Optimal Shill Bidding in the VCG Mechanism

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Abstract

This paper studies shill bidding in the VCG mechanism applied to combinatorial auctions. Shill bidding is a strategy whereby a single decision-maker enters the auction under the guise of multiple identities (Sakurai, Yokoo, and Matsubara 1999). I formulate the problem of *optimal* shill bidding for a bidder who knows the aggregate bid of her opponents. A key to the analysis is a subproblem–the *cost minimization problem (CMP)*–which searches for the cheapest way to win a given package using shills. An analysis of the CMP leads to several fundamental results about shill bidding: (i) I provide an exact characterization of the aggregate bids b such that some bidder would have an incentive to shill bid against b in terms of a new property *Submodularity at the Top*; (ii) the problem of optimally sponsoring shills is equivalent to the *winner determination problem* (for single minded bidders)–the problem of finding an efficient allocation in a combinatorial auction; (iii) shill bidding can occur in equilibrium; and (iv) the problem of shill bidding has an inverse, namely the collusive problem that a coalition of bidders may have an incentive to merge (even after competition among coalition members has been suppressed). I show that only when valuations are additive can the incentives to shill and merge simultaneously disappear.

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1 Introduction

The Vickrey-Clarke-Groves (VCG) mechanism serves as an important benchmark for combinatorial auctions–or in other words, auctions in which many items are sold simultaneously and bidders may submit bids on packages of items. Assuming transferable utility, the VCG mechanism is essentially the unique mechanism that implements efficient outcomes in dominant strategies on smoothly connected domains (Green and Laffont 1977, Holmstrom 1979). The VCG mechanism charges each decision-maker her externality, and thus causes the decision-maker to internalize her effect on others. However, suppose that it were possible for a decision-maker to enter the auction under the guise of multiple identities. For example, if the auction is conducted over the internet, a bidder may use multiple screen names, and it may not be possible for the seller to verify that different screen names correspond to different identities. Alternatively, a decision-maker may instruct several bidders to bid on her behalf. In this case a bidder is no longer a decision-maker. The mechanism can no longer identify the decision-maker and and therefore cannot charge her the externality she imposes. Sakurai, Yokoo, and Matsubara (1999) were the first to show that the strategy of using multiple identities–known as *shill bidding*–could benefit a buyer.

The main contribution of the current paper is to study *optimal* shill bidding. In particular, I introduce and study the *shill bid cost minimization problem* (CMP). The CMP is as follows: suppose that Ann is a buyer who knows the aggregate bid of all others, and who is required to win some package Z. What is the cheapest way that Ann can win Z using shills? For each package Z, the CMP yields a package price for Z. Given the solution to the CMP and Ann's valuation, one may ask which package Ann would like to buy under these package prices.

An analysis of the CMP leads to several fundamental results on the shill bidding problem, including:

(i) I provide an exact characterization of the aggregate valuations against which Ann has no incentive to use shills. This characterization is in terms of a new property called *Submodularity at the Top (SubTop)*, which is a substitutes property. This characterization provides the largest collection of valuations V such that when Ann's beliefs assign probability 1 to V, she will never have an incentive to use shills. The characterization is related to other conditions–in particular, *gross substitutes* and *submodularity*– which have been found to deter shill bidding in the literature (Lehmann, Lehmann, and Nisan 2006, Ausubel and Milgrom 2002).[1] I discuss these relations and also prove new results about submodular and gross substitutes valuations.

(ii) The problem of how to optimally sponsor shills is equivalent to the *winner determination problem*, which is the problem of finding an efficient allocation in a combinatorial

[1] Yokoo, Sakurai, and Matsubara (2004) also present another condition–*bidder submodularity*–which is sufficient to deter shill bidding.

auction.

(iii) Shill bidding can occur in equilibrium. In particular, suppose that there are two players both of whom may use shills, and who know one another's valuation. Player 1 has a strictly *supermodular* valuation, meaning that goods are complements for player 1. Suppose that at the efficient allocation, player 2 wins at least two items. Then there exists a Nash equilibrium *in undominated strategies* in which player 1 bids truthfully, and player 2 wins at least two items (not necessarily the package it would be efficient for her to win). Player 2 sponsors one shill per item that she wins. The shill for a given item bids only for that item. Player 2 achieves a strictly higher payoff than she would under truthful bidding. As a corollary, if both players have strictly supermodular valuations and both win at least two items at the efficient allocation, then the VCG mechanism has multiple equilibria in undominated strategies which are not payoff equivalent. This is in sharp contrast to the VCG mechanism when shills are disallowed, in which there is a unique equilibrium in undominated strategies.

(iv) Shill bidding is intimately related to collusion. Setting aside the ordinary incentive to suppress competition, the disincentive to disintegrate using shills when facing a substitutes valuation translates into an incentive to merge for a coalition facing the same valuation. I show that only when valuations are additive can the incentives to shill and merge simultaneously disappear.

2 Preliminaries

2.1 Combinatorial Auctions

A combinatorial auction is an auction in which bidders may bid for packages of goods. Formally, assume a finite collection N of goods, and a finite collection I of bidders. A **package** Z is simply a subset of N. Each bidder $i \in I$ has a **valuation** $v_i : 2^N \to \mathbb{R}_+$, assigning a value to each package. If i receives package Z for a price p, his utility is $v_i(Z) - p$. We assume $v_i(\emptyset) = 0$ and **monotonicity/free disposal**: $Y \subseteq Z \Rightarrow v_i(Y) \leq v_i(Z)$. The following definition gives an important class of valuations for our purposes:

Definition 1 *Let $Z \subseteq N$ and $r \in \mathbb{R}_+$. Say that a bidder i is* ***single-minded for package Z at value*** *r if $v_i(Y) = r$ if $Y \supseteq Z$ and $v_i(Y) = 0$ otherwise. We write $v_i = v^{Z,r}$. In words, i only values Z; once i receives Z, i's marginal value for additional packages is zero.*

Actual or potential uses of combinatorial actions include auctions for provision of transportation services (Caplice and Sheffi 2006, Cantillon and Pesendorfer 2006) industrial procurement (Bicheler, Davenport, Hohner, and Kalagnanum 2006), arrival and departure times at airports (Rassenti, Smith, and Bulfin 1982), and use of radio spectrum (Milgrom 2000). Complementarities are often important in combinatorial auctions. For example, in a spectrum auction, a firm may need to win several licenses to have a viable business, and therefore may assign a value to a package exceeding sum of values of the stand alone licenses.

2.2 The VCG Mechanism

The VCG Mechanism is an example of a combinatorial auction. In the VCG Mechanism a bid is a valuation. The VCG mechanism then implements an efficient allocation taking the bids at face value. That is, goods are divided among bidders so as to maximize the sum of reported valuations. Formally, let $\mathcal{X} := \{(X_i : i \in I) : \forall i, j \in I, X_i \subseteq N, X_i \cap X_j = \emptyset\}$ be the set of allocations, where X_i is the package received by i. Let $\mathcal{X}_{-i}$ be the allocations excluding i (i.e., $X_i = \emptyset$). Let $(v_i : i \in I)$ be the profile of bids, and let X_i^* be the package which is assigned to i at the efficient allocation. In other words $(X_i^* : i \in I)$ solves $\max\{\sum_{i \in I} v_i(X_i) : (X_i : i \in I) \in \mathcal{X}\}$. If there are multiple efficient allocations, the VCG mechanism selects one using some tie-breaking rule. Bidder i's VCG payment is:

$$p_i = \underbrace{\max\{\sum_{j \in I \setminus i} v_j(X_j) : (X_j : j \in I \setminus i) \in \mathcal{X}_{-i}\}}_{(*)} - \underbrace{\sum_{j \in I \setminus i} v_j(X_j^*)}_{(**)}$$

Term (*) is the value of the efficient allocation when i is excluded, and (**) is the total value to bidders other than i at the efficient allocation including i. The difference between (*) and (**) can be interpreted as bidder i's externality, or alternatively as the opportunity cost of i's package. Truthful bidding–submitting a bid equal to one's value–is a dominant strategy in the VCG mechanism. The VCG mechanism is essentially unique efficient auction in which truthful bidding is a dominant strategy (Green and Laffont 1977, Holmstrom 1979). By essentially unique, I mean that for each bidder i, the payments may be altered by a constant depending on the reports of other bidders. The VCG mechanism suffers from a host of problems (Milgrom 2004, Ausubel and Milgrom 2006, Rothkopf 2007) and is not commonly used for selling multiple heterogenous objects. The motivation for studying this mechanism is that it is the unique auction with certain desirable properties, and we would like to understand how these properties inevitably lead to certain problems.

2.3 Shill Bidding

Imagine that a VCG auction is conducted over the internet, and that a single bidder may enter the auction under multiple screen names. Suppose it is impossible to verify that different screen names correspond to different bidders. Shill bidding is related to collusion (see Section 9). Shill bidding may arise without the internet if a single decision-maker controls multiple bidders. The study of such manipulative strategies was initiated by Sakurai, Yokoo, and Matsubara (1999).[2] The following example shows how shill bidding may be effective.

Example 1 Suppose there are two bidders, Ann and Bob, and two goods, 1 and 2. Bob values a single good at \$1, but the package containing both at \$4. Ann values each good at \$1.50 and the package at \$3. It is efficient to give the package to Bob. Under truthful bidding, the VCG mechanism gives the package to Bob and charges Bob \$3. Suppose instead that Ann enters auction under two identities, Carol and Dan. Carol claims only to value good 1 at \$3.50, and Dan claims only to value good 2 at \$3.50. Taking these bids at face value, it would be efficient to give Carol good 1 and Dan good 2. Thus ultimately Ann would receive both items. Carol's VCG payment would be \$1 = \$4.50 − \$3.50. \$4.50 is the total value to all bidders if Carol is excluded: Dan would still win good 2, and Bob would receive good 1. \$3.50 is the combined value to Dan and Bob in the efficient allocation including Carol. Similar reasoning shows that Dan's VCG payment would also be \$1. Since Ann is responsible for the payments of all of her shill bidders, Ann's total payment is \$2. Since Ann receives both goods, her utility is $3 - 2 = 1$, which is higher than her utility of zero from truthful bidding.

The main problem is that under shill bidding, a decision-maker no longer corresponds to a single bidder. The idea behind the VCG mechanism is that each *decision-maker* is charged his externality. However, with shill bidding, it is no longer possible to identify the decision-maker.

This paper will focus on two characters, Ann and Bob. Ann is the character who shill bids. Rather than submitting a single bid, Ann may submit a profile of bids $(v_j : j \in J)$, selecting both the valuations v_j and the (finite) set J of identities. When Ann submits $(v_j : j \in J)$, the VCG auction is run as if these were distinct legitimate bids. Bob is not interpreted as Ann's only opponent, but rather as Ann's *aggregate* opponent. That is, Ann faces a set O of opponents, who submit bids $(v_i : i \in O)$. Bob's bid v_b is defined by:

$$v_b(Z) := \max\{\sum_{i \in I} v_i(X_i) : \bigcup_{i \in O} X_i = Z, i \neq j \Rightarrow X_i \cap X_j = \emptyset\}$$

So Bob's value to any package is equal to the sum of values to Ann's opponents if Z is allocated efficiently to Ann's opponents. Ann's payment and allocation–whether or not she

[2] Rastegari, Condon, and Leyton-Brown (2007) and Day and Milgrom (2008) show that shill bidding can be a problem not just in the VCG mechanism, but for a larger class of auctions. Several papers have proposed solutions to the shill bidding problem, either in the form of limited verification of identities (Conitzer 2007) or in the form of proposed auctions which do not suffer from the shill bidding problem (Ausubel and Milgrom 2002, Matsuo, Takayuki, Day, and Shintani 2006, Yokoo 2006, Yokoo and Iwasaki 2007, Day and Milgrom 2008). Of course, as these papers point out, verification of identities may sometimes be infeasible, and the auctions which avoid the shill bidding problem do not have all of the attractive features of the VCG mechanism.

uses shills–depends only on the aggregate bid of her opponent.[3] So the analysis applies whenever Ann faces an arbitrary (finite) number of opponents.

2.4 Informational and Behavioral Assumptions

Below, I conduct a sort of *worst-case analysis.* Without shills, the VCG mechanism has a strong worst-case guarantee: no matter what knowledge a bidder has, there is never an incentive to manipulate the mechanism. This paper studies violations of this guarantee when shill bidding is allowed.

In analyzing the optimal shill bidding problem, I make the extreme assumption that Ann knows or can correctly guess the aggregate bid of her opponent. While this assumption is unrealistic, it is very useful. The analysis provides Ann's best reply to any deterministic belief. Moreover, the analysis has consequences for situations where Ann is uncertain about the bid of her aggregate opponent. In particular, Theorem 3 provides the the largest collection of valuations V such that when Ann's beliefs assign probability 1 to V, she will never have an incentive to use shills. Another consequence of the analysis which does not assume that bidders know the aggregate bid that they face is Corollary 3, which gives conditions under it is a dominant strategy for a coalition of bidders to merge. Theorems 6 and 7 explore dominant strategies when shill bidding is possible as well as the riskiness of shill bidding, two important issues which are relevant when Ann is considering the use of shills but is uncertain of the aggregate bid she faces.

The study of the optimal shill bidding problem also leads to conclusions about *equilibrium* when both players may use shills. Theorem 8 and Corollary 2 provide properties of equilibrium in undominated strategies when one bidder views goods as complements. The analysis assumes that bidders know one another's valuations but bids are determined endogenously in equilibrium. While the theorems just mentioned assume that both bidders may use shills, many of the other results may be reinterpreted as equilibrium results if it is assumed that only one bidder has the capacity to use shills. This follows from the fact that for a bidder who cannot use shills, truthful bidding is a dominant strategy.

To summarize that argument of this section, the study of optimal shill bidding under strong informational assumptions is justified by its many corollaries.

[3] The one possible exception is when there are multiple efficient allocations and the VCG mechanism selects one. Then, insofar as the tie-breaking rule may depend on individual bids and not just the aggregate bid of Ann's opponents, Ann's utility may be affected by the individual bids–but only if she uses shills. Below, I will ignore this knife-edge case because none of the theorems would be altered by taking this possibility seriously.

3 Optimal Shill Bidding

3.1 Optimal Shill Bidding and Cost Minimization

The main contribution of this paper is to study the problem of *optimal* shill bidding under an extreme informational assumption (See section 2.4). Assume that Ann *knows* or *correctly guesses* the aggregate bid that she is facing. In this case, what is Ann's optimal use of shills?

Optimal Shill Bidding Problem (OSB)

Input 1. A valuation v_b for Bob.

2. A valuation v_a for Ann.

Output A shill bid profile $(v_j : j \in J)$ that maximizes Ann's utility in the VCG mechanism.

It will be useful to focus on the following subproblem:

Shill Bidding Cost Minimization Problem (CMP)

Input 1. A valuation v_b for Bob

2. A package Z that Ann is required to win.

Output A shill bid profile $(v_j : j \in J)$ that wins Z but makes the smallest possible payment among all shill bid profiles that win Z.

Here, Ann knows v_b, and is required to win package Z. The CMP asks for the cheapest way to win Z using shills. Many qualitative conclusions carry over from the CMP to the OSB, and indeed studying the CMP provides a more direct insight into the structure of the shill bidding problem.

3.2 A Central Example

Suppose that there are three goods, $N = \{1, 2, 3\}$, and that Bob has the valuation:

Package	**1**	**2**	**3**	**12**	**13**	**23**	**123**
Bob	11	8	5	19	13	31	34

The top row gives a package, and the bottom row gives Bob's value for that package. For example, Bob's value for the package $\{1, 2\}$ is \$21. In the CMP, let us require Ann to win all goods in N. Observe that if Ann wins all goods under a single identity, she must pay \$34. What is the cheapest way for Ann to win these goods using shills? I start with some suboptimal shill bidding strategies, and gradually improve them, developing some principles of optimal shill bidding along the way.

Suppose that Ann uses two shills, Carol and Dan, with the following valuations:

Package	**1**	**2**	**3**	**12**	**13**	**23**	**123**
Bob	11	8	5	19	13	31	34
Carol	12	12	12	24	24	24	36
Dan	25	20	10	45	35	30	55

(1)

Carol values each item at \$12, and has an additive valuation, so that two goods are valued at \$24 and three at \$36. Dan values item 1 at \$25, item 2 at \$20, and item 3 at \$10, and values packages additively, so that Dan's value for any package is the sum of values of items contained in that package. Against Bob, it is efficient to give Carol package item 3, and Dan package $\{1, 2\}$, for a total utility of $12+45 = 57$. If Carol did not exist, it would be efficient to give Dan item 1 and Bob the package $\{2, 3\}$ for a total utility of $25+31 = 56$. So Carol's payment is $\$56 - \$45 = \$11$. If Dan did not exist, it would be efficient to give Carol good 1 and Bob $\{2, 3\}$ for a total utility of $12 + 31 = 43$. So Dan's payment is $\$43 - \$12 = \$31$. So Ann's total payment through her shills is $\$11 + \$31 = \$42$, which is worse than Ann's payment \$34 under one identity. Notice, however, that when Dan is excluded, Carol gets one of Dan's items, namely good 1. So part of Dan's payment is due to competition with Carol for good 1. To lower payments, each shill should eliminate bids on goods won by the other shill. Eliminating such bids, suppose that instead of (1), Ann submits:

Package	**1**	**2**	**3**	**12**	**13**	**23**	**123**
Bob	11	8	5	19	13	31	34
Carol	0	0	12	0	12	12	12
Dan	25	20	0	45	25	20	45

(2)

Carol has reset her bids for the items 1 and 2 that Dan wins to 0. Carol still values packages additively, which implies–given her values for single items that she values a package at 12 if it contains item 3 and at 0 otherwise. Likewise, Dan has reset his bid for item 3 that Carol wins to 0, but maintains his values for items 1 and 2, and generates values for larger packages additively. Because the value of every allocation is weakly lower under (2) than under (1), but the value of the previously efficient allocation in which Carol gets item 3 and Dan gets $\{1, 2\}$ is unchanged, this allocation is still efficient. Carol's payment is unchanged and remains at \$11. If Dan did not exist, then it would be efficient to give Bob everything for a utility of 34, so Dan's payment is $\$34 - \$12 = \$22$, and Ann's total payment becomes $\$11 + \$22 = \$33$, which improves on Ann's payment of \$43 under (1), and in fact is better than Ann's payment of \$34 when she wins everything under a single identity. So observing the change which led to the improvement in moving from (1) to (2), the first lesson that we should learn is:

Principle 1 *Shill bidders should not compete with one another; a shill bidder should not bid on items won by other shills.*

This is quite straightforward. Let us now return to Carol's payment, and observe that since Dan's bid obeys Principle 1, Carol's payment does not depend on any bid that Dan makes for the item Carol wins. However, when Carol is excluded, it is efficient to give Dan item 1 and Bob the remaining items. So Carol's payment does depend on Dan's losing bid for item 1. Ann would be better off if her shills did not bid for any items except those that they win. With this in mind, suppose that Ann instead submits shill bid profile:

Package	**1**	**2**	**3**	**12**	**13**	**23**	**123**
Bob	11	8	5	19	13	31	34
Carol	0	0	12	0	12	12	12
Dan	0	0	0	45	0	0	45

(3)

Note that Dan's valuation is no longer additive. In particular, Dan values the package $\{1, 2\}$ at more than the sum of its parts. Of course, by monotonicity Dan must still value the package $\{1, 2, 3\}$ at \$45, (as Carol must value the package $\{1, 3\}$ at \$12). However, Dan's marginal value for 3 given that he receives $\{1, 2\}$ is zero, so he will never win 3 at an efficient allocation. Again, for the same reason as above, the efficient allocation remains the one in which Carol wins 3 and Dan wins $\{1, 2\}$. If Carol did not exist, Dan's allocation would not change, but Bob would receive item 3 and a utility of 5. So Carol's payment is \$5. If Dan did not exist, Bob would get everything, for a utility of 34. So Dan's payment is $\$34 - \$12 = \$22$. So Ann's total payment is $\$5 + \$22 = \$27$, which is better than the payment of \$33 under (2).

Noting the change that led to the improvement when moving from (3) to (2), and recalling the definition of a *single-minded valuation* (Definition 1), we have:

Principle 2 *Each shill bidder should bid single-mindedly for the package she wins.*

Notice that Principle 2 generalizes Principle 1. By construction, in (3) Carol and Dan bid single-mindedly for their packages. However notice that when Dan is excluded, Bob receives all goods, meaning that Carol's good–good 3–is reallocated to Bob. This reallocation occurs because of complementarities in Bob's valuation. When Dan receives package $\{1, 2\}$, Bob's marginal value for good 3 is only 5, so at the efficient allocation, this good is allocated to Carol, who values it at 12. However, when Dan releases his package back into the economy, then Bob receives it, and given that Bob has $\{1, 2\}$, his marginal value for good 3 rises to 15, so it is efficient to give Bob Carol's good as well. This reallocation increases Dan's payment. To understand this, recall that Dan's payment consists of two terms:

$$\begin{aligned} \text{Dan's payment} \quad &= \quad \underbrace{\text{total utility in economy without Dan}}_{(*)} \\ \text{minus} \quad & \quad \underbrace{\text{utility to everyone but Dan in economy with Dan.}}_{(**)} \\ &= \quad 34 - 12 \end{aligned}$$

Now suppose that Carol raises her bid by some small amount ϵ. This will not affect the efficient allocation when Dan is excluded, and thus will not affect (*). However this does raise (**) by ϵ, and so reduces Dan's payment; since Carol's value has gone up, and Carol only receives her package when Dan is present, the opportunity cost associated with the awarding Dan $\{1, 2\}$ has gone down because the other bidders are not as badly off when Dan is present. This observation highlights non-monotonicities in the revenues of the VCG mechanism. Notice that as we increase ϵ, Dan's payment will continue to decline until it actually becomes efficient to award Carol good 3 even when Dan is excluded, at which point further increases in Carol's bid will increase (*) and (**) by the same amount, canceling out. Of course, Carol's payment is unchanged throughout this process, because Carol's payment is independent of her bid conditional on winning. So suppose that Ann raises Carol's bid to 17, which is sufficient for Carol to still win good 3 when Dan is excluded:

Package	**1**	**2**	**3**	**12**	**13**	**23**	**123**
Bob	11	8	5	19	13	31	34
Carol	0	0	17	0	17	17	17
Dan	0	0	0	45	0	0	45

(4)

Of course it is still efficient to give Carol good 3 and Dan package $\{1, 2\}$. Carol's payment is unchanged at \$5. On the other hand, if Dan is excluded, Carol now keeps her package, and Bob gets $\{1, 2\}$ for a value of 19. So Dan's payment is now \$19. Ann's total payment is now \$5 + \$19 = \$24, which is better than the payment of \$27 under (3). Noting the improvement, we have:

Principle 3 *Each shill bidder should place a sufficiently high bid on the package she wins so that she would still win the package if some other shill bidder were excluded.*

Corollary 1 *At an optimal shill bid profile, the payment of any shill bidder for her package is equal to Bob's marginal value for that package (given that Bob wins exactly the goods not won by any shills).*

In our example, since the shills collectively win everything, the parenthetical statement can be ignored. Notice that the property in Corollary 1 does not hold for (sub-optimal) single-minded shill bid profiles that fail to satisfy Principle 3. In particular, in (3) which violates Principle 3, Dan's payment is \$22, which is more than Bob's value of \$19 for the package $\{1, 2\}$ that Dan wins.

For any $Z \subseteq N$, let $\Pi(Z)$ be the set of partitions of Z, where a **partition** $\mathcal{P}$ of Z is a collection of subsets of Z, whose union is Z and such that every pair of sets in $\mathcal{P}$ is disjoint. An element of $\mathcal{P}$ (also called a **cell**) is denoted P. For example if $Z = \{1, 2, 3\}$, then the partition $\mathcal{P} = \{\{1, 2\}, \{3\}\}$ belongs to $\Pi(Z)$ and $P = \{1, 2\}$ is a cell of $\mathcal{P}$. It follows from Principle 2 and Corollary 1 that (in the case where Ann is required to win all goods N) finding an optimal shill bid profile (solving the CMP) amounts to finding a partition $\mathcal{P}$ of the set of goods N that minimizes the sum of values that Bob attaches to the packages in $\mathcal{P}$. The following table lists all such partitions:

	1	**2**	**3**	**12**	**13**	**23**	**123**	**Payment**	
Bob	11	8	5	19	13	31	34		
$\mathcal{P}^1$	P_1	P_2	P_3					$11 + 8 + 5$	$= 24$
$\mathcal{P}^2$	P_1					P_2		$11 + 31$	$= 42$
$\mathcal{P}^3$		P_1			P_2			$8 + 13$	$= 21$
$\mathcal{P}^4$			P_1	P_2				$5 + 19$	$= 24$
$\mathcal{P}^5$							P_1	34	$= 34$

(5)

There are five partitions, labeled $\mathcal{P}^i$ for $i = 1, \ldots, 5$ in the first column. In each row, the partition cells are numbered and are drawn beneath the corresponding packages. For example the partition $\mathcal{P}^4$ has two cells $P_1 = \{1, 2\}$ and $P_2 = \{3\}$. Each row corresponds to a shill bid profile in which Ann sponsors one shill for each package corresponding to a partition cell, and has the shill bidder bid single-mindedly for the corresponding cell (Principle 2) and with a value sufficiently high that Principle 3 is satisfied. (4) represents a shill bid profile corresponding to $\mathcal{P}^4$. By Corollary 1, the payment for a shill bid profile corresponding to any partition is just the sum of Bob's value to the partition cells (given in the last column). The table shows that there is a shill bid profile that outperforms (4), namely $\mathcal{P}^3$ in which one shill bidder is sponsored for each of the packages $\{2\}$ and $\{1, 3\}$, bids are sufficiently high to win the packages and also satisfy Principle 3, and the total payment is \$21. The profile corresponding to $\mathcal{P}^3$ with sufficiently high bids solves the CMP.

3.3 Properties of Optimal Shill Bidding

Let $Y, Z \subseteq N$ be such that $Y \cap Z = \emptyset$. Then Bob's **marginal value for** Z **given** Y is:

$$v_b(Z|Y) := v_b(Y \cup Z) - v_b(Y)$$

In other words, $v_b(Z|Y)$ gives Bob's value for Z given that Bob already has Y. For any package Z, $N - Z$ is the complement of Z, or in other words, the set of all goods in N which are not in Z. Recall also the definition of single-mindedness (Definition 1) and the definition of $\Pi(Z)$ (end of Section 3.2). The following theorem presents a program whose solution allows one to derive a solution for the CMP and gives some properties of the optimum.

Theorem 1 *Fix bid v_b for Bob. Suppose $\mathcal{P}^*$ solves:*

$$\min\{\sum_{P\in\mathcal{P}} v_b(P|N-Z) : \mathcal{P} \in \Pi(Z)\} \tag{6}$$

Then there exists a cheapest way to win Z against Bob which has the following properties:

1. *Ann sponsors one shill for every package P in $\mathcal{P}^*$.*
2. *The shill bidder for P bids single-mindedly for P.*
3. *The shill bidder for P bids sufficiently high that:*
   - (a) *The shill bidder actually wins package P.*
   - (b) *The shill bidder would still win P if any other shill bidder were excluded from the auction.*

Moreover, these properties are sufficient. A shill bid profile satisfying all properties except 3b is suboptimal.[4] *Ann's payment when she uses shills optimally to purchase Z is given by the value of (6).*

The proof of the theorem generalizes the logic of the example in Section 3.2 and can be found in the appendix. The optimization problem (6) has as its feasible set, the set of all partitions $\mathcal{P}$ of Z. The objective function to be minimized is a sum over partition cells P of $\mathcal{P}$ of the marginal values $v_b(P|N-Z)$. The example in Section 3.2 concerned the case in which $Z = N$ in which case $v_b(P|N-Z) = v_b(P)$. As in the example, at an optimum different shills sponsored by Ann do not compete with one another. The most interesting property in the theorem is 3b. Property 3b is not implied by property 3a because when a shill releases her package back into the economy and it goes to Bob, this package may be a complement for some other shill's package, increasing Bob's marginal value for the second package, and so it may be efficient to give Bob the second package as well. In the example in Section 3.2, the shill profile (3) satisfies all properties except 3b and so is suboptimal, and moreover the payment induced by (3) does not take the form of the objective in (6);

[4]In condition 3b, for optimality, if there are multiple efficient allocations, it is sufficient that the shill for P wins at *some* efficient allocation when another shill is excluded. This observation only applies to a knife-edge case.

this illustrates that at suboptimal single-minded shill bid profiles (failing to satisfy 3b), a shill bidder may be charged more than $v_b(P|N-Z)$.

Define:

$$p_Z^{VCG} := v_b(Z|N-Z) \tag{7}$$

$$p_Z^{\text{Shill}} := \min\{\sum_{P\in\mathcal{P}} v_b(P|N-Z) : \mathcal{P} \in \Pi(Z)\} \tag{8}$$

p_Z^{VCG} (the "VCG price" of Z) is the amount that Ann would have to pay for package Z if she wins Z under a single identity. So, for example, if Ann wins Z when she bids truthfully, her VCG payment will be p_Z^{VCG}. VCG prices have the property that if known, they induce efficient decisions.[5] By Theorem 1, p_Z^{Shill} (the "shill price" of Z) is the amount that Ann would have to pay for Z if she wins it in the cheapest way using shills. So the possibility of shill bidding also induces a vector of prices, one for each package, which differ from the VCG prices. Since one possible strategy for Ann even when shills are available is to bid under a single identity, we have:

$$p_Z^{\text{Shill}} \leq p_Z^{VCG} \tag{9}$$

Suppose that Z^* solves

$$\max\{v_a(Z) - p_Z^{\text{Shill}} : Z \subseteq N\} \tag{10}$$

Then any solution to the CMP for Z^* solves the OSB (optimal shill bidding problem). In particular, there exists an optimal shill bid profile with the properties described in Theorem 1. In this way, the properties of the CMP carry over to the OSB.

Let $\overline{v} = (v_j : j \in J)$ be a shill bid profile, and let $(X_j^* : j \in J \cup \{b\})$ be an efficient allocation between Bob and the members of J. Then define:

$$\gamma(\overline{v}) := \sum_{j\in J} v_j(X_j^*)$$

So $\gamma(\overline{v})$ is the total amount that Ann's shills claim to value the packages they win.[6] Of course, this is usually more than the amount that they pay. Since it is possible that J contains only one element, $\gamma(v)$ is also defined for a single valuation v. Let $\text{Opt}(Z)$ be the set of shill bid profiles solving the CMP for Z. Let $\text{Win}(Z)$ be the set of bids consisting of a single valuation that win exactly package Z. Since any bid v in $\text{Win}(Z)$ charges Ann $p_Z^{VCG} = v_b(Z|N-Z)$, all bids in $\text{Win}(Z)$ are equally good from the standpoint of how much Ann has to pay. Observe that:

$$\inf\{\gamma(v) : v \in \text{Win}(Z)\} = v_b(Z|N-Z) \tag{11}$$

[5]Note that the VCG price is a personalized price.

[6]In the case of multiple efficient allocations, $\gamma(\overline{v})$ may depend on the tie-breaking rule.

In words, if Ann wins Z under a single identity, the minimum that she can claim to value Z is $v_b(Z|N-Z)$. In fact, this is also the minimum amount that Ann would have to claim to value Z to win Z using shills. Since $p_Z^{VCG} = v_b(Z|N-Z)$. Another way of putting (11) is as follows:

- When bidding under a single identity, the minimum Ann can claim to value Z and still win Z (optimally) is p_Z^{VCG}, the VCG price for Z.

As explained above the qualification "optimally" is vacuous here, since all winning bids are equally good when bidding under a single identity. Notice that not all bids v with $v(Z) = p_Z^{VCG}$ will win Z, since under v, Ann may also claim that other packages are good substitutes for Z, but, for example, the single-minded bid $v^{Z,r}$ with $r = p_Z^{VCG}$ will always win Z (modulo tie-breaking). In contrast, we have:

Theorem 2 *If using shills allows Ann to win Z more cheaply than she could under a single identity, then:*

$$\inf\{\gamma(\overline{v}) : \overline{v} \in \text{Opt}(Z)\} > v_b(Z|N-Z)$$

Proof. In Appendix. □

Another way of putting Theorem 2 is the following:

- Assuming Ann has a strict incentive to use shills, the minimum that Ann's shills can claim to value Z–if Ann uses shills *optimally*–is strictly more than p_Z^{VCG}, which in turn is strictly more than the price that Ann will pay, p_Z^{Shill}.

So relative to bidding without shills where the minimal winning bid is equal to the price paid, optimal shill bidding moves the minimum *optimal* winning bid and the price paid apart. Theorem 11 is closely related to 3b of Theorem 1. The proof shows that property 3b gives a lower bound on the amount that Ann's shills must claim to value Z which is always binding,[7] or in other words, pushes this claim above $v_b(Z|N-Z)$, when shill bidding is worthwhile.

4 Characterization of the Incentive to Shill

The optimization problem (6) immediately generates a necessary and sufficient condition on aggregate bids v_b for there to be no incentive to shill against v_b. More precisely, we answer the question: exactly when does there exist a potential bidder outside of the collection I of bidders who would have an incentive to use shills against I? The answer depends only on the aggregate bid of I (or the aggregate valuation, if the members of I bid truthfully). The characterization is in terms of a new property of valuations *Submodularity at the Top*

[7] More precisely, at least one of the constraints described by property 3b is always binding.

(SubTop), which is of independent interest. Lemma 1 in Section 9 provides evidence that the class of SubTop valuations is a mathematically natural class. The analysis also has consequences for shill bidding when Ann is uncertain about the bids that she faces. In particular, Theorem 3 implies that the set of SubTop valuations is the largest set of valuations S with the property that for all probability measures μ with support contained in S, when Ann's belief over the aggregate valuation that she faces is given by μ, Ann has no incentive to sponsor shills.

Definition 2 *Ann has a **profitable shill bid against** v_b if Ann has some shill bidding strategy that outperforms truthful bidding against v_b.*

Theorem 3 *The following conditions are equivalent:*

1. *(**Submodularity at the Top (SubTop)**): For all $Z \subseteq N$ and $\mathcal{P} \in \Pi(Z)$:*

$$v_b(Z|N-Z) \leq \sum_{P\in\mathcal{P}} v_b(P|N-Z) \tag{12}$$

2. *For all valuations for Ann, there is no profitable shill bid for Ann against v_b in the VCG auction for N.*

Proof. SubTop is equivalent to:

$$v_b(Z|N-Z) \leq \min\{\sum_{P\in\mathcal{P}}[v_b(P|N-Z) : \mathcal{P} \in \Pi(Z)\}, \quad \forall Z \subseteq N$$

which in turn, is equivalent to:

$$p_Z^{VCG} \leq p_Z^{\text{Shill}}, \quad \forall Z \subseteq N$$

Theorem 1 now implies that the price that Ann has to pay for any package without shills is no more than the price that she has to pay with shills. (In fact, the two prices must be equal by (9)). So SubTop implies that Ann does not have a profitable shill bid.

Going in the other direction, suppose that SubTop fails. This means that there exists $Z \subseteq N$ such that $p_Z^{\text{Shill}} < p_Z^{VCG}$. Then if Ann values Z sufficiently, and values nothing outside of Z, she will have an incentive to use shills. □

SubTop is a substitutes property. SubTop says that *if Bob already has all items outside of* Z, then his marginal value for Z is less than the sum of his marginal values for its parts, regardless of how Z is broken into parts. So the whole is less than the sum of its parts, a substitutes condition. Notice that the italicized qualification above is important. Because of this qualification, SubTop does not imply subadditivity of the marginal valuation $v_b(\cdot|N-Z)$.

SubTop is related to other substitutes conditions–in particular, *gross substitutes* and *submodularity*– which have been found to deter shill bidding in the literature (Ausubel and Milgrom 2002, Lehmann, Lehmann, and Nisan 2006).[8] Sections 10.1-10.2 provide a detailed comparison with these results. In the process, I derive new results about both submodular valuations and gross substitutes valuations. Submodular valuations in particular are closely related to SubTop valuations (explaining the name); as explained in Section 10.1, the set of SubTop valuations is strictly larger than the set of submodular valuations, showing that SubTop is a weaker substitutes condition than those usually studied. Section 10.1 uses Theorem 3 to derive a sort of converse to a result by Lehmann, Lehmann, and Nisan (2006) and thereby *characterizes* submodularity in terms of the incentive to shill in certain auctions.

5 Pure Complements

The previous section explored the case of substitutes–in which there was no incentive to use shills. When goods are not substitutes, there are two possibilities: (i) goods may be pure complements, or (ii) there is a mix of complements and substitutes. This section explores the case of pure complements, which is the polar opposite of the case of substitutes, and it is the case in which the incentive to shill takes its purest form. In this case, the shill bidding cost minimization problem has a very simple solution.

Definition 3 *For any $Y \subseteq N$ and valuation v, $v(\cdot|Y)$ is* ***superadditive*** *if, for all nonempty $C, D \subseteq N - Y$:*

$$C \cap D = \emptyset \Rightarrow v_b(C \cup D|Y) \geq v_b(C|Y) + v_b(D|Y) \tag{13}$$

A valuation v is ***supermodular*** *if for all $Y \subseteq Z \subseteq N$ and $x \notin Z$:*

$$v(x|Y) \leq v(x|Z)$$

Both superadditivity and supermodularity represent notions of complements. Superadditivity of $v(\cdot|Y)$ means that the whole package $C \cup D$ is worth more than the sum of its parts C and D when one already has Y. Supermodularity is the dual of submodularity (See Definition 6); v is supermodular if it exhibits increasing marginal utility of additional goods as the set of goods already acquired increases.

[8] Yokoo, Sakurai, and Matsubara (2004) also present another condition–*bidder submodularity*–which is sufficient to deter shill bidding. This is a condition on the coalition value function which is a function which takes as arguments collections of bidders rather than collections of goods.

Theorem 4 *1. Assume that $v_b(\cdot|N-Z)$ is superadditive. Then a solution to the CMP for Z is as follows:*

(a) Ann sponsors a shill for each item x in Z.

(b) The shill bidder for each item x bids single-mindedly for x with a sufficiently high bid (as explained in Theorem 1).

At the cost minimizing shill bid profile, Ann's payment is:

$$p_Z^{\text{Shill}} = \sum_{x \in Z} v_b(x|N-Z)$$

2. Assume v_b supermodular. Then there is an optimal shill bidding strategy (i.e., solution to OSB) that involves Ann sponsoring one single-minded shill bidder per item won.[9]

Proof. Superadditivity of $v_b(\cdot|N-Z)$ implies that for all $\mathcal{P} \in \Pi(Z)$:

$$\sum_{x \in B} v_b(x|N-Z) = \sum_{P \in \mathcal{P}} \sum_{x \in P} v_b(x|N-Z) \leq \sum_{P \in \mathcal{P}} v_b(P|N-Z)$$

This, together with Theorem 1 implies 1. 2 is an immediate consequence of 1 and the fact that superaddivity of $v_b(\cdot|N-Z)$ for all $Z \subseteq N$ is equivalent to the supermodularity of v_b. □

Remark 1 *Without the assumption of pure complements, there exist situations in which there is a shill bidding strategy that outperforms truthful bidding, but every shill bidding strategy which employs one shill per item won, who bids single-mindedly for that item is worse than truthful bidding.*[10]

[9] Let Z^* be the package that Ann wins at the optimal shill bid. Then if $|Z^*| > 1$ and $v_b(\cdot|N-Z^*)$ is strictly superadditive (i.e., then inequalities (13) are strict when C and D are nonempty for $Y = N - Z^*$), then sponsoring one shill per item with a sufficiently high bid is strictly better than truthful bidding. *Strict* supermodularity of v_b is equivalent to strict superadditivity of $v_b(\cdot|N-Z)$ for all Z, and therefore strict supermodularity is a sufficient condition for shill bidding to strictly outperform truthful bidding.

[10] Let $N = \{1, 2, 3\}$. Suppose that Bob's valuation is given by:

	1	2	3	12	13	23	123
v_b	3	3	1	4	4	4	6

Suppose that Ann's true valuation is single-minded for the package $\{1, 2, 3\}$ at a value greater than 6. Then there is a profitable shill bidding strategy in which Ann sponsors two shills, one of whom bids single-mindedly for $\{1, 2\}$ and the other of whom bids single-mindedly for 3. However, any shill bidding strategy in which Ann sponsors three shills, each of which bid single mindedly for one item, is worse than truthful bidding.

The following table summarizes our conclusions about the incentives to use shills:

Substitutes	Truthful Bidding	1 Identity
Complements	Incentive to Disintegrate	1 Identity per Item Won
Mix of Complements And Substitutes	Partial Incentive to Disintegrate	1 Identity per Package

6 Mixture of Substitutes and Complements

In previous sections, we have seen that when goods are substitutes, there is no incentive to use shills, and when goods are complements, there is an incentive to sponsor one single-minded shill bidder per item won. In the intermediate case, where there is a mix of substitutes and complements, there is a more partial incentive to disintegrate, so that individual shills may bid for packages rather than individual items. In this section, I show that in the general case when there may be a mix of substitutes and complements, the CMP is equivalent to a version of the *winner determination problem*, which is the problem of finding an efficient allocation in a combinatorial auction. This shows that the bidder's problem of optimally sponsoring shills to win a given package most cheaply is equivalent to the auctioneer's problem of efficiently allocating goods given a collection of bids.

6.1 The Winner Determination Problem

Consider a profile $(v^{S_i,r_i} : i \in I)$ of single-minded valuations. As above v^{S_i,r_i} is the single-minded valuation for package S_i at value $r_i \in [0,1]$. We can also express v^{S_i,r_i} as (S_i, r_i). Now consider the problem of finding an efficient allocation given valuation profile $(v^{S_i,r_i} : i \in I)$. This problem is known as the **winner determination problem for single-minded bidders** (WDSMB). For simplicity, let us assume that $i \neq j \Rightarrow S_i \neq S_j$, since if there are multiple bids for the same package, we can easily eliminate all but the highest bid. Also, the problem is not changed in any essential way if all values are assumed to belong to the interval $[0,1]$. Now WDSMB can be expressed as follows:

Input A collection $\{(S_i, r_i) : i \in I\}$ of single minded bids satisfying $i \neq j \Rightarrow S_i \neq S_j$ and $r_i \in [0,1]$ for all i.

Output A subset J of I such that for all $i, j \in J$, $S_i \cap S_j = \emptyset$ which maximizes $\sum_{i \in J} r_i$.[11]

The constraint that $S_i \cap S_j = \emptyset$ when $i, j \in J$ represents the impossibility of allocating the same item to to different bidders. Solving WDSMB amounts to finding an efficient allocation given valuations $\{(S_i, r_i) : i \in I\}$. Lehmann, O'Callaghan, and Shoham (2002) has shown that WDSMB is NP-hard.

[11] See Blumrosen and Nisan (2007) for a similar formulation.

6.2 Equivalence of Winner Determination and Shill Bid Cost Minimization

In this section, I establish the equivalence of the CMP and WDSMB under some additional assumptions. We begin by restricting attention to the special case of the CMP in which $Z = N$, so that Ann is required to win all goods. Notice that a valuation requires specification of exponentially many packages (exponential in $|N|$). Let us consider a subproblem which allows us to express a valuation with a smaller number of variables. Allowing for problem instances with fewer variables will be useful for assessing the complexity of the CMP below.[12] Let us restrict attention to valuations v_b such that:

$$\forall S \subseteq N, |S| - 1 \leq v_b(S) \leq |S| \tag{14}$$

In other words, the value of every package is between its cardinality and one less than its cardinality. Observe that (14) implies monotonicity. Next consider a collection $\{(S_i, r_i) : i \in I\}$ for some index set I satisfying $i \neq j \Rightarrow S_i \neq S_j$, and assume the for all $i \in I, S_i \subseteq N$ and $r_i \in [0, 1]$. Above, we used $\{(S_i, r_i) : i \in I\}$ to express a profile of single-minded valuations. Now we use $\{(S_i, r_i) : i \in I\}$ as a shorthand to express a single valuation for Bob through the equation:

$$v_b(Z) := \begin{cases} |Z| - r_i, & \text{if } Z = S_i; \\ |Z|, & \text{if for all } i \in I,\ S \neq Z_i. \end{cases} \tag{15}$$

v_b is well-defined because $i \neq j \Rightarrow S_i \neq S_j$. To summarize, we start with the additive valuation $w(S) = |S|$, and then declare a series of exceptions $\{(S_i, r_i) : i \in I\}$ where for each S_i, the value of S_i is not $w(S_i) = |S_i|$, but instead is $v_b(S_i) = |S_i| - r_i$. When an exception is not declared, then $v_b(S) = w(S) = |S|$. Notice that because we assume $r_i \in [0, 1]$, for all $\{(S_i, r_i) : i \in I\}$, the induced valuation v_b given by (15) satisfies (14). Likewise for any valuation v_b satisfying (14) consider the description $\{(S_i, r_i) : i \in I\}$ where for all $Z \subseteq N$, there exists $i \in I$ such that $S_i = Z$, and $r_i = v_b(S_i)$. Then $\{(S_i, r_i) : i \in I\}$ induces v_b via (15), so all valuations satisfying (14) can be expressed in this way. Given this notation, we can formulate the CMP for N as follows:

Input A description $\{(S_i, r_i) : i \in I\}$ with $i \neq j \Rightarrow S_i \neq S_j$ and $r_i \in [0, 1]$ for all i. This description expresses a valuation v_b via (15).

Output A partition $\mathcal{P} \in \Pi(N)$ that minimizes $\sum_{P \in \mathcal{P}} v_b(P)$.

[12] Indeed, if the description of valuations is always exponential in the number of goods, then there may trivially exist algorithms that have polynomial running time in the size of the description simply because the description is so large. For this reason, we allow for more compact representations of at least some valuations. This is common practice in assessment of complexity in such settings.

Notice that because Ann is required to win $Z = N$, $v_b(P|N - Z) = v_b(P)$ above. Now choose any partition $\mathcal{P} \in \Pi(N)$, and let $I_{\mathcal{P}} := \{i \in I : S_i \in \mathcal{P}\}$. Then observe that:

$$\sum_{P \in \mathcal{P}} v_b(P) = \sum_{i \in I_{\mathcal{P}}} |S_i| - r_i + \sum_{P \in \mathcal{P}: P \neq S_i, \forall i \in I} |P| = |N| - \sum_{i \in I_{\mathcal{P}}} r_i \tag{16}$$

Consider WDSMB and CMP applied to the same input. Then J is feasible in WDSMB exactly if $J \subseteq I_{\mathcal{P}}$ for some $\mathcal{P} \in \Pi(N)$ and J can be optimal in WDSMB only if $J = I_{\mathcal{P}}$ for some $\mathcal{P} \in \Pi(Z)$, because if $J \subsetneq I_{\mathcal{P}}$, for some $\mathcal{P} \in \Pi(N)$, there is a feasible J' strictly containing J.[13] It follows from (16) that $\sum_{P \in \mathcal{P}} v_b(P)$ is minimized whenever $\sum_{i \in I_{\mathcal{P}}} r_i$ is maximized, which in turn implies:

Theorem 5 *For v_b satisfying (14), the CMP for N is equivalent to the WDSMB.*

Of course, the theorem presupposes a certain way of expressing the valuations v_b. The assumptions that the valuation satisfies (14) (as well as the weaker assumption that all values are in $[0, 1]$), and that $Z = N$ in the CMP are inessential, and only simplify the translation.

It is important to note that in the case of the CMP, the pairs $\{(S, v_b(S)) : S \subseteq N\}$ are interpreted as *exclusive bids* since the auctioneer can serve at most one of them. In contrast in the WDSMB, the pairs $\{(S_i, r_i) : i \in I\}$ are interpreted as *non-exclusive bids* because the auctioneer can serve more than one. The main point in the translation is the re-interpretation of the exclusive bids of Bob as non-exclusive bids of a collection of single-minded bidders. The fact that in the translation we turn the minimization in the CMP into a maximization in the WDSMB is of course very superficial, since we could just as well think of the winner determination problem in a procurement setting where the auctioneer's objective is to minimize the sum of the costs; indeed, the translation can be interpreted in this way. To understand the translation heuristically in an example, re-interpret Bob's valuation in table (5)–which is a collection of exclusive bids–as a collection of non-exclusive single-minded bids to supply Z in a procurement auction, in which the auctioneer would like to select the collection of bids that supply Z at the lowest cost. Then the partitions in (5) can be reinterpreted as the set of possible ways of allocating the supply among the bidders, and the problem of finding the cost minimizing way of supplying the goods is mathematically equivalent to Ann's shill bid cost minimization problem when the same numbers are interpreted as Bob's valuation for various packages.

One consequence of Theorem 5 is that the CMP is NP-hard. This is because the theorem provides a reduction of WDSMB to CMP, and incidentally, also a reduction back from the CMP for N satisfying (5) back to WDSMB. The upshot is that the auctioneer and the bidder sponsoring shills face exactly the same hard problem. That optimal shill bidding is

[13] Here we assume that for all $i \in I$, $r_i > 0$, since any bid with $r_i = 0$ can be eliminated at the outset.

NP-hard also follows from previous results by Sanghvi and Parkes (2004). See see section 10.3 for a discussion of the relation to the current work.

7 Dominance and Risk

This section studies dominant strategies in the presence of shill bidding and the riskiness of shill bidding. A discussion of dominance requires a way of dealing with situations with multiple efficient allocations. If Ann bids truthfully and there are multiple efficient allocations, her utility is independent of which efficient allocation the VCG mechanism selects. In contrast, when Ann uses shills (or uses a single non-truthful bid), her utility may depend on which efficient allocation is selected by the VCG mechanism. In what follows, I present a notion of dominance which is independent of the tie-breaking rule employed by the VCG mechanism.

Definition 4 1. *Shill bid profile* $\overline{v}$ ***dominates*** $\overline{v}'$ ***for*** v_a *if:*

  (a) *For all valuations* v_b*, when Ann's valuation is* v_a*, and Bob submits* v_b*, Ann's utility is at least as high when she submits* $\overline{v}$ *as when she submits* $\overline{v}'$*, under two separate assumptions: (i) the VCG mechanism always breaks ties in the way the maximizes Ann's utility, and (ii) the VCG mechanism always breaks ties in the way that minimizes Ann's utility.*[14]

  (b) *There exists a valuation* v_b *such that* $\overline{v}$ *is strictly better than* $\overline{v}'$ *against* v_b *under at least one of the assumptions (i) or (ii).*

2. *Two shill bid profiles* $\overline{v}$ *and* $\overline{v}'$ *are* ***non-equivalent for*** v_a *if there exists a bid* v_b *for Bob such that* $\overline{v}$ *and* $\overline{v}'$ *give Ann different utility under either assumption (i) or (ii). Shill bid profile* $\overline{v}$ *is* ***dominant for*** v_a *if it dominates all other non-equivalent shill bid profiles for* v_a*.*

As an example of equivalence, notice that any single bid v is equivalent to the bid profile $(v, \mathbf{0})$, where $\mathbf{0}$ is the valuation that assigns 0 to all packages. Define a **non-shill bidding strategy** as any strategy under which Ann submits a bid under a single identity.

[14] Of course, actual execution of (i) and (ii) would require more information than is contained in the reports to the VCG mechanism. The purpose here is merely to avoid issues relating to the arbitrary ways in which changing one's bid may affect the allocation through some arbitrary tie-breaking rule. An alternative definition of dominance which would also be sufficient for the results to be presented below would be to simply restrict attention to profiles v_b for which the allocation given both reports $\overline{v}$ and $\overline{v}'$ would be unique.

Theorem 6 *1. Consider any $Z \subseteq N$ with $v_a(Z) > 0$, any $\mathcal{P} \in \Pi(Z)$ with $|\mathcal{P}| \geq 2$ and any profile of positive numbers $(r_P : P \in \mathcal{P})$. Let $\overline{v} = (v^{P,r_P} : P \in \mathcal{P})$. There exists neither a non-shill bidding strategy which dominates $\overline{v}$ for v_a, nor one which is equivalent to $\overline{v}$ for v_a. In particular, truthful bidding does not dominate $\overline{v}$ for v_a.*

2. *There does not exist a shill bidding strategy that dominates truthful bidding.*

3. *When shill bidding is possible, there is no dominant strategy in the VCG mechanism.*

The proof is in the appendix. In interpreting part 1 of the theorem, recall that from our notational conventions, v^{P,r_P} is a single-minded valuation for package P at value r_P. Part 1 of the theorem shows that there is a large class of shill bidding strategies which are sometimes better than truthful bidding. Part 3 is in sharp contrast to the standard analysis the VCG mechanism without shills, in which truthful bidding is a dominant strategy.

The previous theorem shows that it is impossible to rank shill bidding against truthful bidding in terms of dominance. Another potential criterion is risk. Let $U_a(\overline{v}, v_b; v_a)$ be Ann's utility when her true value is v_a, Bob's bid is v_b, and Ann uses shill bid profile $\overline{v}$.[15] Suppose that Ann has made a guess about Bob's valuation but there is a small probability that she may be wrong. What is the worst possible outcome for Ann? If Ann bids truthfully, the worst possible outcome is that she receives a utility of 0. Formally:

$$\min_{v_b} U_a(v_a, v_b; v_a) = 0$$

In contrast, notice that the optimal shill bidding strategies studied above require Ann to place high bids on the packages she wins. These bids may be higher than Ann's true value for these packages, and hence shill bidding in this way may be quite risky for Ann in the sense that in the worst case, Ann would have to pay more for these packages than they are worth to her. In other words, it may be that $\min_{v_b} U_a(\overline{v}, v_b; v_a) < 0$. One might think that whenever Ann bids in such a way as to upset the efficient allocation, so that she wins goods she would not win if she were to bid truthfully, she must bid above her value for certain packages and hence incur a worst case risk in which her utility is negative. However, the following theorem shows that this is incorrect:

Theorem 7 *There exist situations in which:*

1. *Ann has a strict incentive to use shills,*

2. *optimal shill bidding leads to an inefficient allocation, and*

3. $\min_{v_b} U_a(\overline{v}, v_b; v_a) = 0$ *for some optimal shill bid profile $\overline{v}$.*

[15] As explained above, $U_a(\overline{v}, v_b; v_a)$ may depend on the tie-breaking rule. However, for our purposes this will not matter, so henceforth interpret $U_a(\overline{v}, v_b; v_a)$ under any fixed tie-breaking rule.

The proof is in the appendix. The reason that the informal argument preceding the theorem is incorrect is that Ann can sometimes alter the allocation by distorting the relative values that she assigns to packages without exaggerating the absolute value of any package, and shill bidding can make this worthwhile. It follows that in general we can invoke neither dominance nor worst-case risk to rule out shill bidding.

8 Equilibrium

This section uses the preceding analysis to show that shill bidding is consistent with equilibrium.

Theorem 8 *Consider a VCG auction with two bidders who know each others' valuations. Both bidders have the option to use shills. Suppose that bidder 1 has a strictly supermodular valuation, and that bidder 2 receives at least two items at the efficient allocation. Then there is a Nash equilibrium in undominated strategies in which*

1. *Bidder 1 bids truthfully.*
2. *Bidder 2 wins some package Z containing at least two items. Bidder 2 sponsors one shill for each item x in Z. The shill for x bids single-mindedly for x.*
3. *Bidder 2's equilibrium payoff is strictly higher than her payoff if both bidders were to bid truthfully.*

Proof. In Appendix. □

Observe that in the above theorem, the package Z that bidder 2 wins is not necessarily the package that bidder 2 would win at an efficient allocation. I now sketch the main idea. Since bidder 1 has a supermodular valuation v_1, Theorem 4 tells us that bidder 1 has a best reply in which she sponsors one shill per item she wins. *Strict* supermodularity of v_1 implies that bidder 2 will do strictly better against the truthful bid v_1 by using shills than by bidding truthfully herself. Observe that if bidder 2 sponsors one shill per item she wins, the aggregate valuation faced by bidder 1 is additive. Additive valuations satisfy SubTop. So Theorem 3 implies that bidder 1 does not have an incentive to use shills against bidder 2's shills.

The difficult part of the proof of Theorem 8 consists in proving that bidder 2's strategy is undominated. (That bidder 1's strategy is undominated follows from Theorem 6, but Theorem 6 does not imply that bidder 2's strategy is undominated.) The difficulty stems from the fact that bidder 2 has a very large strategy space, and one must show that bidder 2's strategy is sometimes better than every (nonequivalent) strategy in this large space, based only on the limited information about bidder 2's preferences which is implicit in the theorem. For the theorem to have significance, it is important to prove that the equilibrium is in undominated strategies. To see this, note that even when shills are impossible, a second price auction for a single object (which is the VCG mechanism in that case) with perfect information has many equilibria in dominated strategies. For example, there is an equilibrium in which the bidder with the lowest value submits a very high bid and all other bidders submit a bid of zero. In contrast the unique perfect information equilibrium in undominated strategies in the VCG mechanism when shill bidding is disallowed is the strategy profile in which all bidders bid truthfully. The following corollary therefore also sharply contrasts the situation with no shills:

Corollary 2 *Whenever there are two bidders both of which (i) know one another's valuations, (ii) have the option to use shills, (iii) have strictly supermodular valuations, and (iv) win at least two items at the efficient allocation, there exist multiple Nash equilibira in undominated strategies which are not equivalent in terms of the bidders' payoffs.*

Theorem 8 and Corollary 2 apply when at least one of the bidders have a supermodular valuation. Recall that the case of supermodular valuations is that in which there is the purest incentive to use shills. Matters become much more complicated when a bidder's valuation fails to satisfy SubTop but is not supermodular. Then if the bidder sponsoring shills employs the canonnical strategies described by Theorem 1, he may employ a shill that bids single-mindedly for a package containing more than one item. However, this would create create an incentive for his opponent to use shills against him.

9 Collusion

I now use the above analysis of shill bidding to draw conclusions about collusion. The main result of this section is Theorem 9 which shows in a stark way that there is almost always either an incentive for a potential bidder to use shills or an incentive for a potential coalition of bidders to merge.

Consider a coalition J with valuations $\overline{v} = (v_j : j \in J)$ and aggregate valuation v_J. Assume away internal problems of enforcing collusive arrangements, so that J can efficiently collude. J bids against Bob (with valuation v_b), who is J's aggregate opponent. Assume–for expositional simplicity–that there is a unique efficient allocation; Z_j^* is the package assigned to bidder j at the efficient allocation, and $Z^* = \bigcup\{Z_j^* : j \in J\}$ is the set of items assigned to members of J collectively. For any $j \in J$, let $K^j = (J - j) \cup b$, where b is shorthand for Bob, and let v_{K^j} be the aggregate valuation of K^j. For any $j \in J$ and $Z \subseteq N$,

$$p_Z^{VCG,j} := v_{K^j}(Z|N - Z)$$

is the VCG price for Z if all others bid truthfully.

$$p_{Z^*}^{VCG,J} := \sum_{j \in J} p_{Z_j^*}^{VCG,j}$$

is J's payment if all bid truthfully. Since truthful bidding is a dominant strategy (assuming away the possibility of shill bidding), $p_{Z^*}^{VCG,J}$ represents J's payment under noncooperative behavior (given any assumptions about players' knowledge of others' bids). For any package $Z \subseteq N$,

$$p_Z^{\text{Merged}} := v_b(Z|N - Z)$$

would be the VCG price for Z if J merged to become a single entity with valuation v_J.

Theorem 3 implies that in a VCG auction for a single item (i.e., a second price auction), there is no incentive to use shills. Nevertheless, even in this simple case, it is well known that the VCG mechanism is prone to collusion (Graham and Marshall 1987). When it is efficient for some member of J to win the single item x, the incentive for J to collude is expressed by the inequality:

$$p_x^{\text{Merged}} \leq p_x^{VCG,J}, \tag{17}$$

The inequality is strict whenever both the bidders with the highest and second highest value reside in J. If J always submits v_J, which in a single item auction is a bid equal to the highest value for the item within J, then J will pay p_x^{Merged} whenever it wins. Moreover:

Fact 1 *Abstracting away from internal problems of enforcing collusive behavior, in a second price auction for a single item, submitting the single aggregate bid v_J is J's dominant strategy.*

Here, the benefit to merging comes from suppression of competition; merging eliminates losing bids, potentially reducing the opportunity cost of the item. However, in a combinatorial VCG auction, merging does more than just suppress competition. Recall the definitions of Z^* and Z_j^* above. Define:

$$p_{Z^*}^{\text{Suppressed}} := \sum_{j \in J} v_b(Z_j^*|N - Z^*)$$

Proposition 1 $p_{Z^*}^{\text{Supressed}}$ *is the smallest payment possible for coalition J when facing aggregate bid v_b subject to the constraint that for all $j \in J$, bidder j wins Z_j^*.*

$p_{Z^*}^{\text{Suppressed}}$ can be achieved if each member j of J for whom $Z_j^* \neq \emptyset$ submits a single-minded bid for Z_j^* at a sufficiently high value (where "sufficiently high" is explained by condition 3 of Theorem 1, where "shill bidder" is replaced by "coalition member" and P is replaced by Z_j^*). The proof of the proposition is similar to the proof of Theorem 1 and follows a logic which is similar to that in the example in section 3.2. The superscript "Suppressed" refers to suppressed competition. We can now decompose the effect of merging on the price the coalition pays into two terms:

$$p_{Z^*}^{\text{Merged}} - p_{Z^*}^{VCG,J} = \underbrace{(p_{Z^*}^{\text{Merged}} - p_{Z^*}^{\text{Suppressed}})}_{\text{Integration Effect}} + \underbrace{(p_{Z^*}^{\text{Suppressed}} - p_{Z^*}^{VCG,J})}_{\text{Competition Effect}}$$

The Competition Effect represents the reduction in payment due to suppressing competition. The Integration Effect represents the change in payment due to merging once competition has already been suppressed.

Definition 5 *A valuation v_b satisfies **Supermodularity at the Top (SupTop)** if for all $Z \subseteq N$ and $\mathcal{P} \in \Pi(Z)$:*

$$v_b(Z|N-Z) \geq \sum_{P \in \mathcal{P}} v_b(P|N-Z) \tag{18}$$

SupTop is the dual SubTop. Just as SubTop is weaker than submodularity, SupTop is weaker than supermodularity.

Proposition 2 *The Competition Effect is always nonpositive. The Integration Effect can be positive or negative. If goods are substitutes (i.e., v_b satisfies SubTop), then the Integration Effect is nonpositive and hence $p_{Z^*}^{\text{Merged}} \leq p_{Z^*}^{VCG,J}$. If goods are complements (i.e., v_b satisfies SupTop) then the Integration Effect is nonnegative.*

Proof. That the Competition Effect is always nonpositive follows from Proposition 1. The question of whether the Integration Effect is negative or positive is equivalent to the resolution of the following inequality:

$$v_b(Z^*|N-Z^*) \quad \lesseqqgtr \quad \sum_{j \in J} v_b(Z_j^*|N-Z_j^*)$$

SubTop implies that $\lesseqqgtr$ becomes $\leq$, and SupTop implies that $\lesseqqgtr$ becomes $\geq$. □

Corollary 3 *If coalition J knows a priori that Bob's bid v_b satisfies SubTop, then merging (i.e., submitting the single bid v_J) is a dominant strategy for J. If v_b fails to satisfy SubTop, then there exists some coalition J (i.e., some profile of valuations) that would be better off playing noncooperatively than merging.*

The proof is in the appendix. Corollary 3 shows exactly how far (17) and Fact 1 can be generalized when moving from an auction with a single item to a combinatorial auction. Merging automatically suppresses competition which is generally helpful to the coalition; however, the coalition can suppress competition without merging, and then whether or not merging is worthwhile depends on whether the coalition's opponent has a substitutes or complements valuation.

To sharpen the analysis it is useful to eliminate the Competition Effect–which is present even in an auction for a single item and does not depend on Bob's valuation–and to focus on the Integration Effect, which depends on Bob's valuation. Say that a coalition of bidders J with valuations $(v_j : j \in J)$ **is minimally competitive when facing** v_b if $p_{Z^*}^{VCG,J} = p_{Z^*}^{\text{Suppressed}}$, where Z^* is the package that would be efficiently allocated to J when J faces Bob with valuation v_b. So a minimally competitive coalition is one which has already suppressed competition, nullifying the Competition Effect. Say that v_b is **additive** if for all $Z \subseteq N$, $v(Z) = \sum_{x \in Z} v(\{z\})$. If v_b is additive, then Bob views all goods as being independent. Say that a coalition J has an **incentive to merge against** v_b, if the coalition J could do better by submitting a single bid on behalf of all of its members than by having all of its members bid truthfully in the VCG mechanism.[16]

Theorem 9 *The following are equivalent:*

1. *v_b is additive.*
2. *There exists neither (a) a bidder who has an incentive to shill against v_b, nor (b) a minimally competitive coalition that that has an incentive to merge against v_b.*

In order to prove this theorem, it is necessary to prove the following lemma:

Lemma 1 *The following are equivalent:*

1. *v_b is additive.*
2. *v_b satisfies both SubTop and SupTop.*

It is well known that among valuations v_b such that $v_b(\emptyset) = 0$, the additive valuations are the intersection of the submodular and supermodular valuations. Since SupTop is strictly weaker than supermodularity and SubTop is strictly weaker than submodularity, one might imagine that the intersection of the SubTop and SupTop valuations is larger than the set of additive valuations. However, Lemma 1 shows that indeed the intersection of SubTop and SupTop valuations is exactly the set of additive valuations. The proofs of Lemma 1 and Theorem 9 are in the appendix.

Theorem 9 is related to the following observation: *Merging is the inverse of Shill Bidding.*[17] That is to say, the disincentive to shill against a substitutes valuation translates into

[16]It is worth noting that it is possible that a coalition J may not have an incentive to merge against Bob, while some sub-coalition K of J may have an incentive to merge against the aggregate of Bob's bid and the truthful bids of the players in $K \setminus J$. The following theorem considers only merger of the entire coalition J, and not of its subcoalitions as just described.

[17]Milgrom (2004) presents an example which illustrates the incentive to merge in the VCG mechanism when goods are substitutes. The relationship between the incentive to split into multiple identities and the incentive for multiple identities to merge has been studied in other contexts; for instance, Moulin (2008) studies split-proofness and merge-proofness in scheduling problems.

an incentive to merge for a coalition, and the incentive to shill translates into a disincentive to merge for a coalition. Only for additive valuations do both of these incentives disappear simultaneously.

One should note that there is somewhat of an asymmetry between shill bidding and merging, insofar as merging–unlike shill bidding–does not lead to inefficient allocations among the bidders; however merging will reduce the seller's revenue.[18]

I conclude the section by mentioning that the problem of *optimal* collusion in this setting has essentially the same structure as the problem of optimal shill bidding (assuming the coalition can reallocate the goods after the auction). In particular, the cheapest way for a coalition to win a given package Z is:

$$p_Z^{\text{Shill},|J|} := \min\{\sum_{P \in \mathcal{P}} v_b(P|N-Z) : \mathcal{P} \in \Pi(B), |\mathcal{P}| \leq |J|\}$$

In comparison to the solution to the CMP p_Z^{Shill}, $p_Z^{\text{Shill},|J|}$ incorporates the additional constraint that the coalition cannot manufacture members, but must employ its own members. If there are as many coalition members as goods on auction, this additional constraint is not binding, and optimal collusion reduces to optimal shill bidding.

10 Related Literature and Extensions

In this section, I discuss the relations of some of the above results to the most related results in the literature. In the course of the discussion, I use the preceding analysis to derive new results which extend existing results. Sections 10.1 and 10.2 describe results of Lehmann, Lehmann, and Nisan (2006) (concerning submodular valuations) and Ausubel and Milgrom (2002) (concerning gross substitutes valuations) respectively which are related to Theorem 3 in that they provide substitutes conditions which deter shill bidding. In these sections, I also derive new results about submodular valuations and gross substitutes valuations. Section 10.3 discusses previous analyses by Sanghvi and Parkes (2004) and Conitzer and Sandholm (2006) which bear on the complexity of shill bidding strategies and are related to the analysis of Section 6.

[18] Ausubel and Milgrom (2002) show that if bidders satisfy gross substitutes valuations then there is no profitable coalitional deviation from the VCG outcome for losing bidders, but for any larger collection of valuations one can find a counter-example, in the same spirit as a result of theirs which I reproduce as Theorem 12 below (see the discussion in Section 10.2). The fact that a profitable coalitional deviation of losing bidders is not available when goods are substitutes is related to the fact that coalitions do not have an incentive to produce an inefficient allocation when goods are substitutes.

10.1 Submodularity

The section discusses the relation of Theorem 3 to a result of Lehmann, Lehmann, and Nisan (2006) about submodular valuations. In the process, I derive a qualified converse of the Lehmann, Lehmann, and Nisan (2006) result which *characterizes* submodularity in terms of the incentive to sponsor shills in certain auctions (Theorem 11). I also explain the general relationship between submodularity and SubTop.

Definition 6 *A valuation v is **submodular** if for all $Y \subseteq Z \subseteq N$ and $x \in N - Z$:*

$$v(x|Y) \geq v(x|Z)$$

Submodularity means that there is a decreasing marginal utility of additional goods as the set of goods already acquired increases. Therefore submodularity states that the different goods in N are substitutes for one another.

Theorem 10 *(Lehmann, Lehmann, and Nisan 2006) If v_b submodular, then, regardless of her valuation, there is no profitable shill bid for Ann against v_b in the VCG auction for N.*[19]

While submodularity is a sufficient condition for eliminating the incentive to use shills in the VCG auction for N, it is not a necessary condition (See Proposition 3). It then follows from Theorem 3 that SubTop is strictly weaker than submodularity. However what is the precise mathematical relationship between submodularity and SubTop? The answer follows from the following established mathematical fact (see for example, Fujishige (2005)):

Fact 2 *The following conditions are equivalent:*

1. *v_b is submodular.*
2. *For all $Y, Z \subseteq N$ with $Y \cap Z = \emptyset$ and all $\mathcal{P} \in \Pi(Z)$:*

$$v_b(Z|Y) \leq \sum_{P \in \mathcal{P}} v_b(P|Y)$$

Usually this result is not expressed in terms of partitions as above; a more common but equivalent expression is that v_b is submodular if and only if for all $Y \subseteq N$, the function $v_b(\cdot|Y)$ is subadditive. In other words, an alternative definition of submodularity of a valuation is that the marginal value of additional goods is subadditive conditional on any package already acquired. We can now directly compare SubTop and submodularity:

[19] Lehmann, Lehmann, and Nisan (2006) discuss the relation of this result to results in Sakurai, Yokoo, and Matsubara (1999) and Yokoo, Sakurai, and Matsubara (2000), the papers which originally introduced the problem of shill bidding in the VCG mechanism. These results have been updated and summarized recently in Yokoo, Sakurai, and Matsubara (2004) and Yokoo (2006).

- **Submodularity** For all $Y, Z \subseteq N$ with $Y \cap Z = \emptyset$ and all $\mathcal{P} \in \Pi(Z)$:

$$v_b(Z|Y) \leq \sum_{P \in \mathcal{P}} v_b(P|Y) \tag{19}$$

- **SubTop** For all $Z \subseteq N$ and $\mathcal{P} \in \Pi(Z)$:

$$v_b(Z|N-Z) \leq \sum_{P \in \mathcal{P}} v_b(P|N-Z)$$

Both conditions impose a set of inequalities of the form (19) for Y, Z with $Y \cap Z$ and $\mathcal{P} \in \Pi(Z)$, but whereas submodularity imposes these inequalities for all such Y and Z, SubTop only imposes these inequalities when $Y = N - Z$. Unlike submodularity, SubTop does not imply that $v_b(\cdot|Y)$ is subadditive for any Y with $|Y| < |N| - 2$. As a special case, when $|N| \geq 3$, and setting $Y = \emptyset$, unlike submodularity, SubTop does not imply that v_b is subadditive. For example, let $N = \{1, 2, 3\}$ and consider:

$$v_b(Z) := \begin{cases} 3, & \text{if } |Z| \geq 2; \\ 1, & \text{if } |Z| = 1; \\ 0, & \text{if } Z = \emptyset. \end{cases} \tag{20}$$

v_b satisfies SubTop, but v_b is not subadditive: $v_b(1) + v_b(2) = 2 < 3 = v_b(\{1, 2\})$.

For any $D \subseteq N$, define the **VCG auction for D** as the application of the VCG mechanism for allocating all goods in D. In this case we may assume either that bidders submit valuations v_b for all packages $Z \subseteq N$, but that the marginal value for goods outside of D are ignored, or that bidders submit the restriction of v_b to packages contained within D. In light of the fact that SubTop corresponds to satisfaction of the submodularity-like inequalities (19) only when $Y = N - Z$, or in words, interpreting Z as the package that Ann wins through her shills, these inequalities are required only when Bob already has *all goods outside of* Z. Noting that for all Y, Z with $Y \cap Z = \emptyset$, we have $Y = D - Z$ when $D = Y \cup Z$. So we can make Y into all goods outside of Z if we eliminate all goods outside of $D = Y \cup Z$. With this observation in mind, Theorem 3 and Fact 2 together imply:

Theorem 11 ***(The Incentive to Shill in SubAuctions)*** *The following conditions are equivalent:*

1. *v_b is submodular.*
2. *For all $D \subseteq N$, and for all valuations for Ann, there is no profitable shill bid for Ann against v_b in the VCG auction for D.*

This theorem exactly *characterizes* submodularity in terms of shill bidding, and provides a sort of converse to the result of Lehmann, Lehmann, and Nisan (2006). However, notice that the hypothesis of this converse is that Ann does not have a profitable shill bid in the VCG auction for D, *for all* $D \subseteq N$, not just in the VCG auction for N. If one wants instead the exact equivalent of the statement that Ann does not have a profitable shill bid in the VCG auction for a given set of goods N, then one needs to look to Theorem 3.

I conclude this section by formally stating a fact which was alluded to earlier:

Proposition 3 *SubTop is strictly weaker than submodularity; as a consequence, there exist auctions in which there is no incentive to use shills, but in which there may be an incentive to use shills in some subauction (i.e., if some goods are eliminated).*

This proposition follows from Fact 2 and the observation that the valuation presented in (20) satisfies SubTop but not submodularity.

10.2 Gross Substitutes

Theorem 3 characterizes Ann's incentive to use shills in terms of the aggregate valuation of her opponents. This subsection relates this to a characterization due to Ausubel and Milgrom (2002) in terms of the domain from which valuations are drawn–in terms of a well-studied class of valuations known at the *gross substitutes* valuations. To help explain this relationship, I present a new result–Theorem 13–which establishes an interesting relationship between the gross substitutes valuations and both the submodular and the SubTop valuations.

For any valuation v, define the demand correspondence induced by v by:

$$D(p; v) := \text{argmax}\{v(Y) - \sum_{x \in Y} p_x : Y \subseteq N\},$$

where $p \in \mathbb{R}^N_+$ is a price vector. Say that v is a **gross substitutes** valuation if for all $p, p' \in \mathbb{R}^N_+$ with $p \leq p'$ (i.e., $p_x \leq p'_x, \forall x \in N$) and $Y \in D(p)$, there exists $Z \in D(p')$ such that $\{x \in Y : p_x = p'_x\} \subseteq Z$. We can think of the set of all valuations

$$V = \{v \in \mathbb{R}^{2^N \setminus \{\emptyset\}}_+ : \forall \text{ nonempty } Y, Z \subseteq N, Y \subseteq Z \Rightarrow v(Y) \leq v(Z)\}$$

as a subset of $\mathbb{R}^{2^N \setminus \{\emptyset\}}_+$ (We do not need to specify $v(\emptyset)$ because $v(\emptyset)$ always equals 0). Let V_{GS} be the set of gross substitutes valuations.

Theorem 12 *(Ausubel and Milgrom 2002) Suppose that* $(v_1, \ldots, v_n) \in V^n_{GS}$*. Then assuming that all bidders other than 1 bid truthfully, bidder 1 has no incentive to use shills. On the other hand, for any* V_* *with* $V \supseteq V_* \supsetneq V_{GS}$*, there exists a natural number n and a profile* $(v_1, v_2, \ldots, v_n) \in V^n_*$ *such that bidder 1 has an incentive to use shills.*[20]

[20] Actually, Ausubel and Milgrom (2002) show something stronger, namely that every set V_* of valuations that contains either the additive valuations or the unit demand valuations and at least one valuation which

In contrast to the characterization of the incentive to shill in terms of the aggregate valuation of Ann's opponent, the result of Ausubel and Milgrom (2002) provides a maximal domain such that if valuations are drawn from this domain, there is no incentive to shill. Ausubel and Milgrom (2006) argue that this sort of a maximal domain characterization is superior to a characterization in terms of the aggregate valuation of Ann's opponents because the a priori knowledge that one is likely to have about bidders is likely to concern the domain from which valuations are drawn rather than the aggregate valuation of the opponent.

With a view to providing a counter-argument, consider the following facts: (i) every gross substitutes valuation is submodular (Gul and Stacchetti 1999), (ii) the set of gross substitutes valuations have zero Lebesgue measure in $\mathbb{R}_+^{2^N\setminus\{\emptyset\}}$, (iii) the set of submodular valuations have positive Lebesgue measure in $\mathbb{R}_+^{2^N\setminus\{\emptyset\}}$, and (iv) the aggregate valuation formed out of a collection of gross substitutes valuations is also a gross substitutes valuation ((ii), (iii), and (iv) are from Lehmann, Lehmann, and Nisan (2006)). Recall also that the SubTop valuations strictly contain the submodular valuations (Proposition 3). It follows from these considerations that there are many more ways of forming SubTop (also submodular) aggregate valuations from valuations failing to satisfy gross substitutes than from valuations satisfying gross substitutes. For example, if n valuations are chosen independently from V with an atomless full support probability measure on V, then the probability of choosing n gross substitutes valuations is zero, but the probability of choosing valuations with aggregate valuation that is either submodular or satisfies SubTop is positive. It follows that there are many possible circumstances in which bidders fail to have gross substitutes preferences but in which the VCG mechanism is immune to shills.

I now derive a result that sheds more light on the relationship between the gross substitutes valuations and the submodular and SubTop valuations. Say that a set $U \subseteq V$ of valuations is a **maximal domain** for a set $W \subseteq V$ of valuations if (i) for all natural numbers n and all $(u_1, \ldots, u_n) \in U^n$, the aggregate valuation,

$$w(Y) := \max\{\sum_{i=1}^{n} u_i(X_i) : \forall i, j = 1, \ldots, n, i \neq j \Rightarrow X_i \cap X_j \neq \emptyset, \bigcup_{k=1}^{n} X_i = Y\}, \tag{21}$$

is such that $w \in W$, and (ii) for all $U_* \subseteq V$, with $U_* \supsetneq U$, there exists a natural number n and $(u_1, \ldots, u_n) \in U_*^n$, such that w defined by (21) does not belong to W.

Theorem 13 *1. The gross substitutes valuations are a maximal domain for the submodular valuations.*

2. The gross substitutes valuations are a maximal domain for the SubTop valuations.

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violates the gross substitutes property is such that one can use valuations from V_* to construct a profile of valuations such that some bidder has an incentive to use shills. It is possible to strengthen the statement of Theorem 13 in a similar way. The proof in the appendix already accommodates such a strengthening.

Proof. In Appendix. □

This result explains the relationship of Theorem 3 to the result of Ausubel and Milgrom (2002). The theorem is also of independent interest, as it shows that one cannot enlarge the zero-Lebesgue measure set of gross substitutes valuations and be guaranteed to stay within positive Lebesgue measure set of submodular valuations or even the larger set of SubTop valuations when aggregating.

10.3 Complexity

In this section, I briefly discuss some previous results on complexity of manipulative strategies which are related to the analysis in Section 6. Sanghvi and Parkes (2004) studied a decision problem, which they called the *false-name manipulation problem*: given a profile of bids for all agents other than some bidder i (who plays the role of Ann in this paper), does i have a strategy using shills which gives i a utility of ϵ more than truthful bidding? Sanghvi and Parkes (2004) showed that the false-name manipulation problem is NP-hard, by reduction from EXACT-COVER-BY-THREE-SETS, a known NP-hard problem. It is an immediate consequence of this result that the problem of optimal shill bidding is also NP-hard. However, these authors did not analyze the relationship between optimal shill bidding and the winner determination problem, which is the main focus of Section 6.

Conitzer and Sandholm (2006) discuss the complexity of collusive strategies in the VCG mechanism. When all bidders are single-minded, they ask when a cartel can win all goods and make a payment of zero, and show that this problem is NP-complete. Conitzer and Sandholm (2006) assume that there is a fixed cartel with a fixed number of bidders, and what makes their question hard is the question of whether there are enough members in the cartel to win all items for free. If one always has access to an unlimited number of shills, as in this paper, then this question becomes computationally trivial: it is possible to win all items for no payment against a collection of single-minded bidders if and only if there are no bids on individual items.[21][22] So the analysis of that paper has no consequences for the complexity of optimal shill bidding without a bound on the number of shills, as in the scenario studied in Section 6. In contrast, these results do bear on the complexity of collusive strategies discussed in Section 9.

11 Conclusion

This paper has studied the problem of optimal shill bidding in the VCG mechanism. Complementarities create an incentive for bidders to disintegrate. When there is a mixture of

[21] Vincent Conitzer made this point at http://www.cs.duke.edu/courses/fall06/cps296.2/homework3.pdf.

[22] More generally, one can show that allowing for valuations which are not single-minded, it is possible to win all items for free if and only if the aggregate bid of the opponent assigns a value of zero to each package containing only a single item.

complements and substitutes, this incentive is attenuated, and bidders would like to partially disintegrate. As discussed in Section 6, the problem of exactly how to optimally disintegrate–in other words, the problem of how shill bidders should divide the package among themselves to minimize the payment–is a computationally hard problem, and is in fact equivalent to the problem of efficiently allocating goods among a collection of real bidders (Theorem 5). In the extreme case where goods are pure complements, there is a pure incentive to totally disintegrate so that each shill bidder bids for exactly one item (Theorem 4). In the case where goods are pure substitutes, these incentives are reversed, and there is an incentive for a collection of bidders to merge (Corollary 3). Shill bidding–and the inverse problem of merging–highlights a sense in which the VCG mechanism is unstable to the creation or combination of identities among the bidders. This instability exists except in the case when valuations are additive, or in other words, when the values for the different goods are independent (Theorem 9).

To arrive at a more precise assessment of this instability, it would be desirable to analyze further the risks of shill bidding when a bidder is uncertain about the aggregate bid that she faces. This paper has provided results which would be useful in an assessment of these risks. Theorem 3 provided conditions on a bidder's beliefs such that she would have no incentive to use shills (namely, if she assigns probability 1 to the aggregate bid of her opponent satisfying SubTop). Moreover, Section 7 undertook a detailed analysis of of dominance relations when shill bidding was possible. It was shown on the one hand, that there is a large class of shill bidding strategies which always outperform truthful bidding, but on the other hand, every shill bidding strategy is sometimes inferior to truthful bidding (Theorem 6); there is no clever shill bidding strategy which always at least matches and sometimes outperforms truthful bidding. So, in general, when an agent is uncertain about the opposing bids that she faces one cannot determine whether she will use shills on the basis of dominance. Moreover, Theorem 7 showed that surprisingly, there are situations where optimal shill bidding outperforms truthful bidding and upsets the efficient allocation, but the worst case risk from shill bidding is no worse than for truthful bidding. These results provide some information about shill bidding under uncertainty. To advance the analysis, it would be interesting to study Ann's optimization problem when she has an arbitrary belief (probability distribution) over the aggregate bid that she faces.

A further question is whether in equilibrium when all bidders anticipate that others may use shills, equilibrium behavior will endogenously generate bids which discourage the use of shills. Theorem 8 provides a negative answer to this question in the two bidder case when one bidder views the goods as complements.

12 Appendix

Proof of Theorem 1

The proof strategy is as follows. We start with an arbitrary shill bid profile $(v_j : j \in J)$ winning Z, and construct another profile satisfying properties 1-3 of Theorem 1 that wins Z but causes Ann to pay (weakly) less. This implies the existence of an optimum satisfying 1-3. Single-mindedness (property 2) and 3b imply that when the shill for P is excluded, the only reallocation involves giving P to Bob. In turn, this implies that this shill's payment is $v_b(P|N-Z)$, explaining the objective in (6). To see that shill profiles satisfying properties 1-3a but not 3b are suboptimal, observe that in the course of the proof, it is shown that any such profile can be strictly improved upon.

So let us consider a shill profile $\overline{v} = (v_j : j \in J)$ that wins Z against v_b. Then there must exist a partition $(P_j : j \in J) \in \Pi(Z)$ such that bidder j wins package P_j.[23] Let us tentatively set $r_{P_j} := v_j(P_j)$ and suppose that instead, Ann submitted profile $\overline{v}' = (v^{P_j, r_{P_j}} : j \in J)$ (i.e., bidder j is single minded for package P_j at value r_{P_j}). Notice that for all packages Y, $v^{P_j, r_{P_j}}(Y) \leq v_j(Y)$, implying that the value of every allocation is lower under $\overline{v}'$ than under $\overline{v}$. However, the value of the efficient allocation X selected by the VCG mechanism when Ann submits $\overline{v}$ is unchanged. Therefore X is still efficient under $\overline{v}'$. I will ignore the possibility that there are multiple efficient allocations as this case complicates the proof slightly without presenting any real problems.[24] Given that X is the unique efficient allocation, the VCG mechanism will still select X under $\overline{v}'$, and so shill bidder j will still win P_j. Let $\hat{v}_j$ be either v_j or $v^{P_j, r_{P_j}}$. Then j's VCG payment–under both $\overline{v}$ and $\overline{v}'$–takes the form:

$$p_j = \underbrace{\max\{v_b(X_b) + \sum_{\ell \in I \setminus j} \hat{v}_\ell(X_\ell) : (X_\ell : \ell \in (J \cup b) \setminus j) \in \mathcal{X}_{-j}\}}_{(*)} - \underbrace{[v_b(N-Z) + \sum_{\ell \in J \setminus j} \hat{v}_\ell(P_\ell)]}_{(**)},$$

where $\mathcal{X}_{-j}$ is the set of allocations when j is excluded from the auction. Notice that term (**) has the same value under both $\overline{v}$ and $\overline{v}'$. However term (*) is (weakly) lower under $\overline{v}'$ than under $\overline{v}$ because–as explained above–the value of every allocation is weakly lower under $\overline{v}'$. Therefore, p_j is lower for all shills j under $\overline{v}'$, and so Ann's total payment is lower. Henceforth, interpret $\hat{v}_j$ as $v^{P_j, r_{P_j}}$. Now suppose that for some shill bidder k, r_{P_k} is not large enough to win P_k in marginal economies excluding shills j exactly in some set $H \subseteq J \setminus k$.[25]

[23] We may assume that each member j of J wins at least one item, since otherwise we could eliminate the members of J who do not win any items. This would simply eliminate possible allocations and therefore reduce the value of the optimal allocation in the marginal economy excluding any shill bidder j who actually wins some items, and thus reduce j's payment.

[24] If there are multiple efficient allocations, then Ann may instead submit the profile $(v^{P_j, r_{P_j}+\epsilon} : j \in J)$ for small $\epsilon > 0$ so that X becomes the unique efficient allocation and the proof would proceed similarly.

[25] Here I mean that H is the set of shill bidders j such that r_{P_k} is not large enough for k to win P_k in *any*

Suppose that Ann raises r_{P_k} slightly. Of course this will not alter the VCG allocation, nor does it alter k's payment, as in the VCG mechanism, a bidder's payment is independent of his bid conditional on the allocation. Let us consider the effect on the payments of the other shill bidders j. There are two cases to consider. First suppose that $j \in H$. Raising r_{P_k} by a sufficiently small amount raises (**), but does not alter (*), because if r_{P_k} is sufficiently small, it will still be inefficient to allocate P_k to k in the marginal economy excluding k, and so the efficient allocation in this marginal economy is unchanged. It follows that in this case p_j is lowered. Next consider shill bidders $j \in J \setminus (H \cup k)$. Then raising r_{P_k} raises (*) and (**) by the same amount and therefore leaves p_j unaltered. It follows that if the components of the profile $(r_{P_j} : j \in J)$ are not initially large enough to satisfy property 3b, we may raise them until they do, and Ann's payment will be lowered in the process. After this change, $\overline{v}'$ will satisfy properties 1-3, completing the proof. □

Proof of Theorem 2

First we may argue–using an argument similar to one used in the proof of Theorem 1–that for every shill bid profile $\overline{v} \in \mathrm{Opt}(Z)$, and every $\epsilon > 0$, there exists a shill bid profile $\overline{v}' = (v^{P,r_P} : P \in \mathcal{P}) \in \mathrm{Opt}(Z)$ with $\mathcal{P} \in \Pi(Z)$ and $|\gamma(\overline{v}) - \gamma(\overline{v}')| < \epsilon$.[26] In words, $\overline{v}'$ contains one shill bidder for each package $P \in \mathcal{P}$ who bids single-mindedly for P at value r_P. The above observation shows that we may restrict attention to shill bid profiles of this form. We know that $\inf\{\gamma(\overline{v}) : \overline{v} \in \mathrm{Opt}(Z)\} \geq v_b(Z|N-Z)$. Assume for contradiction that $\inf\{\gamma(\overline{v}) : \overline{v} \in \mathrm{Opt}(Z)\} = v_b(Z|N-Z)$. Since $\Pi(Z)$ has only finitely many elements, there exists $\mathcal{P} \in \Pi(Z)$ and a sequence $\{\epsilon^k\}_{k=1,\ldots,\infty}$ with $\epsilon^k \geq 0$ for all k and $\epsilon^k \to 0$ as $k \to \infty$, such the for all k there exists exists $\overline{v}^k = (v^{P,r_P^k} : P \in \mathcal{P}) \in \mathrm{Opt}(Z)$ with:

$$\sum_{P \in \mathcal{P}} r_P^k = v_b(Z|N-Z) + \epsilon^k$$

Since $\overline{v}^k$ results in Ann winning Z, it follows that for all $P \in \mathcal{P}$, $r_P^k \geq v_b(P|N-Z)$. Choosing a subsequence if necessary, there exists $(r_P : P \in \mathcal{P})$ such that $(r_P^k : P \in \mathcal{P}) \to (r_P : P \in \mathcal{P})$ as $k \to \infty$. First suppose that for all $P \in \mathcal{P}, r_P = v_b(P|N-Z)$. Then $v_b(Z|N-Z) = \sum_{P \in \mathcal{P}} r_P = \sum_{P \in \mathcal{P}} v_b(P|N-Z)$, which implies–via Theorem 1–that for any k, Ann's payment under $\overline{v}^k$ is the same as for bidding under a single identity, implying that bidding under a single identity is optimal, a contradiction. So for some $P' \in \mathcal{P}$, $r_{P'} > v_b(P'|N-Z)$. Then:

$$\sum_{P \in \mathcal{P}-P'} r_P < v_b(Z|N-Z) - v_b(P'|N-Z) = v_b(Z-P'|(N-Z) \cup P')$$

efficient allocation in the marginal economy excluding j.

[26] Depending on the tie-breaking rule, it may not be possible to have $\gamma(\overline{v}) = \gamma(\overline{v}')$ when $\overline{v}$ induces multiple efficient allocations.

It follows that when the shill bidder for P' is excluded (for sufficiently large k), it is better (in terms of efficiency) to give everything to Bob than to let all shill bidders keep their packages. So in any efficient allocation in the marginal economy excluding P', some other shill for package P must lose his package. Theorem 1 implies that $\overline{v}^k$ (for sufficiently large k) is suboptimal, a contradiction. This completes the proof. $\square$

Proof of Theorem 6

1: Since truthful bidding dominates any non-shill bidding strategy, it is sufficient to show that truthful bidding does not dominate $\overline{v}$. In particular, suppose that Ann has two opponents, Ellen and Fred, so that Bob's valuations is the aggregate of the valuations of these two. Ellen values each item x in $N - Z$ at a value greater than $v_a(N)$, has zero value for all items in Z. Ellen's valuation extends to packages additively. Fred's valuation is $v^{Z,r}$, where $0 < r < \min\{\sum_{P \in \mathcal{P}} r_P, v_a(Z)\}$. If Ann bids truthfully, she will win package Z and pay r.[27] In contrast, if Ann submits shill profile $(v^{P,r_P} : P \in \mathcal{P})$, she will win B and make a zero payment.

2: Fix v_a, and assume for contradiction that there exists some shill bidding strategy $\overline{v} = (v_j : j \in J)$ that dominates truthful bidding. Let v_J be the aggregate valuation induced by $\overline{v}$. First suppose that there exists some $Z \subseteq N$ and some $x \in Z$ such that $v_J(Z) - v_J(Z - x) < v_a(Z) - v_a(Z - x)$. Suppose that Ann has two opponents, Ellen and Fred, so that Bob's valuation is the aggregate of these two. Ellen's valuation is v^{N-Z,r^*} where $r^* > \max\{v_J(N), v_a(N)\}$. Frank's valuation is $v^{x,r}$, where $v_J(Z) - v_J(Z - x) < r < v_a(Z) - v_a(Z - x)$. Then if Ann bids truthfully, her utility will be $v_a(Z) - r > v_a(Z - x)$, whereas $v_a(Z - x)$ will be an upper bound on Ann's utility if she reports $\overline{v}$, a contradiction. Next suppose that for all $Y \subseteq N$ and all $x' \in Y$, $v_J(Y) - v_J(Y - x') \geq v_a(Y) - v_a(Y - x')$, and for some $Z \subseteq N$ and $x \in Z$, $v_J(Z) - v_J(Z - x) > v_a(Z) - v_a(Z - x)$. Then suppose that Ellen is as above, and that Frank's utility is $v^{x,r}$ where $v_J(Z) - v_J(Z - x) > r > v_a(Z) - v_a(Z - x)$. Then Bob's utility from truthful bidding is exactly $v_a(Z - x)$ and Bob's utility from submitting $\overline{v}$ is exactly $v_a(Z) - r < v_a(Z - x)$,[28] a contradiction. The only case that remains is when $v_J(Z) - v_J(Z - x) = v_a(Z) - v_a(Z - x)$ for all $Z \subseteq N$ and all $x \in Z$. Since $v_J(\emptyset) = v_a(\emptyset) = 0$, it follows that $v_J = v_a$. The fact that $\overline{v}$ dominates v_a implies that there must exist at least one package Z with $v_a(Z) > v_a(Z - x)$ for all $x \in Z$ and such that there is an efficient allocation of Z to the shills in J (taking their bids at face value) in which at least two shills receive items. If this were not the case, then for

[27] More precisely, Ann will win a package Y such that $v_a(Y) = v_b(Z)$ and pay r. It is possible that $Y \neq Z$ if there exists $x \in Z$ such that $v_a(Z - x) = v_a(Z)$.

[28] Here we have used the fact that for all $Y \subseteq N$ and all $x' \in Y$, $v_J(Y) - v_J(Y - x') \geq v_a(Y) - v_a(Y - x')$ to ensure that whenever some package has positive marginal value for Ann, v_J also assigns positive marginal value for that packages, so that goods in Z that Ann truly values are allocated to Ann when she submits shill bid profile $\overline{v}$.

every bid of Bob v_b, and every efficient allocation between Ann and Bob given this bid,[29] Ann's payment would be at least as high under $\overline{v}$ as under v_a for the same allocation, a contradiction. Suppose that for each shill bidder $j \in J$, Z_j is the package that shill bidder j wins under an efficient allocation of Z to the members of J in which at least to members of J receive items. Note that $\bigcup_{j \in J} Z_j = Z$. Suppose again that Ann has two opponents, Ellen and Frank, whose valuations are aggregated by Bob. Ellen's valuation is as above. Frank's valuation v_f is such that $0 < v_f(Z) < \min\{v_a(Z) - v_a(Z - x) : x \in Z\}$ and that $\sum_{j \in J} v_f(Z_j) > v_f(Z)$. Notice that when Ann bids against Ellen and Frank with valuations as just described, and submits shill bid profile $\overline{v}$, it is efficient to allocate Z_j to shill bidder j for all $j \in J$. Ann's payment will then be at least $\sum_{j \in J} v_f(Z_j)$. On the other hand, if Ann bids truthfully, at the unique efficient allocation, she will win Z and her payment will be exactly $v_f(Z)$, which is lower, and so she will be better off, a contradiction.
3: Part 1 of the theorem implies that if Ann has a dominant it must involve the use of shills, and cannot be equivalent to any non-shill bidding strategy. It then follows from Part 2 of the theorem that any such strategy cannot be dominant. □

Proof of Theorem 7

Suppose that $N = \{1, 2, 3\}$ and Ann and Bob's valuations are given by the following table:

Package	1	2	3	12	13	23	123
Ann	2	2	4	5	4	4	5
Bob	0	0	3	5	3	3	5

Then the efficient allocation assigns item 3 to Ann and the package $\{1, 2\}$ to Bob. Ann's payment is 0, and hence her utility is 4. Suppose that instead of presenting her true valuation, Ann sponsors two shills, Carol and Dan. Carol bids single-mindedly for item 1 at a value of 2, and Dan bids single-mindedly for item 2 at a value of 2. Then Ann will win items 1 and 2 through her shills, and Bob will receive item 3, an inefficient allocation with respect to Ann and Bob's true valuations. Ann's payment will be 0, and so her utility will be 5, which is better than under truthful bidding. The aggregate value of Ann's shills for any package is no greater than Ann's actual value for that package, implying that $\min_{v_b} U_a(\overline{v}, v_b; v_a) = 0$, where $\overline{v}$ is the shill bid profile described above. One can verify that the $\overline{v}$ is an optimal shill bidding strategy in this case. □

[29] by assumption, the set of efficient allocations between these two players does not depend on whether Ann submits v_a or $\overline{v}$.

Proof of Theorem 8

Let Y^*–which contains at least two elements–be the package bidder 2 wins at the efficient allocation (given the true valuations), and let v_1 (resp. v_2) be bidder 1's (resp., bidder 2's) valuation. Assume bidder 1 bids truthfully. Because v_1 is supermodular, Theorem 4 provides a shill bidding strategy under which bidder 2 wins Y^* for a weakly lower payment than under truthful bidding, and since v_1 is *strictly* supermodular, this payment is strictly lower than under truthful bidding. So bidder 2 has a best reply $\overline{v}^*$ to bid v_1 which outperforms truthful bidding and under which bidder 2 wins some package Z^*. This implies that $\overline{v}^*$ must involve multiple bidders and $|Z^*| \geq 2$. Theorem 4 implies that $\overline{v}^*$ may be chosen so that $\overline{v}^* = (v^{x,r^*_x} : x \in Z^*)$ for sufficiently large $r^* := (r^*_x : x \in Z^*) \in \mathbb{R}^{Z^*}_+$. We may assume that Z^* is chosen to be *minimal* in the sense that there does not exist Y strictly included in Z^* such that bidder 2 wins Y at some best reply to v_1. Because the aggregate valuation formed from the valuations in $\overline{v}^*$ is additive, this aggregate valuation satisfies SubTop, and Theorem 3 implies that bidding truthfully is indeed a best reply for bidder 1 against $\overline{v}^*$.

Theorem 6 implies v_1 is not dominated for bidder 1. It remains only to show that $\overline{v}$ is not dominated for bidder 2 if r^* is chosen appropriately. First we argue for $\overline{v}^*$ in the previous paragraph to have been a best reply to v_1 in the previous paragraph, it is sufficient to choose r^* to solve:

$$\min \sum_{x \in Z^*} r_x \tag{22}$$

$$\text{s.t.} \quad \sum_{x \in Y} r_x \geq v_1(Y|N - Z^*), \ \forall Y \subseteq Z^*, \tag{23}$$

$$\sum_{x \in Y} r_x \geq v_1(Y|(N - Z^*) \cup y), \ \forall y \in Z^*, \forall Y \subseteq Z^* - y, \tag{24}$$

$$\sum_{x \in Y} r_x \geq v_2(Y), \ \forall Y \subseteq Z^*, \tag{25}$$

Any $r' = (r'_x : x \in Z^*)$ that satisfies all constraints (23) with strict inequality is such that $\overline{v}' = (v^{x,r'_x} : x \in Z^*)$ wins Z^* against v_1. Strict supermodularity of v_1 implies that if r' satisfies all constraints (24), r' satisfies constraints (23) for all Y strictly contained in Z^* with strict inequality. Theorem 1 implies that any r' that satisfies all constraints (23)-(24) with strict inequality is such that $\overline{v}'$ belongs to $\text{Opt}(Z^*)$ (evaluated with respect to v_1). It then follows from Theorem 2 that any r' that satisfies (23)-(24), satisfies (24) for $Y = Z^*$ with a strict inequality. Continuity of payments in opponents' bids[30] conditional on a fixed allocation then implies that for any r' satisfying (23)-(24), $\overline{v}' \in \text{Opt}(Z^*)$. It follows that r^* can indeed be chosen to solve (22)-(25). Notice that we have shown that the constraints (23) in the program (22)-(25) are not binding.

[30] Here the relevant opponents are the other shill bidders for bidder 2.

In what follows, for any $Y \subseteq N$, define $r^*(Y) = \sum_{x \in Y \cap Z^*} r^*_x$. Next define:

$$\tilde{v}_1(Y) := v_1(Y - (N - Z^*)|N - Z^*) + 2|(N - Z^*) \cap Y|v_1(N) \tag{26}$$

Because v_1 is strictly supermodular, $v_1(N) > 0$. It is straightforward to show that:

$$\forall W \supseteq N - Z^*, \qquad v_1(\cdot|W) = \tilde{v}_1(\cdot|W) \tag{27}$$

$$\forall W \not\supseteq N - Z^*, \forall Y \subseteq N - W, \qquad v_1(Y|W) < \tilde{v}_1(Y|W) \tag{28}$$

Since Z^* was chosen to be minimal (as explained above), it follows from (27)-(28) that in any best reply to $\tilde{v}_1$, bidder 2 wins exactly Z^*, and (27) implies that $\overline{v}^*$ is a best reply for bidder 2 to $\tilde{v}_1$.

Consider an arbitrary shill bid profile $\overline{w} = (w_j : j \in J)$ for bidder 2. Let w be the aggregate valuation of the bidders in J, and for any $j \in J$, let w_{-j} be the aggregate valuation of all bidders other than j.

Lemma 2 *Assume that $\overline{w}$ dominates $\overline{v}^*$. Then at every efficient allocation of Z^* to the members of J with valuations $\overline{w}$, for all $z \in Z^*$, there exists $j_x \in J$ who receives exactly item x.*

Proof. Let $X = (X_j : j \in J)$ be an efficient allocation of Z^* to J, where X_j is the package received by j. Assume for contradiction that for all such allocations X there exists $k \in J$ with $|X_k| > 1$. Because $\overline{w}$ dominates $\overline{v}^*$, $\overline{w}$ wins exactly Z^* against $\tilde{v}_1$. Under $\overline{w}$, bidder 2's payment is at least $\sum_{j \in J} \tilde{v}_1(X_j|N - Z^*)$. Moreover $\sum_{j \in J} \tilde{v}_1(X_j|N - Z^*) = \sum_{j \in J} v_1(X_j|N - Z^*) > \sum_{x \in Z^*} v_1(x|N - Z^*) = \sum_{x \in Z^*} \tilde{v}_1(x|N - Z^*)$, where the last term is bidder 2's payment under $\overline{v}^*$, and the strict inequality follows from strict supermodularity of v_1. So $\overline{v}^*$ outperforms $\overline{w}$ against $\tilde{v}_1$, contradicting the assumption that $\overline{w}$ dominates $\overline{v}^*$. It follows that there exists at least one efficient allocation $X = (X_j : j \in J)$ of Z^* to J under $\overline{w}$ in which $|X_j| \leq 1$. For all $x \in Z^*$, let j_x be the unique member of J for which $X_j = \{x\}$. Moreover, for at least one such efficient allocation–and in particular any allocation of Z^* to J under $\overline{w}$ when $\overline{w}$ at least matches the performance of $\overline{v}^*$ against $\tilde{v}_1$–letting $r_x = w_{j_x}(x)$ for all $x \in Z^*$, $(v^{x,r_x} : x \in Z)$ must belong to $\text{Opt}(Z^*)$ with respect to $\tilde{v}_1$, which implies that $r = (r_x : x \in Z^*)$ must satisfy all inequalities (23) for $Y^* \subsetneq Z^*$ with a strict inequality, and hence $w_{j_x}(x) = r_x > \tilde{v}_1(x|N - Z^*) = v_1(x|N - Z^*)$. Assume for contradiction that there exists another efficient allocation $(X'_j : j \in J)$ of Z^* to J under $\overline{w}$ such that $|X'_k| > 1$ for some $k \in J$. Again, if $\overline{w}$ dominates $\overline{v}^*$, the VCG mechanism must allocate Z^* to bidder 2 under $\overline{w}$ against $\tilde{v}_1$. If the VCG mechanism gives Z^* to bidder 2 via allocation X', then an argument exactly as above shows that $\overline{v}^*$ with outperform $\overline{w}$ against $\tilde{v}_1$, a contradiction. So suppose the VCG mechanism gives Z^* to bidder 2 via allocation X. There must be at least one identity $\ell = j_y \in J$ such that $X_\ell \neq \emptyset$ but $X'_\ell = \emptyset$. If ℓ is excluded from the auction, then one feasible allocation is to give each $j \in J \setminus \ell$, X'_j, and to give bidder 1 $N - Z^*$. So ℓ's payment must be at least $w(Z^*) - \sum_{x \in Z^* - y} w_{j_x}(x) = w_{j_y}(y) > \tilde{v}_1(y|N - Z^*)$. Every other $j_x \in J$ must pay at least $\tilde{v}_1(x|N - Z^*)$. So, again, it follows that against $\tilde{v}_1$, the payment under $\overline{w}$ is greater than the payment under $\overline{v}^*$, a contradiction. □

Lemma 3 *Assume that $\overline{w}$ dominates $\overline{v}^*$. For some efficient allocation of Z^* to J under $\overline{w}$, let j_x be as in Lemma 2. Then $w_{-j_x}(x) = 0$.*

Proof. Consider a valuation u such that $u(Y) := 2|Y - x| \max\{w(N), r^*(N)\}$. If bidder 2 submits $\overline{v}^*$ against u, she will win x and pay nothing. If bidder 2 submits $\overline{w}$, she will win x and pay $w_{-j_x}(x)$. If $\overline{w}$ dominates $\overline{v}^*$, we must have $w_{-j_x}(x) = 0$. □

Corollary 4 *Assume that $\overline{w}$ dominates $\overline{v}^*$. Then there exists a unique efficient allocation of Z^* to the members of J with valuations $\overline{w}$, in which for all $z \in Z^*$, there exists $j_x \in J$ who receives exactly item x.*

Proof. Lemma 2 implies that if there are two distinct efficient allocations X and X' of Z^*, then under X (resp., X'), for all $x \in Z^*$, there exists $j_x \in J$ (resp., $j'_x \in J$) such that j_x wins x (resp., j'_x wins x). If $j_x \neq j'_x$, then by Lemma 3, $w_{j_x} = w_{j'_x} = 0$. Recall that $|Z^*| \geq 2$ and choose $y \in Z^* - x$. It follows that there is an efficient allocation of Z^* to J in which for all $z \in Z^* - \{x, y\}$, j_z wins z, and j_y wins $\{x, y\}$, contradicting Lemma 2. □

To complete the proof, I consider a number of cases which exhaust all possibilities and in each case, I construct a valuation against which $\overline{v}^*$ outperforms $\overline{w}$, so that $\overline{w}$ does not dominate $\overline{v}^*$. In light of Corollary 4, I will henceforth assume that at the unique efficient allocation of Z^* to the members of J under valuation $\overline{w}$, for each $x \in Z^*$, shill bidder j_x receives exactly x.

Case 1: $\exists \mathbf{Y}' \subseteq \mathbf{Z}^*, \mathbf{r}^*(\mathbf{Y}') < \mathbf{w}(\mathbf{Y}')$. Choose $\epsilon > 0$ so that $r^*(Y') + |Y'|\epsilon < w(Y')$. Consider a valuation u with $u(x) = r^*_x + \epsilon$ if $x \in Y'$, and $u(x) = 2\max\{w(N), r^*(N)\}$ if $x \notin Z$. Extend u additively to all subsets Y of N. If bidder 2 uses $\overline{v}^*$ against u, then bidder 2 will win nothing and pay nothing, attaining a utility of 0. If bidder 2 uses $\overline{w}$ against u, bidder 2 will not win any items outside of Y'. Moreover bidder 2 will win some nonempty subset Y of Y', because $w(\emptyset) + u(N) < w(Y') + u(N - Y')$, implying that allocating nothing to the identities in J is inefficient. However, additivity of u implies that bidder 2 will have to pay at least $u(Y) = r^*(Y) + |Y|\epsilon > v_2(Y)$, where the inequality is from (25), and hence will attain a negative utility, and so $\overline{v}^*$ outperforms $\overline{w}$ in this case.

Case 2: $\forall \mathbf{Y} \subseteq \mathbf{Z}^*, \mathbf{w}(\mathbf{Y}) \leq \mathbf{r}^*(\mathbf{Y})$ and $\exists \mathbf{Y}' \subseteq \mathbf{Z}^*, \mathbf{w}(\mathbf{Y}') < \mathbf{r}^*(\mathbf{Y}')$. Since the constraints (23) are not binding, there are two possibilities: (i) $w(Y') < v_2(Y')$ or (ii) $w(Y') < v_1(Y'|(N - Z^*) \cup y)$ for some $y \in Z^* - Y'$. First assume (i). Choose $\epsilon > 0$ so that

$w(Y') + |Y'|\epsilon < v_2(Y')$. Define valuation u' by:

$$u(Y) = \begin{cases} w(Y') + \epsilon + 2\max\{w(N), r^*(N)\}, & \text{if } Y = N; \\ 2\max\{w(N), r^*(N)\}, & \text{if } N - Y' \subseteq Y \subsetneq N; \\ 0, & \text{otherwise.} \end{cases}$$

Then if bidder 2 submits $\overline{w}$ against u, she will win nothing an pay nothing. If bidder 2 submits $\overline{v}^*$ against u, then she will win Y'. For each $x \in Y'$, the shill for x will pay $w(Y') + \epsilon - r^*(Y' - x)$ if $r^*(Y' - x) < w(Y') + \epsilon$ and 0 otherwise. Let $Y'' := \{x \in Y' : r^*(Y' - x) < w(Y') + \epsilon\}$. Then bidder 2's payment is equal to:

$$\sum_{x \in Y''} (w(Y') + \epsilon - r^*(Y' - x)) = |Y''|(w(Y') + \epsilon - r^*(Y' - Y'')) - (|Y''| - 1)r^*(Y'')^{31}$$

$$\leq |Y''|(w(Y') + \epsilon) - (|Y''| - 1)r^*(Y') \leq |Y''|(w(Y') + \epsilon) - (|Y''| - 1)w(Y') = w(Y') + |Y''|\epsilon$$

So $w(Y') + |Y''|\epsilon$ is an upper bound on bidder 2's payment and $w(Y') + |Y''|\epsilon < v_2(Y')$, so bidder 2 receives a positive payoff under $\overline{v}^*$, and so $\overline{v}^*$ outperforms $\overline{w}$ in this case.

Next assume (ii). Suppose that bidder 1 submits $\tilde{v}_1$. Recall that $\overline{v}^*$ is a best reply to $\tilde{v}_1$. Also, at any best reply to $\tilde{v}_1$, bidder 2 wins Z^*. If bidder 2 submits $\overline{w}$, for each item $x \in Z^*$, j_x that wins exactly x against $\tilde{v}_1$. It follows that the bid profile $\overline{v}' = (v^{x, w_{j_x}(x)} : x \in Z^*)$ would win Z^* and cause bidder 2 to make no larger a payment than under $\overline{w}$. However notice that (ii) and the fact that the aggregate value of bidder 2's shill bidders for any package is weakly lower under $\overline{v}'$ than under $\overline{w}$ implies that $\sum_{x \in Y'} w_{j_x}(x) < v_1(Y'|(N - Z^*) \cup y) = \tilde{v}_1(Y'|(N - Z^*) \cup y)$, where the last equality follows from (27). However this implies that when shill bidder j_y is excluded, giving $Y' \cup y \cup (N - Z^*)$ to bidder 1 is more efficient than giving only $y \cup (N - Z^*)$ to bidder 1, and hence some shill bidder did not bid high enough to retain his package when j_y is excluded. So by Theorem 1, $\overline{v}'$–and hence $\overline{w}$–is a suboptimal shill bid profile against v_1'. So $\overline{v}^*$ outperforms $\overline{w}$ against $\tilde{v}_1$.

Case 3: $\forall \mathbf{Y} \subseteq \mathbf{Z}^*, \mathbf{w}(\mathbf{Y}) = \mathbf{r}^*(\mathbf{Y})$ **and** $\exists \mathbf{Y}' \subseteq \mathbf{N}, \mathbf{w}(\mathbf{Y}') \neq \mathbf{r}^*(\mathbf{Y}')$. Choose Y' to be minimal according to inclusion. Then there exists $k \in J$ such that $w_k(Y') = w(Y') > r^*(Y')$. Since $|Z^*| \geq 2$, there exists $y \in Z^*$ such that $i_y \neq k$. Let $S =: Y' - Z^*$. Then $S \neq \emptyset$. Let $T := N - (S \cup Z^*)$. The following lemma will be useful:

Lemma 4 *In Case 3, if $\overline{w}$ dominates $\overline{v}^*$, $w_{-j_x}(Z^*) = r^*(Z^* - x)$ for all $x \in Z^*$.*

Proof. Note that in Case 3, $w_{-j_x}(Z^*) \geq r^*(Z^* - x)$. Assume for contradiction that $w_{-j_x}(Z^*) > r^*(Z^* - x)$. Then suppose that bidder 1 submits utility function u' with $u'(Y) = 2|Y - Z^*|w(N)$. Then if bidder 2 submits $\overline{v}^*$, she will win Z^* and pay nothing. If bidder 2 submits $\overline{w}$, she will win Z^* and pay $w_{-j_x}(Z^*) - r^*(Z^* - x) > 0$, contradicting the assumption that $\overline{w}$ dominates $\overline{v}^*$. □

[31] Notice that $r^*(Y'') = 0$ if $|Y''| = 0$.

Given the lemma, we may assume that $w_{-j_x}(Z^*) = r^*(Z^* - x)$ for all $x \in Z^*$. Let γ be a very large number; throughout the argument, I will omit qualifications of the form "if γ is sufficiently large", but such qualifications will often be implicitly assumed. Likewise let $\epsilon > 0$ be some very small number. Let

$$u(Y) = \begin{cases} \gamma|T \cup (Z^* - Y')| + \max\{0, w(Y') + \epsilon - r^*(Z^*)\}, & \text{if } T \cup (Z^* - Y') \cup S \subseteq Y; \\ \gamma|T \cup (Z^* - Y')| + w(Y') + \epsilon - r^*(Z^*), & \text{if } T \cup S \subseteq Y \\ & \text{and } T \cup (Z^* - Y') \not\subseteq Y; \\ \gamma|T \cup (Z^* - Y')|, & \text{if } T \cup (Z^* - Y') \subseteq Y \\ & \text{and } T \cup S \not\subseteq Y; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $T \cup (Z^* - Y') = \emptyset$ only if $Y' = N$, in which case $w(Y') > r^*(Y') = r^*(Z^*)$. So u is monotone and non-negative valued even in this case. Suppose that bidder 1 submits u. u is constructed in such a way that in what follows it is sufficient to consider only two allocations: (A) bidder 1 gets $T \cup S$, and bidder 2 gets Z^*, and (B) bidder 2 gets $T \cup (Y' - Z^*)$ and bidder 1 gets Y'.[32] Suppose that bidder 2 submits $\overline{v}^*$. Then under allocation (A), the total utility is $\gamma|T \cup (Z^* - Y')| + w(Y') + \epsilon - r^*(Z^*) + r^*(Z^*) = \gamma|T \cup (Z^* - Y')| + w(Y') + \epsilon$. Under allocation (B), total utility is $\gamma|T \cup (Z^* - Y')| + r^*(Y') < \gamma|T \cup (Z^* - Y')| + w(Y')$, which implies that allocation (A) is superior, so that bidder 2 will win Z^*, and the shill bidder for x will pay $\max\{0, r^*(Y' - x) - (w(Y') + \epsilon - r^*(Z^*) + r^*(Z^* - x))\} = \max\{0, r^*(Y' \cup x) - w(Y') - \epsilon\}$. Next, suppose that bidder 2 submits $\overline{w}$. Under allocation (A), total utility will be $\gamma|T \cup (Z^* - Y')| + w(Y') + \epsilon$ just as before. Under allocation (B), total utility will be $\gamma|T \cup (Z^* - Y')| + w(Y')$, so again allocation (A) is efficient, and bidder 2 wins Z^*. Notice also that r^*_{-x}–the aggregate valuation from $\overline{v}^*$ when the shill for x is excluded–is such that $r^*_{-x}(Y) \leq w_{-i_x}(Y)$ for all $Y \subseteq N$. This implies that for all $x \in Z^*$, the shill for x under $\overline{v}^*$ will pay no more than i_x under $\overline{w}$. Next consider i_y and recall that $i_y \neq k$. Suppose that bidder 2 submits $\overline{w}$, and consider the efficient allocation when i_y is excluded. Under allocation (A), the total utility is $\gamma|T \cup (Z^* - Y')| + w(Y') + \epsilon - r^*(Z^*) + w_{-j_y}(Z^*) = \gamma|T \cup (Z^* - Y')| + w(Y') + \epsilon - r^*(Z^*) + r^*(Z^* - y) = \gamma|T \cup (Z^* - Y')| + w(Y') - r^*_y + \epsilon$. Under allocation (B), the total utility is $\gamma|T \cup (Z^* - Y')| + w_{-j_y}(Y') = \gamma|T \cup (Z^* - Y')| + w(Y')$, where the equality follows from the fact that $i_y \neq k$, and so allocating Y' to k is feasible when i_y is excluded, and $w_k(Y') = w(Y')$. Since–as established above–$r^*_y > 0$, it follows that if ϵ is sufficiently small, allocation (B) is strictly more efficient than allocation (A). So i_y's payment is: $w(Y') - (w(Y') + \epsilon - r^*(Z^*) + r^*(Z^* - y)) = r^*_y - \epsilon > 0$. Moreover, notice that $r^*_y - \epsilon = r^*_y + w(Y') - (w(Y') + \epsilon) > r^*_y + r^*(Y') - (w(Y') + \epsilon) \geq r^*(Y' \cup y) - (w(Y') + \epsilon)$. It follows under $\overline{w}$, i_y pays strictly more than the shill for y under $\overline{v}^*$, and hence $\overline{v}^*$ outperforms

[32]Strictly speaking bidder 1 may get either Y' or a subset of Y' that gives him the same utility given his reported valuation in case he doesn't value some goods in Y'. A similar comment applies to Z^* under allocation (A).

$\overline{w}$ against u.

Case 4: $\forall \mathbf{Y} \subseteq \mathbf{N}, \mathbf{w}(\mathbf{Y}) = \mathbf{r}^*(\mathbf{Y})$.

Lemma 5 *In Case 4, if* $\overline{w}$ *dominates* $\overline{v}^*$*, for any* $Y \subseteq Z^*$*, the unique efficient allocation of* Y *to* J *given valuation profile* $\overline{w}$ *is to assign each* $x \in Z^*$ *to* j_x*.*

The proof of the lemma is similar to arguments presented above and so is omitted. Fix a bid profile $\overline{u}$ submitted by bidder 1. It is efficient for bidder 2 to win Y' if 2 submits $\overline{v}^*$ if and only if it is efficient for bidder 2 to win Y' against $\overline{w}$. Observe that under both $\overline{v}^*$ and $\overline{w}$, bidder 2's payment is the same if he wins Y' as if he wins $Y' \cap Z^*$. The unique efficient allocation of $Y' \cap Z^*$ to bidder 2 under $\overline{v}^*$ is to assign each $x \in Z^* \cap Y'$ to the shill for x, and by Lemma 5 we may assume that the unique efficient allocation to 2 under $\overline{w}$ assigns each $x \in Z^* \cap Y'$ to i_x. Since $r^*_{-x}(Y) \leq w_{-i_x}(Y)$ for all $x \in Z^*$, the shill for x under $\overline{v}^*$ pays no more than i_x under $\overline{w}$. □

Proof of Corollary 3

The first statement follows from the same logic as Theorem 3. Bidding under separate real identities in Corollary 3 corresponds to sponsoring shills in Theorem 3, and merging in the former corresponds to bidding under a single identity in the latter. For the second statement, there exists $Z \subseteq N$ and $\mathcal{P} \in \Pi(Z)$ violating (12). Then if J consists of one bidder per cell $P \in \mathcal{P}$ who is single-minded for P at a sufficiently high value, then it will be efficient for J to win Z, the Competition Effect will be zero, and bidding noncooperatively will be strictly better than merging for J. □

Proof of Lemma 1

Throughout the course of this proof, I will relax the assumptions that the valuation v_b is monotone and that $v_b(Z) \geq 0$ for all $Z \subseteq N$. If the equivalence holds without these assumptions, then of course it holds with these assumptions as well. Note that I will maintain the assumption that $v_b(\emptyset) = 0$.

It is well known that a valuation v_b (with $v_b(\emptyset) = 0$) is additive if and only if it is both submodular and supermodular. Since SubTop is weaker than submodularity and SupTop is weaker than supermodularity, it follows that any additive valuation satisfies both SubTop and SupTop.

If v_b satisfies both SubTop and SupTop, then for all $Z \subseteq N$ with $1 \leq |Z| \leq |N| - 2$:

$$v_b(N) - v_b(Z) = \sum_{x \in N-Z} [v_b(Z \cup x) - v(Z)] \tag{29}$$

(29) is equivalent to:

$$v_b(Z) = \frac{[\sum_{x \in N-Z} v_b(Z \cup x)] - v_b(N)}{|N-Z| - 1} \tag{30}$$

Applying (30) recursively, it follows that given $(v_b(Z) : Z \subseteq N, |Z| = |N|-1)$, it is possible to derive $(v_b(Z) : Z \subseteq N, 1 \leq |Z| \leq |N|-2)$. Moreover, if v_b satisfies SubTop and SupTop, it must also satisfy (29) for $Z = \emptyset$, and since $v_b(\emptyset) = 0$, this means that:

$$v_b(N) = \sum_{x \in N} v_b(x),$$

so this means that if we know $(v_b(Z) : Z \subseteq N, |Z| = |N|-1)$, we can derive $v_b(N)$ as well.

Next observe that for any additive valuation v_b and $x \in N$:

$$v_b(x) = \frac{[\sum_{Z \ni x : |Z| = |N|-1} v_b(Z)] - (|N|-2)v_b(N-x)}{|N|-1} \tag{31}$$

On the other hand, if given $(v_b(Z) : Z \subseteq N, |Z| = |N|-1)$, we define $v_b(x)$ by (31), then a simple calculation shows that for all $Y \subseteq N$ with $|Y| = |N|-1$, $\sum_{x \in Y} v_b(x) = v_b(Y)$. It follows that for every profile $(v_b(Z) : Z \subseteq N, |Z| = |N|-1)$, there exists exactly one additive valuation w such that for all $Z \subseteq N$ with $|Z| = |N|-1$, $w(Z) = v_b(Z)$.

Because (i) any profile $(v_b(Z) : Z \subseteq N, |Z| = |N|-1)$ uniquely determines an additive valuation and also uniquely determines a valuation satisfying both SubTop and SupTop, and (ii) any additive valuation satisfies both SubTop and SupTop, it follows that the set of additive valuations equals the set of valuations satisfying both SubTop and SupTop. □

Proof of Theorem 9

If v_b is additive, then it satisfies SubTop, so from Theorem 3, there is no incentive to shill. Next consider the incentive to merge. However, if a coalition is minimally competitive against v_b, the Competition Effect is null. Moreover, any additive valuation satisfies SupTop. It now follows from Proposition 2 that the Integration Effect is nonnegative, so there is no benefit to merging.

Next suppose that v_b is not additive. Then Lemma 1 implies that v_b violates either SubTop or SupTop. If v_b violates SubTop, then by Theorem 3, there is an incentive to shill. If v_b violates SupTop, then there exists Z and partition $\mathcal{P} \in \Pi(Z)$ such that (18) is violated. Then a noncompetitive coalition $J = \mathcal{P}$ such that it would be efficient for each $j = P \in J$ to win package P would have an incentive to merge. □

Proof of Theorem 13

Theorem 6 of Gul and Stacchetti (1999) shows that the aggregate valuation w formed by a collection of gross substitutes valuations is submodular. Since SubTop is weaker than submodularity, w also satisfies SubTop. Since SubTop is weaker than submodularity in order to establish both parts 1 and 2 of Theorem 13, it is sufficient to show that for any strict superset of the gross substitutes valuations, one can construct an aggregate valuation which violates SubTop. This is achieved by the following two lemmas.

Lemma 6 *Suppose that v satisfies SubTop. Then for all $Y \subseteq N$, if $Y, N \in D(p; v)$, then for all Z with $Y \subseteq Z \subseteq N$, $Z \in D(p; v)$.*

Proof. That $Y, N \in D(p; v)$ implies that:

$$v(N) - v(Y) = \sum_{x \in N-Y} p_x \tag{32}$$

SubTop implies that for all P, Q such that $\{P, Q\} \in \Pi(N - Y)$,

$$v(N) - v(Y) \leq [v(Y \cup P) - v(Y)] + [v(Y \cup Q) - v(Y)] \tag{33}$$

In particular, we may choose P so that $Y \cup P = Z$. (32) and (33) imply that:

$$\sum_{x \in N-Y} p_x \leq [v(Y \cup P) - v(Y)] + [v(Y \cup Q) - v(Y)], \tag{34}$$

Moreover, $Y \in D(p; v)$ implies:

$$\sum_{x \in P} p_x \geq v(Y \cup P) - v(Y) \quad \text{and} \quad \sum_{x \in Q} p_x \geq v(Y \cup Q) - v(Y) \tag{35}$$

(34) and (35) together imply:

$$\sum_{x \in P} p_x = v(Y \cup P) - v(Y) \quad \text{and} \quad \sum_{x \in Q} p_x = v(Y \cup Q) - v(Y)$$

which in turn implies that $Z = Y \cup P \in D(p; v)$. □

Lemma 7 *Suppose that v is not a gross substitutes valuation. Then there exists a gross substitutes valuation u such that:*

$$w(T) := \max\{v(S) + u(T - S) : S \subseteq T\} \tag{36}$$

violates SubTop.

Proof. Suppose that v is not a gross substitutes valuation. It follows from Theorem 1 of Gul and Stacchetti (1999) that there exists $p \in \mathbb{R}^N_+$, $Y, Z \in D(p; v)$, and $X \subseteq Y - Z$ such that:

$$\forall C \subseteq Z - Y, (Y - X) \cup C \notin D(p; v). \tag{37}$$

Define valuation u by:

$$u(T) := \sum_{x \in T \cap [N-Y]} p_x$$

u is an additive valuation and hence is a gross substitutes valuation. Define w via (36). Then, for all $T \subseteq N$:

$$\begin{aligned} w(T) - \sum_{x \in T} p_x &= \max\{v(S) + \sum_{x \in T-S} p_x : Y \cap T \subseteq S \subseteq T\} - \sum_{x \in T} p_x \\ &= \max\{v(S) - \sum_{x \in S} p_x : Y \cap T \subseteq S \subseteq T\} \end{aligned} \tag{38}$$

It follows that:

$$\begin{aligned} \max\{w(T) - \sum_{x \in T} p_x : T \subseteq N\} &= \max\{\max\{v(S) - \sum_{x \in S} p_x : Y \cap T \subseteq S \subseteq T\} : T \subseteq N\} \\ &= \max\{v(S) - \sum_{x \in S} p_x : S \subseteq N\}, \end{aligned} \tag{39}$$

where the least inequality follows from the fact that for all $S \subseteq N$, $Y \cap T \subseteq S \subseteq T$ when $T = S$. Next observe that for any $T \in D(p; v)$:

$$\begin{aligned} w(T) - \sum_{x \in T} p_x &\geq v(T) - \sum_{x \in T} p_x \\ &= \max\{v(S) - \sum_{x \in S} p_x : S \subseteq N\} \\ &= \max\{w(S) - \sum_{x \in S} p_x : S \subseteq N\} \end{aligned}$$

So $D(p; v) \subseteq D(p; w)$. In particular $Y \in D(p; w)$, which implies that:

$$w(N) - \sum_{x \in N} p_x = \max\{v(S) - \sum_{x \in S} p_x : Y \subseteq S \subseteq N\} = v(Y) - \sum_{x \in Y} p_x = w(Y) - \sum_{x \in Y} p_x$$

It follows that $N \in D(p; w)$. Notice also that since $Z \in D(p; v)$, $Z \in D(p; w)$. Now assume for contradiction that w satisfies SubTop. Then by Lemma 6, and the fact that $Z \subseteq (Y \cup Z) - X \subseteq N$, it follows that $(Y \cup Z) - X \in D(p; w)$. But:

$$\begin{aligned} w((Y \cup Z) - X) - \sum_{x \in (Y \cup Z) - X} p_x &= \max\{v(S) - \sum_{x \in S} p_x : Y - X \subseteq S \subseteq (Y \cup Z) - X\} \\ &< \max\{v(S) - \sum_{x \in S} p_x : S \subseteq N\} \\ &= \max\{w(S) - \sum_{x \in S} p_x : S \subseteq N\}, \end{aligned} \tag{40}$$

where the first equality follows from (38), the inequality from (37), and the last equality from (39). (40) contradicts $(Y \cup Z) - X \in D(p; w)$. It follows that w violates SubTop. □

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