CHAPTER IX

SOLUTIONS TO THE EXERCISES

9.1 Chapter 1

1. We use Theorem 1.3.1, where now we deal with the column vector $\begin{bmatrix} p_t \\ q_t \end{bmatrix}$ in place of the row vector (x, y). We need to compute the mean and variance of this column vector. The mean is given by

$$E \begin{bmatrix} p_t \\ q_t \end{bmatrix} = \begin{bmatrix} \beta_2 \\ \beta_1 + \gamma_1 \beta_2 \end{bmatrix}$$

hence

$$p_t - E\{p_t\} = \varepsilon_{t2} \quad \text{and} \quad q_t - E\{q_t\} = \varepsilon_{t1} + \gamma_1 \varepsilon_{t2}.$$

The variance is therefore

$$\begin{aligned}
&\operatorname{Var} \begin{bmatrix} p_t \\ q_t \end{bmatrix} \\
&= \begin{bmatrix} E\{(p_t - E\{p_t\})^2\} & E\{(p_t - E\{p_t\})(q_t - E\{q_t\})\} \\ E\{(q_t - E\{q_t\})(p_t - E\{p_t\})\} & E\{(q_t - E\{q_t\})^2\} \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{22} & \sigma_{12} + \gamma_1 \sigma_{22} \\ \sigma_{12} + \gamma_1 \sigma_{22} & \sigma_{11} + 2\gamma_1 \sigma_{12} + \gamma_1^2 \sigma_{22} \end{bmatrix} \\
&\equiv \begin{bmatrix} \sigma_{pp} & \sigma_{pq} \\ \sigma_{qp} & \sigma_{qq} \end{bmatrix}.
\end{aligned}$$

(a) From formulas (1.3.6) and (1.3.5) for the wide-sense conditional expectation of q_t given p_t we have

$$\begin{aligned}
\hat{E}(q_t | p_t) &= E\{q_t\} + \sigma_{qp} \sigma_{pp}^{-1} (p_t - E\{p_t\}) \\
&= \beta_1 + \gamma_1 \beta_2 + \frac{\sigma_{12} + \gamma_1 \sigma_{22}}{\sigma_{22}} (p_t - \beta_2) \\
&= \beta_1 + \gamma_1 p_t + \frac{\sigma_{12}}{\sigma_{22}} (p_t - \beta_2).
\end{aligned}$$

(b) This is equal to the given expression when $\sigma_{12} = 0$.

(c) From Theorem 1.4.1, if the joint distribution of p_t and q_t is *normal*, the wide-sense conditional expectation is equal to the ordinary conditional expectation.

2. Define the residual $1 \times m$ random vector $\delta_t = z_t - \zeta$ and the $(1+m) \times (1+m)$ variance-covariance matrix

$$E \left\{ \begin{bmatrix} \varepsilon_t \\ \delta_t' \end{bmatrix} \begin{bmatrix} \varepsilon_t & \delta_t \end{bmatrix} \right\} = \begin{bmatrix} \sigma^2 & \rho' \\ \rho & \Theta \end{bmatrix}$$

where

$$\sigma^2 = E\{\varepsilon_t^2\}, \quad \Theta = E\{\delta_t'\delta_t\}, \quad \text{and} \quad \rho = E\{\delta_t'\varepsilon\},$$

Θ being positive-definite by assumption. Now we need to compute the variance matrix of

$$\big(y_t - E\{y_t\}, z_t - E\{z_t\}\big) = (\delta_t\gamma + \varepsilon_t,\ \delta_t),$$

which is

$$\begin{aligned}\Sigma &= E\left\{\begin{bmatrix}\gamma'\delta_t' + \delta_t \\ \delta_t'\end{bmatrix}[\,\delta_t\gamma + \varepsilon_t \quad \delta_t\,]\right\} \\ &= \begin{bmatrix}\gamma'\Theta\gamma + 2\gamma'\rho + \sigma^2 & \gamma'\Theta + \rho' \\ \Theta\gamma + \rho & \Theta\end{bmatrix} \equiv \begin{bmatrix}\Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{zy} & \Sigma_{zz}\end{bmatrix}.\end{aligned}$$

Then using (1.3.5) and (1.3.4) we have

$$\begin{aligned}\hat{E}(y_t \mid z_t) &= E\{y_t\} + (z_t - \zeta)\Sigma_{zz}^{-1}\Sigma_{zy} \\ &= \zeta\gamma + x_t\beta + \delta_t\Theta^{-1}[\Theta\gamma + \rho] \\ &= \zeta\gamma + x_t\beta + \delta_t\gamma + \delta_t\Theta^{-1}\rho \\ &= z_t\gamma + x_t\beta + \delta_t\Theta^{-1}\rho.\end{aligned}$$

This is equal to the given expression if and only if the last term is equal to zero. But it is impossible for $\Theta^{-1}\rho$ to be equal to zero unless $\rho = 0$, which signifies lack of correlation between z_t and ε_t. This condition is clearly necessary and sufficient.

5. The mean-square error of any estimator $\check{\mu}$ of μ in this case is necessarily the scalar measure

$$\text{Risk}\{\check{\mu}\} = E\{(\check{\mu} - \mu)^2\}.$$

For the sample mean $\bar{y}$ this computes to

$$\text{Risk}\{\bar{y}\} = E\left\{\left(\sum_{t=1}^{n}\varepsilon_t \Big/ n\right)^2\right\} = \sigma^2/n.$$

For the alternative estimator, $\hat{\mu} = [n/(n+a)]\bar{y}$, the mean-square error is

$$\text{Risk}\{\hat{\mu}\} = E\left\{\left[\left(\sum_{t=1}^{n}\varepsilon_t - a\mu\right) \Big/ (n+a)\right]^2\right\} = \frac{n\sigma^2 + a^2\mu^2}{(n+a)^2}.$$

(a) Thus,

$$\begin{aligned}\text{Risk}\{\hat{\mu}\} \leqq \text{Risk}\{\bar{y}\} &\iff \frac{n\sigma^2 + a^2\mu^2}{(n+a)^2} \leqq \frac{\sigma^2}{n} \\ &\iff n^2\sigma^2 + na^2\mu^2 \leqq n^2\sigma^2 + 2an\sigma^2 + a^2\sigma^2 \\ &\iff na\mu^2 \leqq (2n+a)\sigma^2 \\ &\iff \frac{\mu^2}{\sigma^2} \leqq \frac{2}{a} + \frac{1}{n}.\end{aligned}$$

This holds for all sample sizes n if and only if (1.9.5) holds.

(b) Let the random variable μ be distributed independently of the ε_ts and have prior mean $\bar{\mu}$ and prior variance τ^2, and let our estimator have the affine form

$$\hat{\mu}(y) = c_0 + \sum_{t=1}^{n} c_t y_t.$$

The deviation of this estimator from μ decomposes as

$$\hat{\mu} - \mu = c_0 + \sum_{t=1}^{n} c_t \varepsilon_t + \left(\sum_{t=1}^{n} c_t - 1\right)(\mu - \bar{\mu}) + \left(\sum_{t=1}^{n} c_t - 1\right)\bar{\mu}.$$

Accordingly, since we have assumed that $E\{\mu - \bar{\mu}\}\varepsilon_t = 0$, the mean-square error decomposes as

$$\begin{aligned} E\{(\hat{\mu} - \mu)^2\} &= E\left(\sum_{t=1}^{n} c_t \varepsilon_t\right)^2 + \left(\sum_{t=1}^{n} c_t - 1\right)^2 E\{(\mu - \bar{\mu})^2\} \\ &\quad + \left[c_0 + \left(\sum_{t=1}^{n} c_t - 1\right)\bar{\mu}\right]^2 \\ &= \sigma^2 \sum_{t=1}^{n} c_t^2 + \tau^2 \left(\sum_{t=1}^{n} c_t - 1\right)^2 + \left[c_0 + \left(\sum_{t=1}^{n} c_t - 1\right)\bar{\mu}\right]^2. \qquad (9.1.1) \end{aligned}$$

This is minimized with respect to c_0 when the third expression on the right of (9.1.1) vanishes, i.e., when

$$c_0 = \left(1 - \sum_{t=1}^{n} c_t\right)\bar{\mu}.$$

Differentiating the sum of the first two terms of (9.1.1) with respect to c_t and setting the result equal to zero, we obtain

$$\sigma^2 c_t + \tau^2 \left(\sum_{t'=1}^{n} c_{t'} - 1\right) = 0. \qquad (9.1.2)$$

Summing these equations over t from 1 to n and solving for $\sum_{t=1}^{n} c_t$ we obtain

$$\sum_{t=1}^{n} c_t = \frac{n\tau^2}{\sigma^2 + n\tau^2}.$$

Substituting this expression back into (9.1.2) we obtain

$$c_t = \frac{1}{n + \sigma^2/\tau^2} \quad \text{for all } t = 1, 2, \ldots n.$$

Our Bayes estimator of μ is then

$$\hat{\mu} = \frac{\sigma^2}{\sigma^2 + n\tau^2}\bar{\mu} + \frac{\tau^2}{\sigma^2 + n\tau^2}\sum_{t=1}^{n} y_t = \frac{1}{1 + n\tau^2/\sigma^2}\bar{\mu} + \frac{1}{n + \sigma^2/\tau^2}\sum_{t=1}^{n} y_t. \qquad (9.1.3)$$

Setting $\bar{\mu} = 0$ and $\tau^2 = \sigma^2/a$ this furnishes the desired estimator.

(c) This follows by simple algebra. Solving

$$\frac{1}{1 + n\tau^2/\sigma^2}\bar{\mu} = \bar{\mu} + c$$

for c we obtain

$$c = -\frac{n\tau^2/\sigma^2}{1 + n\tau^2/\sigma^2}\bar{\mu}.$$

Then (9.1.3) becomes

$$\begin{aligned}
\hat{\mu} &= \bar{\mu} + \frac{1}{n + \sigma^2/\tau^2}\sum_{t=1}^{n} y_t - \frac{n}{n + \sigma^2/\tau^2}\mu \\
&= \bar{\mu} + \frac{n}{n + \sigma^2/\tau^2}\bar{y} - \frac{n}{n + \sigma^2/\tau^2}\mu \\
&= \bar{\mu} + \frac{1}{1 + \sigma^2/n\tau^2}(\bar{y} - \bar{\mu}).
\end{aligned}$$

□