This Online Appendix addresses several issues mentioned but not discussed in depth in the paper. Section I discusses various extensions of the basic model. Section II shows that for our identification arguments, it is sufficient to observe aggregate demand where it is single-valued. Section III provides a proof of part 3 of Proposition 4. Section IV provides an example that shows that the set of $M^2$-concave valuations is not convex. Section V shows that our identification result for multi-unit demand (Theorem 2), which assumed at least two goods ($n \geq 2$), would fail in the case of only a single good ($n = 1$). Section VI proves Theorem 2, our identification result for multi-unit demand. In order to maintain unique equation numbers, the numbering of equations in the Online Appendix does not start with (1), but rather starts with (67), which is where the numbering in the main paper left off. Similarly, the numbering of lemmas and propositions continues where the numbering in the main paper left off.

I Extensions

We discuss various extensions of our main identification results.

I.1 Unbounded Individual Demand

The multi-unit demand identification result (Theorem 2) of Section 5, incorporates a bound on the number of units of each good demanded by each consumer. The motivation was this allows valuation to be represented as points in Euclidean space, and so simplifies the required notion of genericity. Almost all of the lemmas supporting the proof of Theorem 2 would continue to hold for unbounded demand, but we still require an appropriate notion of genericity, which leave for future work. We now elaborate.

With unbounded demand (i.e., $N_j = \infty, \forall j \in J$ or equivalently, $\mathbb{B} = \mathbb{Z}_+^n$) we require the following assumption:
Marginal Utility Approaches Zero (MU0) For all $i \in I, j \in J$, and $z \in \mathbb{Z}_n^+$,

$$\lim_{{h \to \infty \atop h \in \mathbb{Z}_+}} v_i(z + (h + 1)e^j) - v_i(z + he^j) = 0.$$  

This says that starting from any consumption bundle $z$, and adding additional units of the same good $j$, the incremental value of additional units of $j$ eventually decreases to zero. The purpose of the MU0 assumption is to guarantee the existence of nonempty demand (i.e., nonempty set of optimal bundles) at all strictly positive price vectors, which follows when MU0 is combined with $M^2$-concavity.

With bounded demand, our generic identification result is proven by finding a set $\hat{S}$ of valuation profiles which is generic in the appropriate sense, and which also satisfies condition 3 of Proposition 6. In the case of unbounded demand, the set $\hat{S}$ is defined in almost exactly the same way as in the case of bounded demand: specifically via an extension of conditions (78-81) and (83-84) to the case where $\mathbb{B} = \mathbb{Z}_n^+$.1 (See Section VI.2 of the paper). Given the set $\hat{S}$, we can prove the analogue of condition 3 of Proposition 2,2 thus establishing identification for all valuation profiles in $\hat{S}$. The arguments are almost identical to those in the proof of Proposition 6. It only remains to show that $\hat{S}$ is generic in an appropriate sense. We believe that an appropriate notion of genericity for which such a result obtains can be supplied,3 but leave this as a task for future work. To summarize, with unbounded demand, we are able to establish identification under conditions analogous to the case of bounded multi-unit demand, but have yet to show that these conditions are generic relative to our qualitative assumptions on valuations.

I.2 Identification of the Number of Consumers $m$

In our identification results, we assumed the number $m$ of consumers was fixed and known to the econometrician. If $m$ were not known, then it could be identified. First assume 1UPG, and consider Theorem 1. Send the prices of some good $j$ to zero, and observe demand for that good. Because generically, the incremental value of any good to any package which does not contain it is strictly positive, the demand for good $j$ will equal the number of consumers in the limit as the price of good $j$ goes to zero.4 Next, relax 1UPG and consider Theorem 2. As the

---

1We do not require an analogue of or substitute for (82) in the case of unbounded demand.

2Observe that in the case of bounded demand, when the true valuation profile is in $\hat{S}$, the price vectors $p(z, +)$ have components equal to zero corresponding only to goods which are at satiation in $z$. With unbounded demand, no goods can be at satiation, so our result that individual demand is non-empty for strictly positive prices is sufficient for our result.

3Note that, on the way to a proof of genericity in an appropriate sense we can prove analogues of Lemmas 8 and 9.

4Notice that the definition of generic identification—Definition 1—depends on the number $m$ of consumers. To study an environment where the number $m$ of consumers must be identified, we modify our definition as follows: valuations are generically identified from demand when there exist a collection of sets of valuation profiles $\{S_m : m = 1, 2, \ldots\}$, such that (i) for all $m$, $S_m \subseteq \mathcal{V}^m$, (ii) $\mathcal{V}^m \setminus S_m$ has zero Lebesgue measure (in the
price of good \( j \) goes to zero, demand for good \( j \) will converge to \( mN_j \). If we assume that the satiation points \( N_j \) are known to the econometrician, then we can identify \( m \). The case where the satiation points are not known to the econometrician or where they are heterogenous is briefly discussed in Section I.4 below.

I.3 Heterogenous Satiation Points and Their Identification

Consider identification with multi-unit demand and assume now that the number \( m \) of consumers is known to the econometrician. Assume however that the satiation points \( N_j \) are not known to the econometrician and assume moreover that they are heterogenous across consumers, so that we must now add a superscript for consumer \( i \), writing \( N_i^j \) for consumer \( i \)'s satiation point for good \( j \). We assume that \( N_i^j \geq 1 \) for all consumers \( i \) and goods \( j \). The interpretation of these satiation points is similar to before: We now assume that for consumer \( i \), additional units of good \( j \) have positive incremental value to any package until \( i \) consumes \( N_i^j \) units, and then additional units confer no additional value. The personalized satiation points \( N_i^j \) are not known to the econometrician.

We now explain in rough outline how the econometrician can identify the satiation points \( N_i^j \). Suppose that the econometrician guesses the satiation points. We denote by \( \tilde{N}_i^j \) the econometrician’s guess for \( i \)'s satiation point for good \( j \) and by \( N_i^j \), \( i \)'s true satiation point for good \( j \). (Note that these guesses are only meaningful up to permutations of the indices \( i \)). Let \( \tilde{B}_i = \{ z \in \mathbb{Z}_+^n : 0 \leq z_j \leq \tilde{N}_i^j, \forall j \in J \} \). It is sufficient to argue that if the econometrician guesses incorrectly, then some property of the demand correspondence will reveal this error. So suppose that the econometrician has made an incorrect guess about the satiation point of some consumer. Since identification is only up to permutations of identities, we may relabel consumers so that the consumer for which the econometrician has made an incorrect guess is \( m \).\(^5\) To simplify the sketch of the argument, assume that the econometrician has already identified the valuations and satiation points of consumers \( 1, \ldots, m-1 \).\(^6\) Then define the set \( Q := \{ v_i(z + e^j) - v_i(z) : i = 1, \ldots, m-1, j = 1, \ldots, n; z, z + e^j \in \tilde{B}_i \} \). So \( Q \) is a collection of marginal values for consumers other than \( m \). We may assume that these marginal values are positive, since otherwise the econometrician would establish that his guess about some satiation point of some consumer other than \( m \) was mistaken so that he had not correctly identified the satiation point of that consumer. By arguments essentially identical to those in Proposition 6, if the econometrician has identified \( m \), then given that she has access to the demand correspondence, she will be able to find collections of price vectors \( P_+ = \{ p(z,+) : z \in \mathbb{B}_m \} \) and

\(^5\)Of course, the econometrician may have made an incorrect guess for other consumers as well.

\(^6\)A full version of the argument justifies this assumption, or to be more precise, makes an argument that plays the role of this assumption.
\( P_+ = \{ p(z, -) : z \in \mathbb{B}_m \} \) that satisfy conditions (62-65) of Proposition 6, and which provide \( m \)'s marginal values of all goods to all packages via relations (66).

Generically,\(^7\)

\[ p_j \notin Q, \quad \forall p \in P_+, \forall j \in J. \quad (67) \]

Intuitively, given discreteness, each consumer has only finitely many marginal values and the probability that the sets of marginal values for any pair of consumers have nonempty intersection is zero.

If the econometrician made an incorrect guess about consumer \( m \)'s satiation points, then either there exists a good \( j \) whose satiation point she set too high so that \( N_j^m < \tilde{N}_j^m \), or there exists a good \( j \) whose satiation point she set too low so that \( \tilde{N}_j^m < N_j^m \). If \( N_j^m < \tilde{N}_j^m \), then using arguments similar to those in the proof of Proposition 6, one could show that if the price vectors in \( P_+ \) and \( P_- \) satisfy (62-65) and (67), it must be that for any bundle \( z \in \tilde{\mathbb{B}}_m \) with \( z_j = N_j^m \), \( p(z,+)j = 0 \). This, in turn, would imply–using arguments similar to those in Proposition 6–that \( v_m(z + e^j) - v_m(z) = 0 \), implying that \( m \)'s satiation point for good \( j \) must be less than or equal to \( N_j^m \), contradicting the econometrician’s assumption that \( \tilde{N}_j^m \) was \( i \)'s satiation point for good \( j \). If instead \( \tilde{N}_j^m < N_j^m \), then one could show that it would be impossible to find price vectors satisfying (62-65) and (67) corresponding to \( i \)'s valuation. In particular, any collections \( P_+ \) and \( P_- \) satisfying all conditions in (62) except \( p(z,+)j = 0, \forall j \in J_+(z) \) and satisfying (63-65) and (67) could not also satisfy \( p(z,+)j = 0, \forall j \in J_-(z) \). So if the econometrician makes a mistaken guess the data will be able to reveal this, and hence the satiation points will be identified.

I.4 Simultaneous Identification of the number of consumers and satiation points

Section I.2 discussed identification of the number of consumers assuming that satiation points are known, whereas Section I.3 discussed identification of satiation points assuming that the number of consumers is known. While we have not verified this, we believe that it would be straightforward to combine the augments and simultaneously identify the number of consumers and their satiation points.

I.5 Consumer Types of Different Masses

Throughout, we have assumed a finite number of consumers. Alternatively, we could have assumed a finite number of types of consumers. A type may be thought of as a continuum of consumers with a certain mass such that all consumers of a given type have the same valuation.

\(^7\)Analogously to the case of an unknown number of consumers, there are some subtle issues associated with genericity when satiation points are unknown. We also deal with the latter case in a manner analogous to the former, for which, see footnote 4.
Different types would typically have different masses. Our results can be re-interpreted in terms of finite number of types of consumers rather than in terms of a finite number of consumers under the assumption that all types of consumers have the same mass. Our assumption then corresponds to the most difficult case for obtaining identification because we cannot use information about the different masses of different types to help us separate types. Assuming that different types have different masses would make the identification exercise much easier because this would allow us to use the mass of indifferent consumers to help us determine whether different prices induce the same type of consumers to become indifferent.

II Sufficiency of Observing Single-Valued Demand Only

This section elaborates on the argument sketched in Section 4.2 that while the argument in the text appeals to observing aggregate demand where it becomes multi-valued, if we observed demand everywhere where it was single valued, we could still obtain identification. As can be verified from an examination of Propositions 2 and 6 and the surrounding discussion, in Sections 4.5 and A, our identification arguments (both Theorems 1 and 2) will require us to observe multi-valued aggregate demand of only two forms

\[ D(p) = y + \{ e^j : j \in K \}, \quad \text{where } p_j > 0 \text{ for all } j \in K, \tag{68} \]

\[ D(p) = y - \{ e^j : j \in K \}, \quad \text{where } p_j > 0 \text{ for all } j \in K, \tag{69} \]

and where \( K \subseteq J \). Observe that aggregate demand takes the form (68) if the individual demand of some consumer \( i \) is multi-valued, taking the form (12), and the individual demands of all other consumers are single-valued (and the prices of goods in \( K \) are positive), provided that we choose \( K = J \setminus J_*(z') \), so that \( V_0(z') = \{ e^j : j \in K \} \) (see (3)).

For any price vector \( p \), define \( B_+(p, \epsilon) \) to be the intersection of the \( \epsilon \)-ball around \( p \) with the non-negative orthant \( \mathbb{R}_+^n \). Then we have:

**Proposition 7. (Sufficiency of observing demand where single-valued)**

Choose \( y \in \mathbb{Z}_+^n \), nonempty \( K \subseteq J \) and \( p \in \mathbb{R}_+^n \) such that \( p_j > 0 \) for all \( j \in K \).

1. Aggregate demand takes the form (68) if and only if there exists \( \epsilon > 0 \) such that for all \( j \in K \) and \( p' \in B_+(p, \epsilon) \):
   
   (a) if \( p'_j - p_j < 0 \) and \( j = \arg \min \{ p'_j - p_j' : j' \in K \} \), then \( D(p) = y + e^j \), and
   
   (b) if \( 0 < \min \{ p'_j - p_j' : j' \in K \} \), then \( D(p) = y \).

2. Similarly, aggregate demand takes the form (69) if and only if there exists \( \epsilon > 0 \) such that for all \( j \in K \) and \( p' \in B_+(p, \epsilon) \):
   
   (a) if \( p'_j - p_j > 0 \) and \( j = \arg \max \{ p'_j - p_j' : j' \in K \} \), then \( D(p) = y - e^j \), and
(b) If \( \max \left\{ p'_{j'} - p_j' : j' \in K \right\} < 0 \), then \( D(p) = y \).

Suppose that demand takes the form (68). This means that aggregate demand is multi-valued at \( p \), containing \( y \) as well as \( y + e^j \) for every good \( j \) in \( K \). Part 1 of the proposition says that for small perturbations in the price vector \( p \) such that the price of good \( j \) in \( K \) falls, and moreover, falls more than the price of any other good in \( K \), aggregate demand becomes single-valued and is equal to the bundle \( y + e^j \). For small perturbations in price such that the prices of all goods in \( K \) rise, aggregate demand becomes single valued and is equal to \( y \). Moreover, these conditions about demand for small perturbations around \( p \) imply that demand takes the form (68). Part 2 of the proposition is similar. So, when aggregate demand has the form (68) or (69), we do not literally have to observe demand at the price vector at which it becomes multi-valued, and statements about multi-valued demand can be taken as shorthand for statements about the pattern of substitution as prices vary around \( p \). Since our identification argument only appeals to multi-valued demand of the form (68) or (69), we do not have to worry about multi-valued demand taking any other form.

Notice finally that the proof of the proposition does not depend on Assumptions 1 or 2, and in particular, does not assume 1UPG or submodularity. It relies only on the definitions of demand and aggregate demand (7-8). Therefore, it applies not only to our first identification result, Theorem 1, but also to our second identification result, Theorem 2, which allows for multi-unit demand.

Proof of Proposition. We prove only Part 1 of the proposition as the proof of Part 2 is similar. So assume that demand takes the form (68). An argument essentially identical to the proof of Lemma 4 establishes that there is exactly one consumer \( i \) with multi-valued demand at price vector \( p \), and all other consumers have single valued demand. Therefore for some \( y' \in \mathbb{Z}_+^n \) and \( z \in \mathbb{B} \), we can write:

\[
D(p) = y' + z + \{ e^j : j \in K \},
\]

where:

\[
\sum_{i' \in I \setminus \{ i \}} D^{i'}(p) = y',
\]

\[
D^i(p) = z + \{ e^j : j \in K \}, \quad \text{and} \quad y' + z = y. \tag{70}
\]

Since there are only finitely many alternatives, and \( i \) strictly prefers every bundle in \( D^i(p) \) to every bundle in \( \mathbb{B} \) but outside of \( D^i(p) \), for sufficiently small \( \epsilon > 0 \), it must be that:

\[
D^i(p') \subseteq D^i(p), \quad \forall p' \in B_+(p, \epsilon), \tag{71}
\]
So suppose that \( p' \in B_+(p, \epsilon) \), and observe that:

\[
\begin{align*}
v_i(z|p') - v_i(z|p) &= \sum_{j=1}^{n} (p_j - p'_j) z_j, \\
v_i(z + e''|p') - v_i(z + e''|p) &= \sum_{j=1}^{n} (p_j - p'_j) z_j + (p_{j''} - p'_{j''}), \quad \forall j'' \in K.
\end{align*}
\]  

(72)

If, as in Part 1a of the proposition, \( j \in K, p'_j - p_j < 0 \) and \( j = \arg \min \left\{ p'_{j'} - p_j' : j' \in K \right\} \), (72) implies that \( i \)'s utility to purchasing package \( z + e^j \) increases, and moreover, increases more than any other package in \( z + \{ e^j : j \in K \} \) when prices change from \( p \) to \( p' \). It follows from (71) that \( D^i(p') = z + e^j \). Since demand was single valued for all consumers \( i' \) other than \( i \) at \( p \), each such consumer \( i' \) strictly preferred his demanded bundle to any other bundle in the finite set \( \mathbb{B} \) at price vector \( p \), so for sufficiently small \( \epsilon \), if \( p' \in B_+(p, \epsilon) \) the demand of consumers \( i' \) other than \( i \) will remain unchanged as prices move from \( p \) to \( p' \). It now follows from (70) that \( D(p') = y + e^j \), establishing Part 1a of the proposition.

If, on the other hand, \( 0 < \min \left\{ p'_{j'} - p_j' : j' \in K \right\} \), then (72) implies that the utility of purchasing each bundle in \( z + \{ e^j : j \in K \} \) goes down when prices go from \( p \) to \( p' \), but the utility of bundle \( z \) is reduced least. So, using an argument similar to the above, for sufficiently small \( \epsilon \), \( D^i(p') = z \) and \( D(p') = y \), establishing Part 1b of the proposition. \( \Box \)

### III Proof of Part 3 of Proposition 4

Here we provide a proof of part 3 of Proposition 4. Murota & Shioura (1999) present the following axiom:

\((M^2-EXC_p)\) For all \( z, z' \in \mathbb{B} \):

\( (i) \) \( \sum_{j=1}^{n} z_j < \sum_{j=1}^{n} z'_j \) \( \Rightarrow \)

\[
v_i(z) + v_i(z') \leq \max_{j' : z'_j > z_j} \left\{ v_i \left( z + e^{j'} \right) + v_i \left( z' + e^{j'} \right) \right\}.
\]

\( (ii) \) \( \sum_{j=1}^{n} z_j \leq \sum_{j=1}^{n} z'_j \) \( \Rightarrow \forall j \in J \) with \( z_j > z'_j \):

\[
v_i(z) + v_i(z') \leq \max_{j' : z'_j > z_j} \left\{ v_i \left( z - e^j + e^{j'} \right) + v_i \left( z' + e^j - e^{j'} \right) \right\}.
\]

\( Murota & Shioura (1999) \) actually present this axiom for \( M^2 \)-convex functions rather than \( M^2 \)-concave functions, but because \( f \) is \( M^2 \)-concave exactly if \( -f \) is \( M^2 \)-convex, the corresponding axiom for \( M^2 \)-concave functions, which we present here, is derived simply by reversing the appropriate inequalities in the axiom of Murota & Shioura (1999).
(iii) \( \sum_{j=1}^{n} z_j > \sum_{j=1}^{n} z'_j \Rightarrow \forall j \in J \text{ with } z_j > z'_j: \)

\[
v_i(z) + v_i(z') \leq \max \left[ v_i(z - e^j) + v_i(z' + e^j), \max_{j': z'_j = z'_j} \left\{ v_i \left( z - e^j + e^j' \right) + v_i \left( z' - e^j + e^j' \right) \right\} \right].
\]

Theorem 4.2 of Murota & Shioura (1999) establishes that \( (M^2\text{-EXC}_p) \) is equivalent to the axiom for \( M^2 \)-concavity which they call \( (M^2\text{-EXC}) \) and which we give in Assumption 3. In other words, Murota & Shioura (1999) prove that \( (M^2\text{-EXC}_p) \) characterizes \( M^2 \)-concavity. \(^9\)

Choose \( j, \ell \in J \) with \( j \neq \ell \), and \( z \in \mathbb{B}_{2\ell} \cap \mathbb{B}_j \) and define \( y := z + e^j \) and \( y' := z + 2e^\ell \). Then \( \sum_{k=1}^{n} y_k \leq \sum_{k=1}^{n} y'_k \) and \( y_j > y'_j \). Moreover, \( \ell \) is the unique element \( j' \) of \( J \) such that \( y'_{j'} > y_{j'} \).

It follows from part (ii) of \( (M^2\text{-EXC}_p) \) that:

\[
v_i(z + e^j) + v_i(z + 2e^\ell) = v_i(y) + v_i(y') \leq v_i(y - e^j + e^\ell) + v_i(y' + e^j - e^\ell) \]

\[
= v_i(z + e^\ell) + v_i(z + e^j + e^\ell). \quad (73)
\]

As (73) is equivalent to part 3 of Proposition 4, this completes the proof. \( \square \)

IV The Set of \( M^2 \)-Concave Valuations is Not Convex.

This section provides an example that shows that the set of \( M^2 \)-concave valuations is not convex. Suppose that there are three goods. As usual, we assume that \( N_j \geq 1 \) for \( j = 1, 2, 3 \). Letting \( z = (z_1, z_2, z_3) \) be an arbitrary bundle, we define three valuations \( v'_i, v''_i, \) and \( v_i \) by:

\[
v'_i(z) := \sqrt{z_1 + z_2}, \quad v''_i(z) := \sqrt{z_2 + z_3}, \quad \text{and} \quad v_i(z) := \frac{1}{2} v'_i(z) + \frac{1}{2} v''_i(z).
\]

\(^9\)Again, the formulation of Murota & Shioura (1999) is in terms of \( M^2 \)-convexity, but applies, through a simple translation, to \( M^2 \)-concavity. See footnote 8.
One can show that \( v'_i \) and \( v''_i \) are both \( M^2 \)-concave.\(^{10}\) On the other hand, letting \( z = (z_1, z_2, z_3) = (0, 1, 0) \) and \( z' = (z'_1, z'_2, z'_3) = (1, 0, 1) \):

\[
v_i(z) + v_i(z') = \left( \frac{1}{2} \sqrt{z_1 + z_2 + \frac{1}{2} \sqrt{z_2 + z_3}} \right) + \left( \frac{1}{2} \sqrt{z'_1 + z'_2 + \frac{1}{2} \sqrt{z'_2 + z'_3}} \right)
\]

\[
> \left( \frac{1}{2} \sqrt{1 + \frac{1}{2} \sqrt{2}} \right) + \left( \frac{1}{2} \sqrt{1 + \frac{1}{2} \sqrt{2}} \right)
\]

\[
= \max \left[ \left( \frac{1}{2} \sqrt{0 + \frac{1}{2} \sqrt{0}} \right) + \left( \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{2}} \right), \max_{j: z'_j > z_j} \{ v_i \left( z - e^2 \right) + v_i \left( z' + e^2 \right) \} \right]
\]

It follows that \( v_i \) is not \( M^2 \)-concave.\(^{11}\) As \( v_i \) is a convex combination of \( v'_i \) and \( v''_i \), it follows that the set of \( M^2 \)-concave valuations is not convex. As \( v'_i \) and \( v''_i \) also satisfy the assumptions of normalization and monotonicity within Assumption 3, it follows that the set \( \mathcal{V} \) is also not convex.

V Failure of Identification for Multi-Unit Demand when \( n = 1 \)

Our identification result for multi-unit demand (Theorem 1) assumes that there are at least two goods \( (n \geq 2) \). In this section, we show that with only one good \( (n = 1) \), when consumers may demand many units of the good, valuations are not generically identified from demand. Here we use the framework of multi-unit demand from Section 5. It is sufficient to restrict attention to the case where \( N_1 = 2 \) so that each consumer wants at most two units of the good.

In this case we can write \( \mathcal{E} = \mathbb{R}^2 \), and we can represent the set of valuations satisfying Assumptions 1 and 3 as \( \mathcal{V} = \{ (r_1, r_2) \in \mathbb{R}^2_+ : r_1 \leq r_2 \leq 2r_1 \} \), where \( r_1 \) is the value that the consumer assigns to a package containing one unit of the good and \( r_2 \) is the value that the consumer assigns to the package containing two units of the good. The inequality \( r_1 \leq r_2 \) is implied by monotonicity and \( r_2 \leq 2r_1 \) is implied by \( M^2 \)-concavity. Unlike the general case, when \( m = 1 \), \( \mathcal{V} \) is both convex and a full dimensional subset of the \( \mathcal{E} \). Therefore the notion of component-wise genericity coincides with the notion of genericity from Section 2.2.2. Suppose there are only two consumers, \( a \) and \( b \), with valuations \( v_a \) and \( v_b \). Since there is only one good, for \( i = a, b \) and \( j = 1, 2 \), we can write \( i \)'s value to consuming \( j \) units of the good as \( v_i(j) \). Let \( c, c', d, d' \) be four distinct positive real numbers, such that both \( c \) and \( c' \) are larger than both

\(^{10}\) \(-v'_i\) and \(-v''_i\) are both special cases of the class of laminar convex functions which are shown to be \( M^3 \)-convex in Note 6.11 of Murota (2003). As the negative of an \( M^2 \)-convex function is \( M^2 \)-concave, it follows that \( v'_i \) and \( v''_i \) are \( M^3 \)-concave.

\(^{11}\) Murota points out that the function \( f(x_1, x_2, x_3) = (x_1 + x_2)^2 + (x_2 + x_3)^2 \) is not \( M^2 \)-convex (Murota 2003, p. 141), which is similar to the fact that \( v_i \) is not \( M^2 \)-concave.
d and \( d' \). Now consider the following two scenarios:

**Scenario 1** \( \nu_a(1) = c, \nu_a(2) = c + d, \) and \( \nu_b(1) = c', \nu_b(2) = c' + d' \).

**Scenario 2** \( \nu_a(1) = c, \nu_a(2) = c + d', \) and \( \nu_b(1) = c', \nu_b(2) = c' + d \).

In both scenarios, valuations belong to \( \tilde{\mathcal{V}} \), (or, equivalently, valuations satisfy Assumptions 1 and 3). Moreover, it is easy to see that both scenarios generate the same aggregate demand correspondence, and therefore, it is impossible to distinguish them on the basis of aggregate demand. There is nothing special or “non-generic” about this example. Indeed, when there are at least two consumers, only one good and consumers value at least two units, not only do valuations fail to be generically identified from demand, but more strongly, valuations are generically not identified from demand. The underlying issue is that our identification argument rests on observing demand at prices at which some consumer becomes indifferent between consuming additional units of different goods, a possibility which is absent with only one good.

**VI Proof of Theorem 2**

As explained in the Appendix of the main paper, to prove Theorem 2, it is sufficient to prove Proposition 6. So this section proves Proposition 6.

**VI.1 The components of \( \tilde{\mathcal{V}} \)**

On the way to proving Proposition 6, we now define the components \( \mathcal{V}_f \) of \( \tilde{\mathcal{V}} \) and \( \mathcal{V}^*_g \) of \( \tilde{\mathcal{V}}^n \), and prove Proposition 5. Define \( Z := \{ (z, z') \in \mathbb{B} \times \mathbb{B} : \sum_{j=1}^n z_j \geq \sum_{j=1}^n z'_j, z \neq z'\} \) as the set of pairs of distinct bundles \( z \) and \( z' \) such that \( z \) contains at least as many total units as \( z' \).

Next, let \( F \) be the set of functions \( f : Z \to J \times J \) satisfying:

\[
f(z, z') = (j, j) \Rightarrow \left( \sum_{k=1}^n z_k > \sum_{k=1}^n z'_k \text{ and } z_j > z'_j \right), \forall (z, z') \in Z, \forall j \in J.
\]

\[
f(z, z') = (j, j') \Rightarrow (z_j > z'_j \text{ and } z'_{j'} > z_j), \forall (z, z') \in Z, \forall j, j' \in J \text{ with } j \neq j'.
\]

The definition of \( Z \) implies for some \((j, j') \in J \times J \) (possibly with \( j = j' \)) \( f(z, z') = (j, j') \), is consistent with (74-75). For each \( f \in F \), let \( \mathcal{V}_f \) be the set of valuations \( v_i \) satisfying normalization and monotonicity (see Assumption 1), and:

\[
v_i(z) + v_i(z') \leq v_i(z - e^j) + v_i(z' + e^j), \forall (z, z') \in Z, \forall j \in J \text{ with } f(z, z') = (j, j).
\]

\[
v_i(z) + v_i(z') \leq v_i(z - e^j + e^{j'}) + v_i(z' + e^j - e^{j'}), \forall (z, z') \in Z, \forall j, j' \in J
\]

\[
\text{with } f(z, z') = (j, j') \text{ and } j \neq j'.
\]
As explained after Proposition 5, the $M^2$-concavity inequalities can be viewed as disjunctions of linear inequalities. The functions $f$ in $F$ can be viewed as ways of selecting inequalities from these disjunctions. So each function $f$ induces a convex polyhedron via the induced linear inequalities (76-77). With this in mind, the following lemma implies Proposition 5.

**Lemma 7.** $\tilde{\mathcal{V}} = \bigcup_{f \in F} \mathcal{V}_f$.

Proof. An $M^2$-convex function is a function $g$ such that $-g$ is $M^2$-concave. Murota & Shioura (1999) present an axiom, which they call (M) present an axiom, which they call ($\bar{M}$), and strong self-substitutability (see Proposition 4). Define:

$$v_i (z + e^j + e^\ell) - v_i (z + e^\ell) \neq v_i (z + e^j) - v_i(z), \forall z \in \mathbb{B}, \forall j, \ell \in J \setminus J_s(z) \text{ with } j \neq \ell. \quad (78)$$

$$v_i (z + e^j + e^\ell) - v_i (z + e^\ell) \neq v_i (z + 2e^j) - v_i (z + e^j), \quad \forall z \in \mathbb{B}, \forall j \in J_{s-2}(z), \forall \ell \in J \setminus J_s(z) \text{ with } j \neq \ell. \quad (79)$$

(78) and (79) rule out equality in the inequalities characterizing, respectively, submodularity and strong self-substitutability (see Proposition 4). Define: $Z_j := \{z \in \mathbb{B} : \exists k \in \mathbb{Z}_+, z = ke^j\}$ and $Z_{-j} := \mathbb{B} \setminus Z_j$. $Z_j$ is the set of packages containing units only of good $j$. Next consider:

$$v_i (z + e^j) - v_i(z) \neq v_i(z' + e^j) - v_i(z'), \quad \forall j \in J, \forall z, z' \in Z_{-j}, \quad \text{with } j \in J \setminus (J_s(z) \cup J_s(z')). \quad (80)$$

$$v_i (y + e^j) - v_i(y + e^j') \neq v_i(z + e^j) - v_i(z + e^j'), \forall j, j' \in J \text{ with } j \neq j', \forall z, y \in \mathbb{B} \text{ with } z_j \neq y_j \text{ and } j, j' \in J \setminus (J_s(z) \cup J_s(y)). \quad (81)$$

$$v_i (\bar{z} - e^j) \neq v_i(\bar{z}), \quad \forall j \in J. \quad (82)$$

(80) says the marginal value of good $j$ to a package containing only units of $j$ differs from the marginal value of $j$ to any package containing some units of other goods. (81) says that the
difference in marginal values of goods \( j \) and \( j' \) is different for packages that contain different numbers of units of good \( j \). (82) says that removing a unit of any good from the satiating bundle (see (1)) alters the consumer’s valuation; it is the multi-unit analog of (35). It is required because we have bounded the number of units of each good a consumer may desire to simplify the required notion of genericity. Unlike the other genericity assumptions, no analog of (82) is required for a related identification argument when demand is unbounded. See Section I.1. Now define \( \bar{W} := \{ v_i \in \bar{V} : v_i \text{ satisfies (78-82)} \} \).

**Proposition 8.** Any valuation \( v_i \) in \( \bar{W} \) satisfies strict monotonicity, strict submodularity, strict component-wise concavity, and strict strong self-substitutability.

Adding “strict” to a property replaces its required weak inequalities by strict inequalities. (See Proposition 4 for these properties; strict monotonicity is \( z < z' \Rightarrow v_i(z) < v_i(z'), \forall z, z' \in \mathbb{B} \).

Proof of Proposition 8: Strict submodularity and strict strong self-substitutability follow from submodularity and strong self-substitutability (Proposition 4) and (78-79). Strict component-wise concavity follows from strict submodularity and strong self-substitutability (using \( n \geq 2 \)). Finally, assume for contradiction that strict monotonicity fails. So for some \( z \in \mathbb{B}, j \in J \setminus J_0(z) \), \( v_i (z + e^j) - v_i(z) = 0. \) (82) implies: \( z \neq z - e^j \). So for some \( \ell \) (possibly equal to \( j \)), \( z + e^j + e^\ell \in \mathbb{B} \). The previous equality and monotonicity with either component-wise concavity (if \( \ell = j \)) or submodularity (if \( \ell \neq j \)) imply: \( v_i (z + e^j + e^\ell) - v_i(z + e^\ell) = 0. \) The two derived equalities contradict either strict component-wise concavity or strict submodularity. □

**VI.2.2 Conditions on relations among valuations**

This section presents conditions relating the valuations of different agents:

\[
v_i(x) - v_i(y) \neq \sum_{j \in J \setminus J_0(z)} [v_i(z + e^j) - v_i(z)] (x_j - y_j), \forall i, i' \in I, \text{ with } i \neq i',
\]

\[
\forall x, y, z \in \mathbb{B} \text{ with } x \neq y, \text{ and } \{ j : x_j \neq y_j \} \subseteq J \setminus J_0(z).
\]

\[
v_i(x) - v_i(y) \neq \sum_{j \in J \setminus J_0(z)} [v_i(z) - v_i(z - e^j)] (x_j - y_j), \forall i, i' \in I, \text{ with } i \neq i',
\]

\[
\forall x, y, z \in \mathbb{B} \text{ with } x \neq y, \text{ and } \{ j : x_j \neq y_j \} \subseteq J \setminus J_0(z).
\]

(83-84) are multi-unit analogs (36-37). We now define the set \( \bar{S} \) referred to in Proposition 6. The definition is \( \bar{S} := \{ v = (v_1, \ldots, v_m) \in \bar{W}^m : v \text{ satisfies (83-84)} \} \).

**Proposition 9.** Any \( v \in \bar{S} \) has the following two properties: (i) \( v_i(z + e^j) - v_i(z) \neq v_i'(z' + e^j) - v_i'(z') \), \( \forall i, i' \in I \text{ with } i \neq i', \forall z, z' \in \mathbb{B}, \forall j \in J \setminus J_0(z) \cup J_0(z') \). (ii) \( v_i(z + e^j) - v_i(z + e^j') \neq v_i'(z' + e^j) - v_i'(z' + e^j') \), \( \forall i, i' \in I \text{ with } i \neq i', \forall z, z' \in \mathbb{B}, \forall j, j' \in J \setminus J_0(z) \cup J_0(z') \text{ with } j \neq j' \).

Proof. (i) follows from setting \( x = z' + e^j \) and \( y = z' \) in (83). (ii) follows from setting \( x = z' + e^j \) and \( y = z' + e^j' \) in (83). □
VI.2.3 Proof that $\tilde{S}$ satisfies condition 2 of Proposition 6.

We now establish that $\tilde{S}$ is component-wise generic in $\mathcal{V}^m$. First, we construct a valuation $v^*_i$ belonging not only to $\mathcal{W}$ but to every component $\mathcal{V}_f$ of $\mathcal{V}$. Let $t \in (0,1)$ be a transcendental number. Define $N_0 := \sum_{j=1}^n N_j$. For $j = 0, \ldots, n$, define: $u_j(0) := 0$, and $u_j(h) := \sum_{k=1}^h t^{k(n+1)+j}, \forall h \in [N_j] \setminus 0$, where $[N_j] := \{0,1,\ldots,N_j\}$. These definitions imply a discrete analogue of strict concavity:

$$h < h' \Rightarrow u_j(h') - u_j(h) = u_j(h) < u_j(h+1) - u_j(h), \quad \forall h, h' \in [N_j], \forall j \in \{0,\ldots,n\}.$$  \hspace{1cm} (85)

Define the valuation $v^*_i$ by: $v^*_i(z) := u_0 \left( \sum_{j=1}^n z_j \right) + \sum_{j=1}^n u_j(z_j), \forall z \in \mathbb{B}$, which is a special case of the class of $M^2$-concave functions given by (59).

**Lemma 8.** $v^*_i \in \bigcap_{f \in F} \mathcal{V}_f$.

**Proof.** It is immediate that $v^*_i$ satisfies normalization and monotonicity. Choose $f \in F$ and $(z,z') \in \mathcal{Z}$ (where $\mathcal{Z}$ and $F$ are defined in Section VI.1). First, assume $f(z,z') = (j,j)$ for some $j \in J$. Then:

$$v^*_i(z) - v^*_i(z') = u_j \left( z'_j + 1 \right) - u_j \left( z'_j \right) - \left[ u_j(z_j) - u_j(z'_j - 1) + u_0(\sum_{k=1}^n z_k^j) \right] \geq 0.$$  \hspace{1cm} (86)

The derivation uses (85), the fact that by (74), $\sum_{k=1}^n z_k^j > \sum_{k=1}^n z_k$ and $z_j > z'_j$, and that packages $z$ and $z'$ are integer-valued. This establishes that $v^*_i$ satisfies (77). Next, assume $f(z,z') = (j,j')$ for $j,j' \in J$ with $j' \neq j$. Then by a justification similar to the above, appealing to (85), (75), and the discreteness of packages:

$$v^*_i(z) - v^*_i(z') = u_j \left( z'_j + 1 \right) - u_j \left( z'_j \right) - \left[ u_j(z_j) - u_j(z'_j - 1) + u_{j'}(z'_{j'} + 1) - u_{j'}(z'_{j'}) \right] \geq 0,$$  \hspace{1cm} (87)

establishing that $v^*_i$ satisfies (77). \hfill $\Box$

**Lemma 9.** $v^*_i \in \mathcal{W}$.

**Proof.** We must prove that $v^*_i$ satisfies (78)-(82). (82) is immediate. Choose $z \in \mathbb{B}$, and $j, k \in J \setminus J_s(z)$ with $j \neq k$. (85) implies:

$$v^*_i(z + e^j) - v^*_i(z) = \left[ u_0 \left( 1 + \sum_{j=1}^n z_j \right) - u_0 \left( \sum_{j=1}^n z_j \right) \right] > 0.$$  \hspace{1cm} (88)

Similarly, for $z \in \mathbb{B}$, $j \in J_s(z)$, and $k \in J \setminus J_s(z)$ with $j \neq k$, (85) implies:

$$v^*_i(z + e^j) - v^*_i(z + e^j) = \left[ u_0 \left( z_j + 1 \right) - u_0 \left( z_j \right) \right] > 0.$$  \hspace{1cm} (89)

Choose $z \in Z_j, z' \in Z_{-j}$ and $j \in J \setminus (J_s(z) \cup J_s(z'))$. Define $z'' := \sum_{k=1}^n z_k$. Because $z \in Z_j, z_j = \sum_{k=1}^n z_k$. The non-equality $v^*_i(z + e^j) - v^*_i(z) \neq v^*_i(z')$ is equivalent to:

$$\left[ u_j(z_j) - u_j(z'_j + 1) \right] \neq u_j(z_j') - u_j(z'_j) + u_0(z'' + 1) - u_0(z'),$$  \hspace{1cm} (90)

which is equivalent to:

$$t(z_j+1)(n+1)+t(z_j+1)(n+1) - (t(z'_j+1)(n+1)+t(z''+1)(n+1)) \neq 0.$$  \hspace{1cm} (91)

Therefore, $z \in Z_{-j}$ implies that $z_j < z''$. So if $z_j = z'_j$, then $z_j \neq z''$. This implies the left-hand side of the
Proposition 10. If $v_i : \mathbb{B} \to \mathbb{R}$ is $M^2$-concave, $v_i$ satisfies the single improvement property:

\[
\forall z, z' \in \mathbb{B}, v_i(z|p) < v_i(z'|p) \Rightarrow \text{either } (\exists j, z'_j > z_j \text{ and } v_i(z + e^j|p) > v_i(z|p)),
\]

or

\[
(\exists j, z'_j < z_j \text{ and } v_i(z - e^j|p) > v_i(z|p)),
\]

or

\[
(\exists j, \exists \ell, z'_j > z_j, z'_\ell < z_\ell \text{, and } v_i(z + e^j - e^\ell|p) > v_i(z|p)).
\]

VI.3 Proof that in Proposition 6, 3b implies 3a

We establish that given the set $\tilde{S}$ defined in Section VI.2.2, 3b of Proposition 6 implies 3a.

**Lemma 10.** Condition 3b of Proposition 6 implies (65).

The proof is identical to the proof of Lemma 1. Next we present a useful result:

**Proposition 10.** If $v_i : \mathbb{B} \to \mathbb{R}$ is $M^2$-concave, $v_i$ satisfies the single improvement property:
This result appears as part of Theorem 11.4 in Murota (2003).\footnote{See also Danilov, Koshevoy & Murota (2001). The single-improvement property was studied under the 1UPG assumption by Gul & Stacchetti (1999).}

**Lemma 11.** For all valuation profiles in $\hat{S}$, if 3b of Proposition 6 holds for consumer $i$:

$$D^i(p(z, +)) = z + V_0(z), \quad \forall z \in B,$$

$$D^i(p(z, -)) = z - V_1(z), \quad \forall z \in B.$$  \hspace{1cm} (86)

$$D^i(p(z, +)) = z + V_0(z), \quad \forall z \in B.$$  \hspace{1cm} (87)

Proof. We begin by establishing

$$v_i(z|p(z, +)) \geq v_i(y|p(z, +))$$  \hspace{1cm} (88)

for $y, z \in B$ in the following cases: 1. $y = z + e^j$ for $j \in J$, 2. $y = z - e^j$ for $j \in J$, 3. $y = z + e^j - e^\ell$ for $j, \ell \in J$ with $j \neq \ell$, 4. $y = z + 2e^j$ for $j \in J$, 5. $y = z + e^j + e^\ell$ for $j, \ell \in J$ with $j \neq \ell$. In case 1, (88) is immediate from condition 3b of Proposition 6. Indeed, (88) holds for $j \in J_s(z)$, then $p(z, +)_j = 0$ by (62), and (89) follows from strict monotonicity (see Proposition 8). So assume $j \in J \setminus J_s(z)$, and observe that: $v_i(z|p(z, +)) > v_i(z - e^j|p(z, +)) \iff v_i(z - p(z, +)_j) > v_i(z - e^j) \iff v_i(z - (v_i(z + e^j) - v_i(z))) > v_i(z - e^j) \iff v_i(z) > v_i(z - e^j) > v_i(z + e^j) - v_i(z).$ We used 3b of Proposition 6 to substitute for $p(z, +)_j$, then $\ell \in J_s(z)$, and observe that:

$$v_i(z|p(z, +)) > v_i(y|p(z, +)).$$  \hspace{1cm} (89)

$$v_i(z|p(z, +)) > v_i(y|p(z, +))$$  \hspace{1cm} (89)

For case 1, if $j \in J_s(z)$, then $p(z, +)_j = 0$ by (62), and (89) follows from strict monotonicity (see Proposition 8). So assume $j \in J \setminus J_s(z)$, and observe that:

$$v_i(z|p(z, +)) > v_i(y|p(z, +)) \iff v_i(z - p(z, +)_j) > v_i(z - e^j) \iff v_i(z - (v_i(z + e^j) - v_i(z))) > v_i(z - e^j) \iff v_i(z + e^j - v_i(z)) > v_i(z + e^j - v_i(z)) \iff v_i(z) > v_i(z + e^j - v_i(z)) > v_i(z + e^j - v_i(z)).$$

The last inequality holds by strict self-substitutability (see Proposition 8), establishing (89) in case 3. For case 4, observe that:

$$v_i(z|p(z, +)) > v_i(z + 2e^j|p(z, +)) \iff v_i(z) > v_i(z + 2e^j - 2p(z, +)_j) \iff v_i(z) > v_i(z + 2e^j - 2p(z, +)_j) \iff v_i(z) > v_i(z + e^j - v_i(z)) \iff v_i(z + e^j - v_i(z) > v_i(z + 2e^j - v_i(z)).$$

The last inequality holds by strict component-wise concavity (see Proposition 8), establishing (89) in case 4. For case 5, observe that:

$$v_i(z|p(z, +)) > v_i(z + e^j + e^\ell|p(z, +)) \iff v_i(z) > v_i(z + e^j + e^\ell - p(z, +)_j - p(z, +)_\ell) \iff v_i(z) > v_i(z + e^j + e^\ell - (v_i(z + e^j) - v_i(z)) - (v_i(z + e^\ell) - v_i(z)) \iff v_i(z + e^j + e^\ell - v_i(z)) > v_i(z + e^j + e^\ell) - v_i(z + e^\ell).$$

The last inequality holds by strict submodularity (see Proposition 8), establishing (89) in case 5.

The single improvement property (Proposition 10) implies that the fact that (88) holds for cases 1-3 implies: $z \in D^i(p(z, +))$. Since (88) holds with equality in case 1, (by condition 3b
of Proposition 6), the preceding condition can be strengthened to:

\[ z + V_0(z) \subseteq D^i(p(z, +)). \]  
(90)

To establish (86), it is now sufficient to establish:

\[ z + V_0(z) \supseteq D^i(p(z, +)). \]  
(91)

The fact that (89) holds in cases 2-5 implies that:

\[ z - e^j, z + e^j - e^\ell, z + 2e^j, z + e^j + e^\ell \notin D^i(p(z, +)), \quad \forall j, \ell \in J \text{ with } j \neq \ell. \]  
(92)

Theorem 11.7 of Murota (2003) implies \( D^i(p(z, +)) \) is an \( M^2 \)-convex set, meaning that: \(^{13}\)

\[ \forall y, y' \in D^i(p(z, +)), \forall j \in J, y_j > y'_j \Rightarrow \text{either } (y - e^j \in D^i(p(z, +)) \text{ and } y' + e^j \in D^i(p(z, +))) \]

or \( (\exists \ell \text{ with } y_\ell < y'_\ell \text{ and } y - e^j + e^\ell \in D^i(p(z, +)) \text{ and } y' + e^j - e^\ell \in D^i(p(z, +))). \)

We now establish (91). Choose \( y \in \mathbb{B} \) such that \( y \notin z + V_0(z). \) If \( z < y, \) then since \( y \notin z + V_0(z), \) it follows that \( z + e^j < y \) for some \( j \in J. \) By (90), \( z + e^j \in D^i(p(z, +)), \) so \( M^2 \)-convexity of set \( D^i(p(z, +)) \) implies that either \( z + 2e^j \in D^i(p(z, +)) \) or \( z + e^j + e^\ell \in D^i(p(z, +)) \) for some \( \ell \in J \setminus j, \) contradicting (92). If \( z \not< y, \) then \( M^2 \)-convexity of set \( D^i(p(z, +)) \) and the fact that \( z \in D^i(p(z, +)) \) imply that either \( z - e^j \in D^i(p(z, +)) \) for some \( j \in J, \) or \( z - e^j + e^\ell \in D^i(p(z, +)) \) for some \( j, \ell \in J \) with \( j \neq \ell, \) again contradicting (92). It follows that \( y \notin D^i(p(z, +)), \) establishing (91), and hence by (90), we have now established (86). The argument for (87) is similar. \( \square \)

**Lemma 12.** For all valuation profiles in \( \tilde{S}, \) whenever (66) in condition 3b of Proposition 6 holds for any consumer \( i, \) relating \( i \)'s marginal values to price vectors \( P_+ \) and \( P_- \), all other consumers will have single-valued demand at price vectors \( P_+ \) and \( P_- \).

The proof appeals to (83-84), and is similar to the proof of Lemma 3.

Lemmas 10-12 imply that within \( \tilde{S}, \) condition 3b of Proposition 6 implies 3a. In condition 3a, \( y(z, +) \) (resp., \( y(z, -) \)) is the sum of \( z \) and the demands of agents other than \( i \) at \( p(z, +) \) (resp., \( p(z, -) \)), where these demands are guaranteed to be single-valued by Lemma 12.

**VI.4 Proof that in Proposition 6, 3a implies 3b**

We establish that given \( \tilde{S} \) defined in Section VI.2.2, 3a of Proposition 6 implies 3b.

**Lemma 13.** If \( z \in \mathbb{B} \setminus \bar{z}, \) and demand satisfies (63) for \( z, \) then there is exactly one consumer with multi-valued demand at price vector \( p(z, +). \) Similarly, if \( z \in \mathbb{B} \setminus 0 \) and demand satisfies (64) for \( z, \) then there is exactly one consumer with multi-valued demand at price vector \( p(z, -). \)

The proof is identical to that of Lemma 4. (Recall that \( \bar{z} \) is the satiating bundle (see (1)).

---

\(^{13}\)See Axiom \((B^2\text{-EXC}[Z])\) on p. 117 of Murota (2003).
Lemma 14. If \( z \in \mathbb{B} \setminus \bar{z} \) and demand satisfies (63) for \( z \), then there exists a consumer \( i = i(z, +) \) and a package \( w(z, +) \in \mathbb{B} \) with \( J_s(w(z, +)) = J_s(z) \) satisfying:

\[
p(z, +)_j = v_i \left( w(z, +) + e^j \right) - v_i \left( w(z, +) \right), \quad \forall j \in J \setminus J_s(z). \tag{93}
\]

Similarly, if \( z \in \mathbb{B} \setminus 0 \) and demand satisfies (64) for \( z \), then there exists a consumer \( i' = i(z, -) \) and a package \( w(z, -) \in \mathbb{B} \) with \( J_0(w(z, -)) = J_0(z) \) satisfying:

\[
p(z, -)_j = v_{i'} \left( w(z, -) \right) - v_{i'} \left( w(z, -) - e^j \right), \quad \forall j \in J \setminus J_0(z). \tag{94}
\]

Proof. (63) and Lemma 13 imply existence of consumer \( i \) and bundle \( w(z, +) \) with \( J_s(w(z, +)) \subseteq J_s(z) \) such that \( D^i(p(z, +)) = w(z, +) + V_0(z) \). By (62), if there exists \( j \in J_s(z) \setminus J_s(w(z, +)) \), then \( p(z, +)_j = 0 \). So \( w(z, +) + e^j \in \mathbb{B} \) and \( v_i(w(z, +) + e^j | p(z, +)) \geq v_i(w(z, +) | p(z, +)) \) so that \( w(z, +) + e^j \in D^i(p(z, +)) \) contradicting \( D^i(p(z, +)) = w(z, +) + V_0(z) \). So \( J_s(w(z, +)) = J_s(z) \). \( D^i(p(z, +)) = w(z, +) + V_0(z) \) now implies the first part of the lemma. Similar arguments hold for \( w(z, -) \). \( \square \)

Lemma 15. For all valuation profiles in \( \bar{S} \), whenever condition 3a of Proposition 6 holds, there exists a unique consumer \( i^* \) such that for all \( z \in \mathbb{B} \setminus \bar{z} \),

\[
p(z, +)_j = v_{i^*} \left( z + e^j \right) - v_{i^*} \left( z \right), \quad \forall j \in J \setminus J_s(z), \tag{95}
\]

and for all \( z \in \mathbb{B} \setminus 0 \),

\[
p(z, -)_j = v_{i^*} \left( z \right) - v_{i^*} \left( z - e^j \right), \quad \forall j \in J \setminus J_0(z). \tag{96}
\]

Lemma 15 implies that within \( \bar{S} \), condition 3a of Proposition 6 implies condition 3b.

Proof of Lemma 15. For each \( k \in \mathbb{Z}_+ \), define \( Z_0(k) := \left\{ z \in \mathbb{B} : \sum_{j=1}^n z_j = k \right\} \), \( Z_1(k) := \left\{ ke^j : j \in J, k \leq N_j \right\} \), and \( Z_2(k) := Z_0(k) \setminus Z_1(k) \). Letting \( \bar{N} := \sum_{j=1}^n N_j \), these definitions imply that:

\[
\bigcup_{k=0}^{\bar{N}-1} Z_0(k) = \mathbb{B} \setminus \bar{z} \quad \text{and} \quad \bigcup_{k=1}^{\bar{N}} Z_0(k) = \mathbb{B} \setminus 0. \tag{97}
\]

Next, observe that if we find \( i^* \) satisfying (95-96), then (i) of Proposition 9 implies that such an \( i^* \) must be unique. Lemma 14 implies that there exists a consumer \( i(0, +) \) satisfying (93) for \( z = 0 \). Define \( i^* := i(0, +) \). We will argue that within \( \bar{S} \), \( i^* \) satisfies (95-96).

We establish Lemma 15 inductively, where induction is on the number \( k = \sum_{j=1}^n z_j \) of units (across all goods) in package \( z \). The argument is split into a base case and an inductive step.

**Base case:** \( b1 \) \( i^* \) satisfies (95) for all \( z \in Z_0(0) \). \( b2 \) \( i^* \) satisfies (96) for all \( z \in Z_0(1) \).

**Inductive step:** \( i1 \) If \( i^* \) satisfies (95) for all \( z \in Z_0(k) \), then \( i^* \) satisfies (96) for all \( z \in Z_0(k) \).
If we establish the base case and the inductive step, then the lemma will follow. First we establish the base case. Let $i = i(e^j, -)$ where $j \in J$. (93) for $i^* = i(0, +)$, (62), and (94) imply: 

$v_i^* (w(0, +) + e^j) = p(0, +)_j = p(e^j, -)_j = v_i (w(e^j, -)) - v_i (w(e^j, -) - e^j)_j.

(i) of Proposition 9 implies that $i = i^*$. So the preceding becomes:

$$v_i^* (w(0, +) + e^j) - v_i^* (w(0, +)) = p(0, +)_j = p(e^j, -)_j = v_i^* (w(e^j, -)) - v_i^* (w(e^j, -) - e^j).

By Lemma 14, $J_0(w(e^j, -)) = J_0(e^j) = J \setminus j$ so that $w(e^j, -) = (h^j + 1)e^j$ for some nonnegative integer $h^j$. (98) and (80) imply that for some nonnegative integer $h$, $w(0, +) = he^j$. Strict component-wise concavity (see Proposition 8) and (98) then imply that $h = h^j$. Since there are at least two goods, it is possible to choose $j' \in J \setminus j$. Since the choice of $j$ in the above argument was arbitrary, an identical argument shows that for some nonnegative integer $h^{j''}$, $w(0, +) = h^{j''}e^{j''}$ so that $h^{j''}e^{j''} = w(0, +) = h e^j$, implying that $h = h^{j''} = 0$. It follows that $w(0, +) = 0$ and $w(e^j, -) = e^j$, so that (98) now implies $b1$ and $b2$.

So now let us prove $i1$, $i2$, and $i3$, starting with $i1$. Suppose that (95) holds for all $z$ with $z \in Z_0(k)$, and consider $z' \in Z_0(k + 2)$. Then by the definition of $Z_0(\cdot)$, there exist $j', j''$ with $j' \neq j''$, $z'_{j'} > 0$ and $z'_{j''} > 0$. Moreover, for all $j', j''$ with $j' \neq j''$, $z'_{j'} > 0$ and $z'_{j''} > 0$, there exists $z'' \in Z_0(k)$ with $z' = z'' + e^{j'} + e^{j''}$. Since (95) holds for all $z \in Z_0(k)$, we have: 

$$p(z^{j', j''}, +)_{j'} = v_i^* (z^{j', j''} + e^{j'}) - v_i^* (z^{j', j''}), \quad \text{and} \quad p(z^{j', j''}, +)_{j''} = v_i^* (z^{j', j''} + e^{j''}) - v_i^* (z^{j', j''}).$$

Subtracting the first equation from the second, this implies:

$$p(z^{j', j''}, +)_{j'} - p(z^{j', j''}, +)_{j''} = v_i^* (z^{j', j''} + e^{j'}) - v_i^* (z^{j', j''} + e^{j''}).

For $i = i(z', -)$, Lemma 14 implies: 

$$p(z', -)_{j'} = v_i (w(z', -)) - v_i (w(z', -) - e^{j'})\quad \text{and} \quad p(z', -)_{j''} = v_i (w(z', -) - e^{j''}).$$

Subtracting the first equation from the second:

$$p(z', -)_{j'} - p(z', -)_{j''} = v_i (w(z', -) - e^{j'}) - v_i (w(z', -) - e^{j''}).

By (65): 

$$p(z^{j', j''}, +)_{j'} + p(z^{j', j''} + e^{j'}, +)_{j'} = p(z^{j', j''}, +)_{j'} + p(z^{j', j''} + e^{j'}, +)_{j''}.\quad \text{So},

\[14\] By $b2$, $i^*$ satisfies (96) for $z \in Z_0(1)$, and by $b1$, $i^*$ satisfies (95) for $z \in Z_0(0)$. Applying $i1$, $i^*$ satisfies (96) for $z \in Z_2(2)$; applying $i3$, $i^*$ satisfies (95) for $z \in Z_0(1)$; applying $i2$, $i^*$ satisfies (96) for $z \in Z_1(2)$; applying $i1$, $i^*$ satisfies (96) for $z \in Z_2(3)$; applying $i3$, $i^*$ satisfies (95) for $z \in Z_0(2)$; applying $i2$, $i^*$ satisfies (96) for $z \in Z_1(3)$ and so on, continuing until we have established (96) for $z \in Z_2(N)$, then (95) for $z \in Z_0(N - 1)$ and finally, (96) for $z \in Z_1(N)$. In this way, we prove that $i^*$ satisfies (95) for all $z \in Z_0(k)$ whenever $k = 0, \ldots, N - 1$, and that $i^*$ satisfies (96) for all $z \in Z_1(k) \cup Z_2(k) = Z_0(k)$ whenever $k = 1, \ldots, N$. By (97), we are done.

\[15\] Note that by the way that $j'$ and $j''$ were chosen, $j'$ and $j''$ both belong to $J \setminus J_0(z')$.\[18\]
For \( v \) and \( y \) together with (101), contradicts (81). So \( w \) and Lemma 14 that there exists \( z' \) such that
\[
\begin{align*}
&v_i \left( z' \bar{j} + e_j \right) - v_i \left( z' \bar{j} + e_j \right) = v_i \left( w (z', -) - e_j \right) - v_i \left( w (z', -) - e_j \right) .
\end{align*}
\]
Defining \( y' \) := \( w (z', -) - e_j - e' \), it follows that: \( v_i \left( z' \bar{j} + e_j \right) - v_i \left( z' \bar{j} + e_j \right) = v_i \left( y' \bar{j} + e_j \right) - v_i \left( y' \bar{j} + e_j \right) . \) (ii) of Proposition 9 now implies that \( i = i^* \), so that:
\[
\begin{align*}
v_i \left( z' \bar{j} + e_j \right) - v_i \left( z' \bar{j} + e_j \right) = v_i \left( y' \bar{j} + e_j \right) - v_i \left( y' \bar{j} + e_j \right) .
\end{align*}
\]

Next assume for contradiction that \( w (z', -) \neq z' \). Then there exists \( \ell \in J \) such that
\[
w (z', -)_{\ell} \neq z' \ell.
\]
Because Lemma 14 implies that \( w (z', -)_{\ell} = 0 \) exactly if \( z' \ell = 0 \), it follows that \( z' \ell > 0 \). Let \( j' = \ell \) and choose \( j'' \neq j' \) such that \( z'_{j''} > 0 \). Such a \( j'' \) exists because \( z' \in Z_2(k + 2) \). The above derivation of (101) remains valid if \( j' \) and \( j'' \) are chosen as just described. \( w (z', -)_{j'} \neq z'_{j'} \) and the definitions of \( z' \bar{j'} \) and \( y' \bar{j'} \) imply \( y' \bar{j'} \neq z'_{j''} \), which together with (101), contradicts (81). So \( y' \bar{j'} = z' \bar{j''} \), implying via the definitions of \( z' \bar{j''} \) and \( y' \bar{j''} \) that \( z' = w (z', -) \). This consequence, (94), and the fact that \( i (z', -) = i^* \), implies i1.

Next we prove i2. So suppose that (95) holds for all \( z \in Z_0(k) \). Consider \( z \in Z_1(k + 1) \). Then \( z = (k + 1)e_j \in Z_1(k + 1) \) for some \( j \in J \) and \( ke_j \in Z_0(k) \). (62) and the inductive hypothesis that (95) holds for all \( z \in Z_0(k) \) imply: \( p((k + 1)e_j, -)_j = p(ke_j, +)_j - v_i(ke_j) \), establishing (96) for \( z = (k + 1)e_j \).

Finally, we prove i3. As the inductive hypothesis, suppose that (96) holds for all \( z \in Z_2(k + 1) \). Choose \( z' \in Z_0(k) \) and \( j' \in J \setminus J_*(z) \). There are two possibilities: (a) \( z' + e_j \in Z_2(k + 1) \), and (b) \( z' + e_j \in Z_1(k + 1) \). First assume (a). (62) and the inductive hypothesis that (96) holds for \( z \in Z_2(k + 1) \) imply that: \( p(z', +)_j' = p(z + e_j, -)_j' = v_i \left( z' + e_j \right) - v_i (z') \), establishing (95) in this case. Next assume (b). Then \( z' = ke_j \). Moreover, because we have already established (95) for \( z \in Z_0(0) \), we may assume that \( k \geq 1 \). It follows from (62) and Lemma 14 that there exists \( i' = i \left( (k + 1)e_j, - \right) \), and a nonnegative integer \( h \) such that \( (h + 1)e_j = w \left( (k + 1)e_j, - \right) \) and:
\[
\begin{align*}
p \left( ke_j, + \right)_j' = p \left( (k + 1)e_j, - \right)_j' = v_i \left( (h + 1)e_j \right) - v_i \left( he_j \right) .
\end{align*}
\]

The assumption that there are at least two goods, we may choose \( j'' \in J \setminus j' \). Moreover, noting that \( N_j \geq 1 \), \( ke_j + e_j' \in Z_2(k + 1) \). So the result we have just established for case (a) implies:
\[
\begin{align*}
p \left( ke_j, + \right)_j'' = v_i \left( ke_j + e_j'' \right) - v_i \left( ke_j \right) .
\end{align*}
\]
For \( i = i \left( ke_j, + \right) \), Lemma 14 implies: \( p \left( ke_j, + \right)_j'' = v_i \left( ke_j + e_j'' \right) - v_i \left( ke_j, + \right) \).

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This, (i) of Proposition 9, and (103) imply that $i = i\left(ke^j, +\right) = i^*$, so that Lemma 14 implies:

$$p\left(ke^j, +\right)_{j'} = v_i^* \left(w\left(ke^j, +\right) + e^j\right) - v_i^* \left(w\left(ke^j, +\right)\right) \quad (104)$$

$$p\left(ke^j, +\right)_{j''} = v_i^* \left(w\left(ke^j, +\right) + e^{j''}\right) - v_i^* \left(w\left(ke^j, +\right)\right) \quad (105)$$

But then (i) of Proposition 9, (102), and (104) imply $i' = i((k+1)e^j, -) = i^*$. So (102) and (104) also imply: $v_i^* \left((h+1)e^{j'}\right) - v_i^* \left(he^{j'}\right) = v_i^* \left(w\left(ke^j, +\right) + e^j\right) - v_i^* \left(w\left(ke^j, +\right)\right). \quad (80)$

and strict component-wise concavity (see Proposition 8) imply that $w\left(ke^j, +\right) = he^{j'}$. (103), (105), and strict submodularity (see Proposition 8) imply $k = h$. Plugging $w\left(ke^j, +\right) = ke^j$ into (104) establishes (95) in case (b), and hence also establishes $i^3$, completing the proof. \(\square\)

References


Murota, K. (2003), *Discrete Convex Analysis*, SIAM.