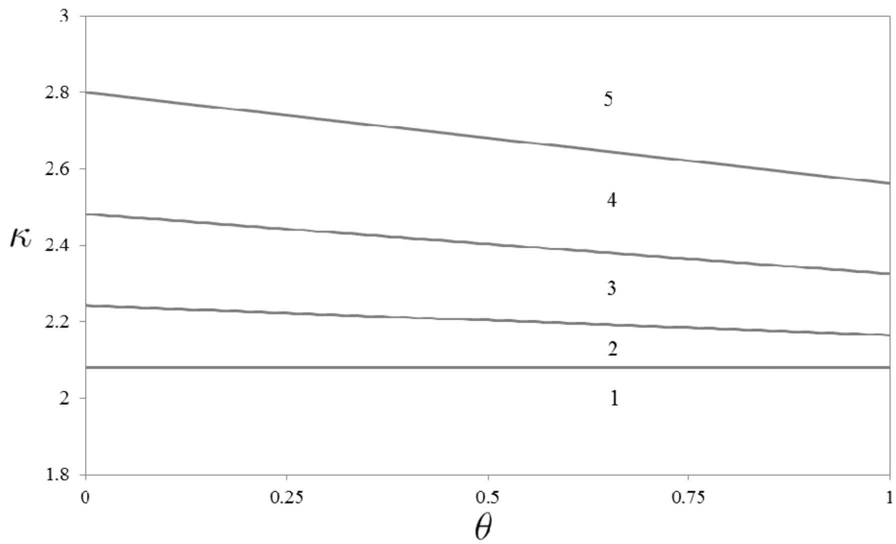
$$\frac{\tilde{\kappa}(\kappa,\theta)}{\kappa_c} \leq g^{\frac{\rho}{1-\rho}}.$$

In the third case, no firms require finance for entry. We will restrict our attention to the first two cases where

$$\frac{\tilde{\kappa}(\kappa,\theta)}{\kappa_c} > g^{\frac{\rho}{1-\rho}}.$$

We will now characterize $\hat{n}(\kappa,\theta)$. We will show that for fixed θ, $\hat{n}(\kappa,\theta)$ is an increasing step function of κ. In addition, for a fixed κ, $\hat{n}(\kappa,\theta)$ is an increasing step function of θ.

Figure 4: $\hat{n}(\kappa,\theta)$



Lemma 3: If $g > 1$, then in any balanced growth path,

(i) $\hat{n}(\kappa,\theta)$ is an increasing step function of κ, for fixed θ, and

(ii) $\hat{n}(\kappa,\theta)$ is an increasing step function of θ, for fixed κ.

Corollary 1:

(i) $P_t(\kappa,\theta)$ is continuous in θ and κ

(ii) $Y_t(\kappa,\theta)$ is continuous in θ and κ

(iii) $\Pi(\kappa,\theta)$ is continuous in θ and κ

(iv) $M(\kappa,\theta)$ is continuous in θ and κ.

Corollary 1 implies that the aggregate variables are continuous even though $\hat{n}(\kappa,\theta)$ is not continuous. This comes from the fact that when there is a jump in $\hat{n}(\kappa,\theta)$, only an infinitesimally small number of firms change the age at which they exit. Corollary 1 is extremely useful for comparative statics. Since $Y_t(\kappa,\theta)$ and $M(\kappa,\theta)$ are continuous, and differentiable everywhere except at the jump points of $\hat{n}(\kappa,\theta)$, it suffices to consider only the points at which $Y_t(\kappa,\theta)$ and $M(\kappa,\theta)$ are differentiable in the comparative statics analysis.

5. Comparative Statics

In this section, we prove that firm entry and output are decreasing in both barriers to entry and financial frictions. Moreover, we show that reforms to entry barriers and financial frictions are substitutes, consistent with the interaction term documented in section 2.

Proposition 2: Suppose that firms use finance. Then

(i) $\log Y_t(\kappa,\theta)$ is decreasing in θ, and

(ii) $\log M(\kappa,\theta)$ is decreasing in θ.

Proof. We will show the case where $\hat{n}(\kappa,\theta)=N$. The proof for the general $\hat{n}(\kappa,\theta)$ case can be found in the appendix.

(i): From proposition 1, we know that when $\hat{n}(\kappa,\theta)=N$,

$$Y_t(\kappa,\theta)=\left(\widehat{\kappa}_e(\kappa_e,\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{a=1}^{n}g^{(1-a)\frac{\rho}{1-\rho}}\right)^{\frac{1}{\gamma}}\left(\rho+\frac{\gamma(1-\rho)-\rho}{\gamma}\widehat{\kappa}_e^{\,-1}\frac{\sum_{a=1}^{n}\kappa_a}{\sum_{a=1}^{n}g^{(1-a)\frac{\rho}{1-\rho}}}\right)^{-\frac{\gamma-\rho}{\gamma\rho}}$$

Then

$$\log Y_t(\kappa,\theta)=\frac{1}{\gamma}\log\left(\tilde{\kappa}(\kappa,\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}\right)-\frac{\gamma-\rho}{\gamma\rho}\log\left(\rho+\frac{\gamma(1-\rho)-\rho}{\gamma}\tilde{\kappa}(\kappa,\theta)^{-1}\frac{\sum_{i=1}^{N}\kappa_i}{\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}}\right)$$

By corollary 1, it suffices to show that $\frac{d\log Y_t(\kappa,\theta)}{d\theta}<0$ where $Y_t(\kappa,\theta)$ is differentiable. We differentiate the expression with respect to θ

$$\frac{d\log Y_t(\kappa,\theta)}{d\theta}=\frac{1}{\gamma}\frac{1}{\tilde{\kappa}(\kappa,\theta)}\frac{d\tilde{\kappa}(\kappa,\theta)}{d\theta}\frac{\gamma(1-\rho)-\rho}{\rho}\left(-1+\frac{\gamma-\rho}{\rho}\frac{\sum_{i=1}^{N}\kappa_i}{\gamma\tilde{\kappa}(\kappa,\theta)\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}+\frac{\gamma(1-\rho)-\rho}{\rho}\sum_{i=1}^{N}\kappa_i}\right)$$

$$=\frac{1}{\gamma}\frac{1}{\tilde{\kappa}(\kappa,\theta)}\frac{d\tilde{\kappa}(\kappa,\theta)}{d\theta}\frac{\gamma(1-\rho)-\rho}{\rho}\left(\frac{-\gamma\tilde{\kappa}(\kappa,\theta)\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}+\gamma\sum_{i=1}^{N}\kappa_i}{\gamma\tilde{\kappa}(\kappa,\theta)\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}+\frac{\gamma(1-\rho)-\rho}{\rho}\sum_{i=1}^{N}\kappa_i}\right)$$

$$=-\frac{1}{\tilde{\kappa}(\kappa,\theta)}\frac{d\tilde{\kappa}(\kappa,\theta)}{d\theta}\frac{\gamma(1-\rho)-\rho}{\rho}\left(\frac{\tilde{\kappa}(\kappa,\theta)\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}-\sum_{i=1}^{N}\kappa_i}{\gamma\tilde{\kappa}(\kappa,\theta)\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}+\frac{\gamma(1-\rho)-\rho}{\rho}\sum_{i=1}^{N}\kappa_i}\right)$$

$$<0$$

which is true since $\frac{d\tilde{\kappa}(\kappa,\theta)}{d\theta}>0$ and $\tilde{\kappa}(\kappa,\theta)>\frac{\sum_{a=1}^{n}\kappa_a}{\sum_{a=1}^{n}g^{(1-a)\frac{\rho}{1-\rho}}}$.

(ii): Similarly, we can show that when $\hat{n}(\kappa,\theta)=N$

$$\log M(\kappa,\theta)=-\log\tilde{\kappa}(\kappa,\theta)+\log\left(L\frac{\gamma(1-\rho)-\rho}{\gamma}\mu^{-1}\frac{1}{\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}}\right)-\log\left(\rho+\frac{\gamma(1-\rho)-\rho}{\gamma}\tilde{\kappa}(\kappa,\theta)^{-1}\frac{\sum_{i=1}^{N}\kappa_i}{\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}}\right)$$

$$\frac{d\log M}{d\theta}=-\frac{\rho\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}}{\rho\tilde{\kappa}(\kappa,\theta)\sum_{i=1}^{N}g^{(1-i)\frac{\rho}{1-\rho}}+\frac{\gamma(1-\rho)-\rho}{\gamma}\sum_{i=1}^{N}\kappa_i}\frac{d\tilde{\kappa}}{d\theta}$$

which is negative since $\frac{d\tilde{\kappa}}{d\theta}>0$. □

Proposition 3: Suppose that firms use finance. Then

(i) $\log Y_t(\kappa,\theta)$ is decreasing in κ, and

(ii) $\log M(\kappa,\theta)$ is decreasing in κ.

The proof of proposition 3 is similar to proposition 2, and can be found in the appendix. Propositions 2 and 3 establish that firm entry and output is decreasing in both financial frictions and entry barriers. The main focus of this paper lies in the interaction between financial frictions and entry barriers. Proposition 4 establishes an important result that reforms to financial frictions and to entry barriers are substitutes, consistent with our empirical findings.[1]

Proposition 4: Suppose that firms use finance. Then

(i) $\frac{\partial^2 \log Y_t(\kappa,\theta)}{\partial\kappa\partial\theta} < 0$, and

(ii) $\frac{\partial^2 \log M(\kappa,\theta)}{\partial\kappa\partial\theta} < 0$.

6. Conclusion

[To be added.]

[1] In the current draft, proposition 4 holds only where $\log Y_t(\kappa,\theta)$ and $\log M(\kappa,\theta)$ are twice differentiable. We plan to address this issue by incorporating heterogeneous operational fixed costs, thereby smoothing the jump points.

References

Amaral, P.S. and E. Quintin (2010), "Limited Enforcement, Financial Intermediation, and Economic Development: A Quantitative Assessment," *International Economic Review*, 51(3), 785–811.

Barseghyan, L. and R. DiCecio (2009), "Entry Costs, Industry Structure, and Cross-country Income and TFP Differences," Federal Reserve Bank of St. Louis Working Paper 2009-005D.

Brandt, L. Van Biesebroeck, J. and Y. Zhang (2009), "Creative Accounting or Creative Destruction? Firm-level Productivity Growth in Chinese Manufacturing," NBER Working Paper 15152.

Buehn, A. and F. Schneider (2007), "Shadow Economies and Corruption All Over the World: Revised Estimates for 120 Countries," *Economics*, 1.

Buera, F.J., Kaboski, J.P., and Y. Shin (2009), "Finance and Development: A Tale of Two Sectors," NBER Working Paper 14914.

Herrendorf, B. and A. Teixeira (2009), "Barriers to Entry and Development," Arizona State University.

Hopenhayn, H. A. (1992), "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, 60(5), 1127–50.

Kehoe, T.J. and E.C. Prescott (2007), "Great Depressions of the Twentieth Century" in T.J. Kehoe and E.C. Prescott eds. *Great Depressions of the Twentieth Century*, Federal Reserve Bank of Minneapolis, 1–20.

Parente, S.L. and E.C. Prescott (1999), "Monopoly Rights: A Barrier to Riches," *American Economic Review*, 89(5), 1216–33.

Parente, S.L. and E.C. Prescott (2002), *Barriers to Riches*, MIT Press.

Petrin, A. White, T., and J.P. Reiter (2009), "The Impact of Plant-level Resource Reallocations and Technical Progression on U.S. Macroeconomic Growth," University of Minnesota.

Restuccia, D. and R. Rogerson (2008), "Policy Distortions and Aggregate Productivity with Heterogeneous Plants," *Review of Economic Dynamics*, 11(4), 707–20.

Appendix

Proof of Lemma 0:

$$
\begin{aligned}
\Pi_t &= \sum_{i=1}^{N} \int_{\hat{x}_{it}}^{\infty} \pi_{it}(x)\, dF_{t-i+1}(x)dx \\
&= \sum_{i=1}^{N} \int_{\hat{x}_{it}}^{\infty} \left(d_{it}(x) + q_{t+1} b_{i,t+1}(x) - b_{i-1,t}(x) \right) dF_{t-i+1}(x)dx \\
&= \sum_{i=1}^{N} \int_{\hat{x}_{it}}^{\infty} d_{it}(x) dF_{t-i+1}(x)dx - \sum_{i=2}^{N} \int_{\hat{x}_{it}}^{\infty} b_{i-1,t}(x) dF_{t-i+1}(x)dx + q_{t+1} \sum_{i=1}^{N-1} \int_{\hat{x}_{it}}^{\infty} b_{i,t+1}(x) dF_{t-i+1}(x)dx \\
&= \sum_{i=1}^{N} \int_{\hat{x}_{it}}^{\infty} d_{it}(x) dF_{t-i+1}(x)dx - \sum_{k=1}^{N-1} \int_{\hat{x}_{k+1,t}}^{\infty} b_{kt}(x) dF_{t-k}(x)dx + q_{t+1} \sum_{i=1}^{N-1} \int_{\hat{x}_{it}}^{\infty} b_{i,t+1}(x) dF_{t-i+1}(x)dx \\
&= D_t + B_t - q_{t+1} B_{t+1}
\end{aligned}
$$

where the last equality comes from equilibrium conditions for aggregate dividends and bond market clearing. □

Proof of Lemma 1:

$$
\begin{aligned}
& V_{j,t+j-1}(b_{j,t+j-1}(\hat{x}_{1t}), \hat{x}_{1t}) \\
&= \pi_{j,t+j-1}(\hat{x}_{1t}) + \sum_{i=j+1}^{n_t(\hat{x}_{1t})} \left(\prod_{s=j}^{i-1} q_{t+s} \right) \pi_{i,t+i-1}(\hat{x}_{1t}) + b_{j,t+j-1}(\hat{x}_{1t}) \\
&= \pi_{j,t+j-1}(\hat{x}_{1t}) + \sum_{i=j+1}^{n_t(\hat{x}_{1t})} \left(\prod_{s=j}^{i-1} q_{t+s} \right) \pi_{i,t+i-1}(\hat{x}_{1t}) + \frac{\pi_{1t}(\hat{x}_{1t}) + \sum_{i=2}^{j-1} \left(\prod_{s=j}^{i-1} q_{t+s} \right) \pi_{i,t+i-1}(\hat{x}_{1t})}{\prod_{s=1}^{j-1} q_{t+s}} \\
&= \frac{1}{\prod_{s=1}^{j-1} q_{t+s}} \left(\pi_{j,t+j-1}(\hat{x}_{1t}) \prod_{s=1}^{j-1} q_{t+s} + \prod_{s=1}^{j-1} q_{t+s} \sum_{i=j+1}^{n_t(\hat{x}_{1t})} \left(\prod_{s=j}^{i-1} q_{t+s} \right) \pi_{i,t+i-1}(\hat{x}_{1t}) + \pi_{1t}(\hat{x}_{1t}) + \sum_{i=2}^{j-1} \left(\prod_{s=j}^{i-1} q_{t+s} \right) \pi_{i,t+i-1}(\hat{x}_{1t}) \right) \\
&= \frac{1}{\prod_{s=1}^{j-1} q_{t+s}} \left(\pi_{1t}(\hat{x}_{1t}) + \sum_{i=2}^{n_t(\hat{x}_{1t})} \left(\prod_{s=j}^{i-1} q_{t+s} \right) \pi_{i,t+i-1}(\hat{x}_{1t}) \right)
\end{aligned}
$$

The enforcement constraint now becomes

$$
\frac{1}{\prod_{s=1}^{j-1} q_{t+s}} \left(\pi_{1t}(\hat{x}_{1t}) + \sum_{i=2}^{n_t(\hat{x}_{1t})} \left(\prod_{s=j}^{i-1} q_{t+s} \right) \pi_{i,t+i-1}(\hat{x}_{1t}) \right) \geq \theta \pi_{j,t+j-1}(\hat{x}_{1t}) \quad \forall j \geq 2
$$

We see that the LHS is increasing in j and the RHS is decreasing in j. Thus, the constraint can only hold with equality in period 2. □

Proof of Lemma 2.

$$V_{2,t+1}(b_{2,t+1}(\hat{x}_{1t}),\hat{x}_{1t}) = \theta\pi_{2,t+1}(\hat{x}_{1t}) \quad \forall t$$

$$\pi_{2,t+1}(\hat{x}_{1t}) + q_{t+2}\pi_{3,t+2}(\hat{x}_{1t}) + ... + \left(\prod_{s=2}^{n_t(\hat{x}_{1t})-1} q_{t+s}\right)\pi_{n_t(\hat{x}_{1t}),t+n_t(\hat{x}_{1t})-1}(\hat{x}_{1t}) + b_{2,t+1}(\hat{x}_{1t}) = \theta\pi_{2,t+1}(\hat{x}_{1t})$$

Now, substituting the expression for b_{2t+1}

$$\pi_{2,t+1}(\hat{x}_{1t}) + q_{t+2}\pi_{3,t+2}(\hat{x}_{1t}) + ... + \left(\prod_{s=2}^{n_t(\hat{x}_{1t})-1} q_{t+s}\right)\pi_{n_t(\hat{x}_{1t}),t+n_t(\hat{x}_{1t})-1}(\hat{x}_{1t}) + \frac{\pi_{1t}(\hat{x}_{1t})}{q_{t+1}} = \theta\pi_{2,t+1}(\hat{x}_{1t})$$

$$\pi_{1t}(\hat{x}_{1t}) + q_{t+1}(1-\theta)\pi_{2,t+1}(\hat{x}_{1t}) + \sum_{i=3}^{n_t(\hat{x}_{1t})}\left(\prod_{s=1}^{i} q_{t+s}\right)\pi_{i,t+i-1}(\hat{x}_{1t}) = 0$$

We can now find the expression for at $\hat{x}_{1t}$

$$\pi_{1t}(\hat{x}_{1t}) + q_{t+1}(1-\theta)\pi_{2,t+1}(\hat{x}_{1t}) + \sum_{i=3}^{n_t(\hat{x}_{1t})}\left(\prod_{s=1}^{i} q_{t+s}\right)\pi_{i,t+i-1}(\hat{x}_{1t}) = 0$$

$$(1-\rho)(L+\Pi_t)P_t^{\frac{\rho}{1-\rho}}(\hat{x}_{1t}\rho)^{\frac{\rho}{1-\rho}} - \kappa + q_{t+1}(1-\theta)\left[(1-\rho)(L+\Pi_{t+1})P_{t+1}^{\frac{\rho}{1-\rho}}(\hat{x}_{1t}\rho)^{\frac{\rho}{1-\rho}} - \kappa_c\right]$$
$$+\sum_{i=3}^{n_t(\hat{x}_{1t})}\left(\prod_{s=2}^{i} q_{t+s-1}\right)\left[(1-\rho)(L+\Pi_{t+i-1})P_{t+i-1}^{\frac{\rho}{1-\rho}}(\hat{x}_{1t}\rho)^{\frac{\rho}{1-\rho}} - \kappa_c\right] = 0$$

Applying the balanced growth path conditions ($P_t = g^{-1}P_{t-1},\ \Pi_t = \Pi_{t-1},\ \beta = q_t$)

$$(1-\rho)(L+\Pi_t)P_t^{\frac{\rho}{1-\rho}}(\hat{x}_{1t}\rho)^{\frac{\rho}{1-\rho}} - \kappa + q_{t+1}(1-\theta)\left[(1-\rho)(L+\Pi_{t+1})P_{t+1}^{\frac{\rho}{1-\rho}}(\hat{x}_{1t}\rho)^{\frac{\rho}{1-\rho}} - \kappa_c\right]$$
$$+\sum_{i=3}^{n_t(\hat{x}_{1t})}\left(\prod_{s=2}^{i} q_{t+s-1}\right)\left[(1-\rho)(L+\Pi_{t+i-1})P_{t+i-1}^{\frac{\rho}{1-\rho}}(\hat{x}_{1t}\rho)^{\frac{\rho}{1-\rho}} - \kappa_c\right] = 0$$

.

$$(1-\rho)(L+\Pi_t)P_t^{\frac{\rho}{1-\rho}}(\hat{x}_{1t}\rho)^{\frac{\rho}{1-\rho}} + \beta(1-\theta)(1-\rho)(L+\Pi_t)g^{-\frac{\rho}{1-\rho}}P_t^{\frac{\rho}{1-\rho}}(\hat{x}_{1t}\rho)^{\frac{\rho}{1-\rho}}$$
$$+\sum_{i=3}^{n_t(\hat{x}_{1t})}\left[\beta^{(i-1)}(1-\rho)(L+\Pi_t)g^{\frac{-(i-1)\rho}{1-\rho}}P_t^{\frac{\rho}{1-\rho}}(\hat{x}_{1t}\rho)^{\frac{\rho}{1-\rho}}\right] = \kappa + \beta(1-\theta)\kappa_c + \sum_{i=3}^{n_t(\hat{x}_{1t})}\left[\beta^{(i-1)}\kappa_c\right]$$

Rearranging terms yields the desired result. □

Proof of Proposition 1.

The expression for aggregate profits is given by

$$\Pi_t = \sum_{i=1}^{N}\mu\int_{\hat{x}_{it}}^{\infty}\pi_{it}(x)\,dF_{t-i+1}(x)$$
$$= \sum_{i=1}^{N}\mu\left\{(1-\rho)(L+\Pi_t)P_t^{\frac{\rho}{1-\rho}}\rho^{\frac{\rho}{1-\rho}}\frac{\gamma g^{\gamma(t-i+1)}(1-\rho)}{\gamma(1-\rho)-\rho}\hat{x}_{i,t}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} - g^{\gamma(t-i+1)}\hat{x}_{i,t}^{-\gamma}\kappa_i\right\}$$

The expression for the composite good price is given by

$$
\begin{aligned}
P_t^{\frac{-\rho}{1-\rho}} &= \sum_{i=1}^{N} \mu \int_{\hat{x}_{it}}^{\infty} p(x)^{\frac{-\rho}{1-\rho}}\, dF_{t-i+1}(x) \\
&= \sum_{i=1}^{N} \mu \int_{\hat{x}_{it}}^{\infty} (\rho x)^{\frac{\rho}{1-\rho}} g^{\gamma(t-i+1)} \gamma x^{-\gamma-1} dx \\
&= \rho^{\frac{\rho}{1-\rho}} \mu \sum_{i=1}^{N} \frac{\gamma g^{\gamma(t-i+1)}(1-\rho)}{\gamma(1-\rho)-\rho} \hat{x}_{it}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \\
&= \frac{\rho^{\frac{\rho}{1-\rho}} \gamma(1-\rho)}{\gamma(1-\rho)-\rho} \mu \sum_{i=1}^{N} g^{\gamma(t-i+1)} \hat{x}_{it}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}
\end{aligned}
$$

Using the expression for cutoffs from lemma 2,

$$
\begin{aligned}
P_t^{\frac{-\rho}{1-\rho}} &= \frac{\rho^{\frac{\rho}{1-\rho}} \gamma(1-\rho)}{\gamma(1-\rho)-\rho} \mu \sum_{i=1}^{N} g^{\gamma(t-i+1)} \hat{x}_{it}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \\
&= \frac{\rho^{\frac{\rho}{1-\rho}} \gamma(1-\rho)}{\gamma(1-\rho)-\rho} \mu \left[\sum_{i=1}^{\hat{n}(\kappa,\theta)} g^{\gamma(t-i+1)} \hat{x}_{1,t-i+1}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(t-i+1)} \hat{x}_{it}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right] \\
&= \frac{\rho^{\frac{\rho}{1-\rho}} \gamma(1-\rho)}{\gamma(1-\rho)-\rho} \mu \left[\begin{array}{l} \sum_{i=1}^{\hat{n}(\kappa,\theta)} g^{\gamma(t-i+1)} \left(\tilde{\kappa}(\kappa,\theta)^{\frac{1-\rho}{\rho}} \left[(1-\rho)(L+\Pi_{t-i+1})\right]^{\frac{1-\rho}{-\rho}} P_{t-i+1}^{-1} \rho^{-1} \right)^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \\ + \sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(t-i+1)} \left(\kappa_c^{\frac{1-\rho}{\rho}} \left[(1-\rho)(L+\Pi_t)\right]^{\frac{1-\rho}{-\rho}} P_t^{-1} \rho^{-1} \right)^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \end{array} \right]
\end{aligned}
$$

Using the balanced growth path conditions ($P_t = g^{-1} P_{t-1},\ \Pi_t = \Pi_{t-1}$) and solving for P_t yields

$$
\begin{aligned}
P_t^{-\gamma} = &\left[(1-\rho)(L+\Pi_t)\right]^{\frac{\rho-\gamma(1-\rho)}{-\rho}} \rho^{\gamma} \frac{\gamma(1-\rho)}{\gamma(1-\rho)-\rho} \mu g^{\gamma t} \\
&\left[\sum_{i=1}^{\hat{n}(\kappa,\theta)} g^{(1-i)\frac{\rho}{1-\rho}} \tilde{\kappa}(\kappa,\theta)^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(1-i)} \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}} \right]
\end{aligned}
$$

Now substitute the balanced growth path conditions ($P_t = g^{-1} P_{t-1},\ \Pi_t = \Pi_{t-1}$) and the cutoff expressions from lemma 2 into the expression for Π_t

$$\Pi_t = \sum_{i=1}^{\hat{n}(\kappa,\theta)} \mu\left\{(1-\rho)(L+\Pi_t)P_t^{\frac{\rho}{1-\rho}}\rho^{\frac{\rho}{1-\rho}}\frac{\gamma g^{\gamma(t-i+1)}(1-\rho)}{\gamma(1-\rho)-\rho}\hat{x}_{it}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} - g^{\gamma(t-i+1)}\hat{x}_{it}^{-\gamma}\kappa_i\right\}$$

$$+\sum_{a=\hat{n}(\kappa,\theta)+1}^{N} \mu\left\{(1-\rho)(L+\Pi_t)P_t^{\frac{\rho}{1-\rho}}\rho^{\frac{\rho}{1-\rho}}\frac{\gamma g^{\gamma(t-i+1)}(1-\rho)}{\gamma(1-\rho)-\rho}\hat{x}_{it}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} - g^{\gamma(t-i+1)}\hat{x}_{it}^{-\gamma}\kappa_i\right\}$$

$$=\mu\left[(1-\rho)(L+\Pi_t)\right]^{\frac{1-\rho}{\rho}\gamma}P_t^{\gamma}\rho^{\gamma}g^{\gamma t}\left(\frac{\gamma(1-\rho)}{\gamma(1-\rho)-\rho}\tilde{\kappa}(\kappa,\theta)^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=1}^{\hat{n}(\kappa,\theta)} g^{(1-i)\frac{\rho}{1-\rho}} - \tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}\kappa_i\right)$$

$$+\mu\left[(1-\rho)(L+\Pi_t)\right]^{\frac{1-\rho}{\rho}\gamma}P_t^{\gamma}\rho^{\gamma}g^{\gamma t}\left(\frac{\gamma(1-\rho)}{\gamma(1-\rho)-\rho}\kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(1-i)} - \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(1-i)}\right)$$

Substituting the expression for P_t to solve for Π_t yields

$$\Pi_t = \left(L+\Pi_t\right)\xi$$

where

$$\xi(\kappa,\theta) = 1-\rho-\frac{\gamma(1-\rho)-\rho}{\gamma\omega(\kappa,\theta)}\left(\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}\kappa_i + \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(1-i)}\right)$$

$$\omega(\kappa,\theta) = \tilde{\kappa}(\kappa,\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)} g^{(1-i)\frac{\rho}{1-\rho}} + \kappa_c^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(1-i)}$$

Thus, our guess has been verified.

$$P_t(\kappa,\theta)^{-\gamma} = g^{\gamma t}\left[(1-\rho)(L+\Pi(\kappa,\theta))\right]^{\frac{\rho-\gamma(1-\rho)}{-\rho}}\rho^{\gamma}\frac{\gamma(1-\rho)}{\gamma(1-\rho)-\rho}\mu\omega(\kappa,\theta)$$

$$M(\kappa,\theta) = \tilde{\kappa}(\kappa,\theta)^{-\frac{\gamma(1-\rho)}{\rho}}\left[L+\Pi(\kappa,\theta)\right]\frac{\gamma(1-\rho)-\rho}{\gamma}\mu^{-1}\omega(\kappa,\theta)^{-1}$$

$$Y_t(\kappa,\theta) = \left[L+\Pi(\kappa,\theta)\right]P_t(\kappa,\theta)^{-1}$$

$$\Pi(\kappa,\theta) = \frac{L\xi(\kappa,\theta)}{1-\xi(\kappa,\theta)} \qquad \tilde{\kappa}(\kappa,\theta) = \frac{\kappa+\beta(1-\theta)\kappa_c + \sum_{i=3}^{\hat{n}(\kappa,\theta)}\beta^{(i-1)}\kappa_c}{1+\beta(1-\theta)g^{-\frac{\rho}{1-\rho}} + \sum_{i=3}^{\hat{n}(\kappa,\theta)}\left[\beta^{(i-1)}g^{\frac{-(a-1)\rho}{1-\rho}}\right]}$$

$$\xi(\kappa,\theta) = 1-\rho-\frac{\gamma(1-\rho)-\rho}{\gamma\omega(\kappa,\theta)}\left(\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}\kappa_i + \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(1-i)}\right)$$

$$\omega(\kappa,\theta) = \tilde{\kappa}(\kappa,\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)} g^{(1-i)\frac{\rho}{1-\rho}} + \kappa_c^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(1-i)}$$

□

Proof of Lemma 3
Part (i):

We will show that there are $N-1$ jump points.

For m such that $2 \le m \le N$, define $\kappa(m,\theta)$ to be

$$g^{(m-1)\frac{\rho}{1-\rho}} = \frac{\kappa(m,\theta) + \beta(1-\theta)\kappa_c + \sum_{i=3}^{m} \beta^{(i-1)}\kappa_c}{1 + \beta(1-\theta)g^{\frac{-\rho}{1-\rho}} + \sum_{i=3}^{m}\left[\beta^{(i-1)} g^{\frac{-(i-1)\rho}{1-\rho}}\right]} \frac{1}{\kappa_c}$$

We show that $\kappa(m,\theta)$ are the jump points of the step function. We show this in two steps.

First, it is clear that for small $\epsilon > 0$

$$\frac{\kappa(m,\theta) - \epsilon + \beta(1-\theta)\kappa_c + \sum_{i=3}^{m} \beta^{(i-1)}\kappa_c}{1 + \beta(1-\theta)g^{-\frac{\rho}{1-\rho}} + \sum_{i=3}^{m}\left[\beta^{(i-1)} g^{\frac{-(i-1)\rho}{1-\rho}}\right]} \frac{1}{\kappa_c} < g^{(m-1)\frac{\rho}{1-\rho}}$$

which violates the condition that the marginal entrant lives to the age of m.

We show that

$$g^{(m-2)\frac{\rho}{1-\rho}} \le \frac{\kappa(m,\theta) - \epsilon + \beta(1-\theta)\kappa_c + \sum_{i=3}^{m-1} \beta^{(i-1)}\kappa_c}{1 + \beta(1-\theta)g^{-\frac{\rho}{1-\rho}} + \sum_{i=3}^{m-1}\left[\beta^{(i-1)} g^{\frac{-(i-1)\rho}{1-\rho}}\right]} \frac{1}{\kappa_c} < g^{(m-1)\frac{\rho}{1-\rho}}$$

which satisfies the condition that the marginal entrant lives to the age of $m-1$.

In the second step, we show that for small $\epsilon > 0$, we show that

$$g^{(m-1)\frac{\rho}{1-\rho}} \le \frac{\kappa(m,\theta) + \epsilon + \beta(1-\theta)\kappa_c + \sum_{i=3}^{m} \beta^{(i-1)}\kappa_c}{1 + \beta(1-\theta)g^{-\frac{\rho}{1-\rho}} + \sum_{i=3}^{m}\left[\beta^{(i-1)} g^{\frac{-(i-1)\rho}{1-\rho}}\right]} \frac{1}{\kappa_c} < g^{m\frac{\rho}{1-\rho}}$$

which satisfies the condition that the marginal entrant lives to the age of m.

Hence, $\hat{n}(\kappa,\theta)$ is an increasing step function of $\theta \in [0,1]$ for any fixed κ

Part (ii):

We first characterize $\kappa(m,\theta)$. Solve for $\kappa(m,\theta)$ from the previous condition

$$\kappa(m,\theta) = \kappa_c \beta(1-\theta)\left(g^{(m-2)\frac{\rho}{1-\rho}} - 1\right) + \kappa_c g^{(m-1)\frac{\rho}{1-\rho}}\left(1 + \sum_{i=3}^{m}\left[\beta^{(i-1)} g^{\frac{-(i-1)\rho}{1-\rho}}\right]\right) - \kappa_c \sum_{i=3}^{m} \beta^{(i-1)}$$

Note that $\kappa(m,\theta)$ is continuous with respect to θ, and that m is held fixed. Differentiate

$$\frac{\partial \kappa(m,\theta)}{\partial \theta} = -\kappa_c \beta \left(g^{(m-2)\frac{\rho}{1-\rho}} - 1 \right)$$

Since $g > 1$, $\kappa(m,\theta)$ is an decreasing function of θ. The desired result follows. □

Proof of Lemma 4.
We know that

$$P_t(\kappa,\theta)^{-\gamma} = g^{\gamma t}\left[(1-\rho)(L+\Pi(\kappa,\theta))\right]^{\frac{\rho-\gamma(1-\rho)}{-\rho}} \rho^{\gamma} \frac{\gamma(1-\rho)}{\gamma(1-\rho)-\rho} \mu\omega(\kappa,\theta)$$

$$M(\kappa,\theta) = \tilde{\kappa}(\kappa,\theta)^{-\frac{\gamma(1-\rho)}{\rho}} \left[L+\Pi(\kappa,\theta)\right] \frac{\gamma(1-\rho)-\rho}{\gamma} \mu^{-1}\omega(\kappa,\theta)^{-1}$$

$$Y_t(\kappa,\theta) = \left[L+\Pi(\kappa,\theta)\right] P_t(\kappa,\theta)^{-1}$$

$$\Pi(\kappa,\theta) = \frac{L\xi(\kappa,\theta)}{1-\xi(\kappa,\theta)}$$

where

$$\tilde{\kappa}(\kappa,\theta) = \frac{\kappa + \beta(1-\theta)\kappa_c + \sum_{i=3}^{\hat{n}(\kappa,\theta)} \beta^{(i-1)}\kappa_c}{1+\beta(1-\theta)g^{-\frac{\rho}{1-\rho}} + \sum_{i=3}^{\hat{n}(\kappa,\theta)} \left[\beta^{(i-1)} g^{\frac{-(a-1)\rho}{1-\rho}}\right]}$$

$$\xi(\kappa,\theta) = 1-\rho - \frac{\gamma(1-\rho)-\rho}{\gamma\omega(\kappa,\theta)} \left(\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma} \sum_{i=1}^{\hat{n}(\kappa,\theta)} \kappa_i + \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}} \sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(1-i)} \right)$$

$$\omega(\kappa,\theta) = \tilde{\kappa}(\kappa,\theta)^{1-\frac{1-\rho}{\rho}\gamma} \sum_{i=1}^{\hat{n}(\kappa,\theta)} g^{(1-i)\frac{\rho}{1-\rho}} + \kappa_c^{1-\frac{1-\rho}{\rho}\gamma} \sum_{i=\hat{n}(\kappa,\theta)+1}^{N} g^{\gamma(1-i)}$$

It suffices to show that $\tilde{\kappa}(\kappa,\theta)$, $\xi(\kappa,\theta)$, and $\omega(\kappa,\theta)$ are continuous in κ,θ at the jump points.

(i) $\tilde{\kappa}(\kappa,\theta)$: We want to show that $\tilde{\kappa}(\kappa(m,\theta),\theta) = \lim_{\epsilon\to 0_+} \tilde{\kappa}(\kappa(m,\theta),\theta-\epsilon)$, $\forall m = 2,...,N$ and $\theta \in [0,1]$.

$$\tilde{\kappa}(\kappa(m,\theta),\theta)=\frac{\kappa(m,\theta)+\beta(1-\theta)\kappa_c+\sum_{i=3}^{m}\beta^{(i-1)}\kappa_c}{1+\beta(1-\theta)g^{-\frac{\rho}{1-\rho}}+\sum_{i=3}^{m}\left[\beta^{(i-1)}g^{\frac{-(i-1)\rho}{1-\rho}}\right]}$$

$$=\frac{\kappa(m,\theta)+\beta(1-\theta)\kappa_c+\sum_{i=3}^{m-1}\beta^{(i-1)}\kappa_c+\beta^{(m-1)}\kappa_c}{1+\beta(1-\theta)g^{-\frac{\rho}{1-\rho}}+\sum_{i=3}^{m-1}\left[\beta^{(i-1)}g^{\frac{-(i-1)\rho}{1-\rho}}\right]+\beta^{(m-1)}g^{\frac{-(m-1)\rho}{1-\rho}}}$$

$$=\frac{\kappa(m,\theta)+\beta(1-\theta)\kappa_c+\sum_{i=3}^{m-1}\beta^{(i-1)}\kappa_c}{1+\beta(1-\theta)g^{\frac{-\rho}{1-\rho}}+\sum_{i=3}^{m-1}\left[\beta^{(i-1)}g^{\frac{-(a-1)\rho}{1-\rho}}\right]}=\lim_{\epsilon\to 0_+}\tilde{\kappa}(\kappa(m,\theta),\theta-\epsilon)$$

since $\dfrac{\beta^{(m-1)}\kappa_c}{\beta^{(m-1)}g^{\frac{-(\bar{n}-1)\rho}{1-\rho}}}=\kappa_c g^{\frac{(m-1)\rho}{1-\rho}}=\tilde{\kappa}(\kappa(m,\theta),\theta)$

Now we show that, $\tilde{\kappa}(\kappa(m,\theta),\theta)=\lim_{\epsilon\to 0_+}\tilde{\kappa}(\kappa(m,\theta)-\epsilon,\theta)$ $\forall m=2,...,N$ and $\theta\in[0,1]$. Using similar logic as before we get that

$$\tilde{\kappa}(\kappa(m,\theta),\theta)=\frac{\kappa(m,\theta)+\beta(1-\theta)\kappa_c+\sum_{i=3}^{m-1}\beta^{(i-1)}\kappa_c}{1+\beta(1-\theta)g^{\frac{-\rho}{1-\rho}}+\sum_{i=3}^{m-1}\left[\beta^{(i-1)}g^{\frac{-(a-1)\rho}{1-\rho}}\right]}=\lim_{\epsilon\to 0_+}\tilde{\kappa}(\kappa(m,\theta)-\epsilon,\theta)$$

(ii) $\omega(\kappa,\theta)$**:**

First, we show that $\omega(\kappa(m,\theta),\theta)=\lim_{\epsilon\to 0_+}\omega(\kappa(m,\theta),\theta-\epsilon)$, $\forall m=2,...,N$ and $\theta\in[0,1]$.

$$\omega(\kappa(m,\theta),\theta)=\tilde{\kappa}(\kappa(m,\theta),\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m}g^{(1-i)\frac{\rho}{1-\rho}}+\kappa_c^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=m+1}^{N}g^{\gamma(1-i)}$$

$$=\tilde{\kappa}(\kappa(m,\theta),\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m-1}g^{(1-i)\frac{\rho}{1-\rho}}+\tilde{\kappa}(\kappa(m,\theta),\theta)^{1-\frac{1-\rho}{\rho}\gamma}g^{(1-m)\frac{\rho}{1-\rho}}+\kappa_c^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=m+1}^{N}g^{\gamma(1-i)}$$

$$=\tilde{\kappa}(\kappa(m,\theta),\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m-1}g^{(1-i)\frac{\rho}{1-\rho}}+\kappa_c^{1-\frac{1-\rho}{\rho}\gamma}g^{\gamma(1-m)}+\kappa_c^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=m+1}^{N}g^{\gamma(1-i)}$$

since $\kappa_c g^{\frac{(m-1)\rho}{1-\rho}}=\tilde{\kappa}(\kappa(m,\theta),\theta)$

since $\kappa_c g^{\frac{(m-1)\rho}{1-\rho}}=\tilde{\kappa}(\kappa(m,\theta),\theta)$. Thus

$$\omega(\tilde{\kappa}(\kappa(m,\theta),\theta)) = \tilde{\kappa}(\kappa(m,\theta),\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m-1} g^{(1-i)\frac{\rho}{1-\rho}} + \kappa_c^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=m}^{N} g^{\gamma(1-i)}$$
$$= \lim_{\epsilon\to 0_+}\omega(\kappa(m,\theta),\theta-\epsilon)$$

Second, we show that $\omega(\kappa,\theta) = \lim_{\epsilon\to 0_+}\omega(\kappa(m,\theta)-\epsilon,\theta)$, $\forall n = 2,...,N$ and $\theta\in[0,1]$. Using similar logic as before we get that

$$\omega(\tilde{\kappa}(\kappa(m,\theta),\theta)) = \tilde{\kappa}(\kappa(m,\theta),\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m-1} g^{(1-i)\frac{\rho}{1-\rho}} + \kappa_c^{1-\frac{1-\rho}{\rho}\gamma} g^{\gamma(1-m)} + \kappa_c^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=m+1}^{N} g^{\gamma(1-i)}$$
$$= \tilde{\kappa}(\kappa(m,\theta),\theta)^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m-1} g^{(1-i)\frac{\rho}{1-\rho}} + \kappa_c^{1-\frac{1-\rho}{\rho}\gamma}\sum_{i=m}^{N} g^{\gamma(1-i)}$$
$$= \lim_{\epsilon\to 0_+}\omega(\kappa(m,\theta)-\epsilon,\theta)$$

(iii) $\xi(\kappa,\theta)$:

First, we show that $\xi(\kappa(m,\theta),\theta) = \lim_{\epsilon\to 0_+}\xi(\kappa(m,\theta),\theta-\epsilon)$, $\forall m = 2,...,N$ and $\theta\in[0,1]$.

$$\xi(\kappa(m,\theta),\theta) = 1-\rho-\frac{\gamma(1-\rho)-\rho}{\gamma\omega(\kappa(m,\theta),\theta)}\left(\tilde{\kappa}(\kappa(m,\theta),\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m}\kappa_i + \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=m+1}^{N} g^{\gamma(1-i)}\right)$$
$$= 1-\rho-\frac{\gamma(1-\rho)-\rho}{\gamma\omega(\kappa(m,\theta),\theta)}\left(\tilde{\kappa}(\kappa(m,\theta),\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m-1}\kappa_i + \tilde{\kappa}(\kappa(m,\theta),\theta)^{-\frac{1-\rho}{\rho}\gamma}\kappa_c + \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=m+1}^{N} g^{\gamma(1-i)}\right)$$
$$= 1-\rho-\frac{\gamma(1-\rho)-\rho}{\gamma\omega(\kappa(m,\theta),\theta)}\left(\tilde{\kappa}(\kappa(m,\theta),\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{n-1}\kappa_i + g^{\gamma(1-m)}\kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=m+1}^{N} g^{\gamma(1-i)}\right)$$

since $\kappa_c g^{\frac{(m-1)\rho}{1-\rho}} = \tilde{\kappa}(\kappa(m,\theta),\theta)$. Thus,

$$\xi(\kappa(m,\theta),\theta) = 1-\rho-\frac{\gamma(1-\rho)-\rho}{\gamma\omega(\kappa(m,\theta),\theta)}\left(\tilde{\kappa}(\kappa(m,\theta),\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m-1}\kappa_i + \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=m}^{N} g^{\gamma(1-i)}\right)$$
$$= \lim_{\epsilon\to 0_+}\xi(\kappa(m,\theta),\theta-\epsilon)$$

Second, we show that $\xi(\kappa,\theta) = \lim_{\epsilon\to 0_+}\xi(\kappa(m,\theta)-\epsilon,\theta)$, $\forall m = 2,...,N$ and $\theta\in[0,1]$. Using similar logic as before we get that

$$\xi(\kappa(m,\theta),\theta) =$$
$$= 1-\rho-\frac{\gamma(1-\rho)-\rho}{\gamma\omega(\kappa(m,\theta),\theta)}\left(\tilde{\kappa}(\kappa(m,\theta),\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m-1}\kappa_i + g^{\gamma(1-m)}\kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=m+1}^{N} g^{\gamma(1-i)}\right)$$
$$= 1-\rho-\frac{\gamma(1-\rho)-\rho}{\gamma\omega(\kappa(m,\theta),\theta)}\left(\tilde{\kappa}(\kappa(m,\theta),\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{m-1}\kappa_i + \kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=m}^{N} g^{\gamma(1-i)}\right) \quad . \square$$
$$= \lim_{\epsilon\to 0_+}\xi(\kappa(m,\theta)-\epsilon,\theta)$$

Proof of Proposition 2:

(i) From Lemma 4, we know that $Y_t(\kappa,\theta)$ is continuous in θ and κ.

It suffices to show that $\frac{d\log Y_t(\kappa,\theta)}{d\theta}<0$ for $\forall\theta\in[0,1]$ and $\kappa\neq\kappa(m,\theta)$, $\forall m=2,...,N$.

Derive expression for $Y_t(\kappa,\theta)$

$$\begin{aligned}Y_t(\kappa,\theta)&=\left(L+\Pi(\kappa,\theta)\right)P_t(\kappa,\theta)^{-1}\\&=\left(L+\Pi(\kappa,\theta)\right)g^t\left[(1-\rho)(L+\Pi(\kappa,\theta))\right]^{\frac{\rho-\gamma(1-\rho)}{-\rho\gamma}}\rho\left(\frac{\gamma(1-\rho)}{\gamma(1-\rho)-\rho}\mu\omega(\kappa,\theta)\right)^{\frac{1}{\gamma}}\\&=g^t\left(\frac{L}{1-\xi(\kappa,\theta)}\right)^{\frac{\gamma-\rho}{\gamma\rho}}(1-\rho)^{\frac{1-\rho}{\rho}}\left(\frac{\gamma\rho^{\gamma}}{\gamma(1-\rho)-\rho}\right)^{\frac{1}{\gamma}}\mu^{\frac{1}{\gamma}}\omega(\kappa,\theta)^{\frac{1}{\gamma}}\end{aligned}$$

Thus,

$$\log Y_t(\kappa,\theta)=-\frac{\gamma-\rho}{\gamma\rho}\log\left(1-\xi(\kappa,\theta)\right)+\frac{1}{\gamma}\log\omega(\kappa,\theta)+\log\left(g^tL^{\frac{\gamma-\rho}{\gamma\rho}}(1-\rho)^{\frac{1-\rho}{\rho}}\left(\frac{\gamma\rho^{\gamma}}{\gamma(1-\rho)-\rho}\right)^{\frac{1}{\gamma}}\mu^{\frac{1}{\gamma}}\right)$$

Furthermore, it suffices to show that

$$\frac{\partial\log Y_t(\kappa,\theta)}{\partial\tilde{\kappa}}<0$$

since

$$\frac{d\log Y_t(\kappa,\theta)}{d\theta}=\frac{\partial\log Y_t(\kappa,\theta)}{\partial\tilde{\kappa}}\frac{\partial\tilde{\kappa}(\kappa,\theta)}{\partial\theta}$$

We can show this condition is true. □

(ii) From Lemma 4, we know that $M(\kappa,\theta)$ is continuous in θ and κ.

It suffices to show that $\frac{d\log M(\kappa,\theta)}{d\theta}<0$ for $\forall\theta\in[0,1]$ and $\kappa\neq\kappa(m,\theta)$, $\forall m=2,...,N$.

Derive expression for $M(\kappa,\theta)$

$$\begin{aligned}M(\kappa,\theta)&=\tilde{\kappa}(\kappa,\theta)^{-\frac{\gamma(1-\rho)}{\rho}}\left[L+\Pi(\kappa,\theta)\right]\frac{\gamma(1-\rho)-\rho}{\gamma}\mu^{-1}\omega(\kappa,\theta)^{-1}\\&=\tilde{\kappa}(\kappa,\theta)^{-\frac{\gamma(1-\rho)}{\rho}}\left[\frac{L}{1-\xi(\kappa,\theta)}\right]\frac{\gamma(1-\rho)-\rho}{\gamma}\mu^{-1}\omega(\kappa,\theta)^{-1}\end{aligned}$$

Thus,

$$\log M(\kappa,\theta) = -\frac{\gamma(1-\rho)}{\rho}\log\tilde{\kappa}(\kappa,\theta) + \log\left(1-\xi(\kappa,\theta)\right) - \log\omega(\kappa,\theta) + \log\left(L\frac{\gamma(1-\rho)-\rho}{\gamma}\mu^{-1}\right)$$

Furthermore, it suffices to show that

$$\frac{\partial \log M(\kappa,\theta)}{\partial\tilde{\kappa}} < 0$$

since

$$\frac{d\log M(\kappa,\theta)}{d\theta} = \frac{\partial \log M(\kappa,\theta)}{\partial\tilde{\kappa}}\frac{\partial\tilde{\kappa}(\kappa,\theta)}{\partial\theta}$$

We can show this condition is true. □

Proof of Proposition 3:

(i) From Lemma 4, we know that $Y_t(\kappa,\theta)$ is continuous in θ and κ.

It suffices to show that $\frac{d\log Y_t(\kappa,\theta)}{d\kappa} < 0$ for $\forall\theta\in[0,1]$ and $\kappa \neq \kappa(n,\theta)$, $\forall n = 2,...,N$.

We know that

$$\frac{d\log Y_t(\kappa,\theta)}{d\kappa} = \frac{\partial\log Y_t(\kappa,\theta)}{\partial\tilde{\kappa}}\frac{d\tilde{\kappa}(\kappa,\theta)}{d\kappa} + \frac{\partial\log Y_t(\kappa,\theta)}{\partial\kappa}$$

We can show that both terms on the right hand side are negative. □

(ii) From Lemma 4, we know that $M(\kappa,\theta)$ is continuous in θ and κ.

It suffices to show that $\frac{d\log M(\kappa,\theta)}{d\kappa} < 0$ for $\forall\theta\in[0,1]$ and $\kappa \neq \kappa(m,\theta)$, $\forall m = 2,...,N$.

We know that

$$\frac{d\log M(\kappa,\theta)}{d\kappa} = \frac{\partial\log M(\kappa,\theta)}{\partial\tilde{\kappa}}\frac{d\tilde{\kappa}(\kappa,\theta)}{d\kappa} + \frac{\partial\log M(\kappa,\theta)}{\partial\kappa}$$

We can show that both terms on the right hand side are negative. □

Proof of Proposition 4:

(i) As before, we know that

$$\log Y_t(\kappa,\theta) = -\frac{\gamma-\rho}{\gamma\rho}\log\left(1-\xi(\kappa,\theta)\right) + \frac{1}{\gamma}\log\omega(\kappa,\theta) + \log\left(g^t L^{\frac{\gamma-\rho}{\gamma\rho}}(1-\rho)^{\frac{1-\rho}{\rho}}\left(\frac{\gamma\rho^{\gamma}}{\gamma(1-\rho)-\rho}\right)^{\frac{1}{\gamma}}\mu^{\frac{1}{\gamma}}\right)$$

Differentiate with respect to θ and we get that

$$\frac{d\left(\log\left(Y_t(\kappa,\theta)\right)\right)}{d\theta}=\frac{d\tilde{\kappa}}{d\theta}\frac{1}{\gamma}\frac{\gamma(1-\rho)-\rho}{\rho}\frac{\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}}{\omega(\kappa,\theta)}Q(\kappa,\theta)$$

where

$$Q(\kappa,\theta)=-1-\frac{\gamma-\rho}{\gamma}\frac{-\frac{1-\rho}{\rho}\gamma\omega(\kappa,\theta)\tilde{\kappa}(\kappa,\theta)^{-1}\frac{\sum_{i=1}^{\hat{n}(\kappa,\theta)}\kappa_i}{\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}}-\left(\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}\kappa_i+\kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=\hat{n}(\kappa,\theta)+1}^{N}g^{\gamma(1-i)}\right)\left(1-\frac{1-\rho}{\rho}\gamma\right)}{\rho\omega(\kappa,\theta)+\frac{\gamma(1-\rho)-\rho}{\gamma}\left(\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}\kappa_i+\kappa_c^{\frac{\rho-\gamma(1-\rho)}{\rho}}\sum_{i=\hat{n}(\kappa,\theta)+1}^{N}g^{\gamma(1-i)}\right)}$$

Differentiate again and we get that

$$\frac{d^2\left(\log\left(Y_t(\kappa,\theta)\right)\right)}{d\theta d\kappa}=$$

$$\frac{1}{\gamma}\frac{\gamma(1-\rho)-\rho}{\rho}Q(\kappa,\theta)\left(\frac{d\tilde{\kappa}}{d\theta}\frac{d\tilde{\kappa}}{d\kappa}\frac{-\frac{1-\rho}{\rho}\gamma\omega(\kappa,\theta)\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma-1}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}-\left(1-\frac{1-\rho}{\rho}\gamma\right)\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}}{\omega(\kappa,\theta)^2}+\frac{d^2\tilde{\kappa}}{d\theta d\kappa}\frac{\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}}{\omega(\kappa,\theta)}\right)$$

$$+\frac{\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}}{\omega(\kappa,\theta)}\frac{1}{\gamma}\frac{\gamma(1-\rho)-\rho}{\rho}\left(\frac{d\tilde{\kappa}}{d\theta}\frac{\partial Q}{\partial\tilde{\kappa}}\frac{d\tilde{\kappa}}{d\kappa}+\frac{d\tilde{\kappa}}{d\theta}\frac{\partial Q}{\partial\kappa}\right)$$

Thus, it suffices to show the following two conditions

$$Q(\kappa,\theta)\left(\frac{d\tilde{\kappa}}{d\theta}\frac{d\tilde{\kappa}}{d\kappa}\frac{-\frac{1-\rho}{\rho}\gamma\omega(\kappa,\theta)\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma-1}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}-\left(1-\frac{1-\rho}{\rho}\gamma\right)\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}}{\omega(\kappa,\theta)^2}+\frac{d^2\tilde{\kappa}}{d\theta d\kappa}\frac{\tilde{\kappa}(\kappa,\theta)^{-\frac{1-\rho}{\rho}\gamma}\sum_{i=1}^{\hat{n}(\kappa,\theta)}g^{(1-i)\frac{\rho}{1-\rho}}}{\omega(\kappa,\theta)}\right)<0$$

$$\frac{d\tilde{\kappa}}{d\theta}\frac{\partial Q}{\partial\tilde{\kappa}}\frac{d\tilde{\kappa}}{d\kappa}+\frac{d\tilde{\kappa}}{d\theta}\frac{\partial Q}{\partial\kappa}<0$$

which we show in the technical appendix. □