

Bickerdike's Theory of Incipient and Optimal Tariffs

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1 Introduction

The “elasticity approach” to the theory of international trade and payments had peculiar beginnings. Its originator, Bickerdike, apparently introduced it for purely technical reasons as a means of demonstrating that a country would gain from the imposition of a tariff. Although he provided the diagram (Bickerdike 1906, p. 533) using Marshall’s offer curves and Edgeworth’s collective indifference curves that made his proposition startlingly obvious, the only tool considered acceptable at the time for analyzing the welfare effects of a tax was Marshall’s consumers’-surplus analysis (Marshall 1879b). Thus, despite Marshall’s warnings (1879a, p. 3) against using partial-equilibrium methods in international trade, Bickerdike proceeded to do so, introducing explicit (and at that time artificial) assumptions to the effect that each country had an inconvertible currency of constant domestic value. Thus was born a model that later came to be applied to the analysis of fluctuating exchange rates (Bickerdike 1920, Robinson 1937, Lerner 1944) and subsequently came under severe criticism (Frenkel & Johnson 1976).

Edgeworth (1908), in his masterly analysis of Bickerdike’s work (but for which it might still have remained in obscurity to the present day), saw the importance of Bickerdike’s monetary assumptions to his methods, but provided only a rather vague interpretation (p. 542):

We might imagine the national money in Mr. Bickerdike’s system to be an inconvertible (or at least unexportable) currency, regulated, as some theorists have proposed, so that its value should remain constant. Constancy of value might be secured by one of the methods of measuring the value of money which I have elsewhere described, preferably the one called Ricardo’s Method, or the Labour Standard.

This was unsatisfactory, since it did not meet the requirements of pure theory so beautifully expressed by Marshall in the opening paragraph of the *Pure Theory of Foreign Trade* (1879a, p. 1):

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The function of a pure theory is to deduce definite conclusions from definite hypothetical premises. The premises should approximate as closely as possible to the facts with which the corresponding applied theory has to deal. But the terms used in the pure theory must be capable of exact interpretation, and the hypotheses on which it is based must be simple and easily handled.

Such an “exact interpretation” of the assumption of a constant value of money in each country was lacking.

A clue was provided by Graaff (1949) who followed Bickerdike in dealing with inverse demand and supply functions, in which import and export prices were dependent variables expressed as functions of the quantities of imports and exports. In a footnote Graaff stated (p. 53): “Since we employ two prices in the argument that follows, we must have at least one domestic good (in addition to the two international ones) to play the role of *numéraire*”.¹ (In fact, for his analysis to have a precise interpretation, there would have to be exactly one domestic good relative to whose price the international prices are expressed.) However, no domestic goods appeared explicitly in Graaff’s formal model.

A more explicit treatment was contained in Johnson (1950) who was the first to derive the formula for the optimal tariff as a function of the elasticity of the foreign offer curve.² Referring to the then-familiar formula of Bickerdike (1907a) which had been revived by Kahn (1947), he stated (p. 30):

It is more usual . . . to express the formula for the optimum welfare tariff in terms of elasticities of demand for exports and supply of imports conceived of as functions of money prices, rather than as functions of the barter terms of trade. To do this, it is necessary to assume the presence of a third good, which can be used as a numéraire in which to express the prices of exports and imports: with only two goods, there can be only one price, and the elasticities of demand for exports and supply of imports are necessarily related by the property [$e_d - e_s = 1$].

Since 1950 the Bickerdike approach to tariff theory has fallen into oblivion, and indeed the elements needed to make it rigorous have been forgotten, as evidenced by a startling subsequent statement by Frenkel and Johnson (1976, p. 27):

The standard elasticity model, like the standard ‘real’ trade theory, assumes that all goods are traded, or in other words can be divided into import and export goods.

¹That Graaff did not have in mind a nontradable produced good, however, is clear from his subsequent statement (1957, p. 133): “we employ two prices, and therefore we must have, in addition to the export and the import, at least one domestic good (e.g. labour) to play the part of numéraire.” This unfortunate example detracted almost entirely from the point being made, since if there were no nontradable produced goods the inverse demand functions would still make no sense unless the supply of labor were elastic, or in other words unless “leisure” played the part of the nontradable. But Graaff’s “domestic transformation function” (1949, p. 58), which is supposed to represent production possibilities, has only the output of tradable commodities as arguments.

²Johnson (1950, p. 29) acknowledged his indebtedness to Graaff for the geometric argument leading to this formula. This geometric treatment subsequently appeared in Graaff (1957, p. 134).

In a previous paper (Chipman 1978), I developed a general-equilibrium framework that furnished the needed explicit assumptions making possible an exact interpretation of Bickerdike's theory. This was applied to problems connected with exchange rates and the balance of payments. In the present paper the same approach is applied to Bickerdike's theory of incipient and optimal tariffs.

2 Formulation of the model

Let us suppose there are two countries each consuming three commodities—two tradables and one nontradable. The home country (country 1) is initially exporting commodity 1 to and importing commodity 2 from the foreign country (country 2). The consumption, production, and net import of commodity j in country k are denoted x_j^k , y_j^k , and $z_j^k = x_j^k - y_j^k$; thus, $z_k^k < 0$, $z_j^k > 0$ ($j \neq k$, $j \neq 3$), and $z_3^k = 0$ in equilibrium.

Let each country have an aggregate utility function $U^k(x^k) = U^k(x_1^k, x_2^k, x_3^k)$, assumed monotone increasing, twice continuously differentiable, and strongly quasi-concave,³ and a closed, bounded, and convex production-possibility set \mathcal{Y}^k consisting of all technically feasible bundles $y^k = (y_1^k, y_2^k, y_3^k)$ of output. We define country k 's *net-utility function* as

$$(1) \quad \tilde{U}^k(z^k) = \max\{U^k(x^k) : x^k \in \mathcal{Y}^k + z^k\}.$$

Geometrically this may be visualized as a three-dimensional version of Meade's (1952) trade-indifference map. The trade-utility function is defined as $\hat{U}^k(z_1^k, z_2^k) = \tilde{U}^k(z_1^k, z_2^k, 0)$.⁴ Like most of his contemporaries (except Edgeworth), Bickerdike tacitly assumed net-utility functions to be additively separable, and in this case he applied such an assumption to (1). Thus, we may write his assumed net-utility function as

$$(2) \quad \tilde{U}^k(z^k) = \varphi_j^k(z_j^k) - \varphi_k^k(-z_k^k) + \varphi_3^k(z_3^k) \quad \text{where } (j, k) = (1, 2) \text{ or } (2, 1).$$

It is important to note that additive separability of the original utility function $U^k(x^k)$ does not imply additive separability of the net-utility function $\tilde{U}^k(z^k)$.⁵ Thus, it is not an innocent simplifying assumption, but quite a restrictive and artificial one. However, in what follows I will show that Bickerdike's basic approach does not depend on this assumption and can easily be generalized. Formulation (2) will then be taken up only as a special case.

³Defining $U_i^k = \partial U^k / \partial x_i^k$ and $U_{ij}^k = \partial^2 U^k / \partial x_i^k \partial x_j^k$, strong quasi-concavity means that the nested principal minors of the bordered Hessian Υ^k of U^k oscillate in sign, i.e.,

$$\begin{vmatrix} 0 & U_i^k & U_j^k \\ U_i^k & U_{ii}^k & U_{ij}^k \\ U_j^k & U_{ji}^k & U_{jj}^k \end{vmatrix} > 0 \quad \text{and} \quad |\Upsilon^k| = \begin{vmatrix} 0 & U_1^k & U_2^k & U_3^k \\ U_1^k & U_{11}^k & U_{12}^k & U_{13}^k \\ U_2^k & U_{21}^k & U_{22}^k & U_{23}^k \\ U_3^k & U_{31}^k & U_{32}^k & U_{33}^k \end{vmatrix} < 0.$$

⁴Cf. Chipman 1979. In Chipman 1978 the term "trade-utility function" was applied to \tilde{U} , but it is better to limit this term to the function \hat{U} of amounts of traded goods only. The contours of \hat{U} correspond to Meade's trade-indifference curves.

⁵The converse is also true; however, if (1) is to be additively separable *for all* (including box-shaped) production-possibility sets \mathcal{Y}^k , then the original utility function must also be separable. This is shown in the Appendix.

The inverse net-demand functions are defined as the demand (or supply) prices of tradables, equal to the marginal rates of substitution (according to the net-utility function) between the tradables and the nontradable:

$$(3) \quad \frac{p_j^k}{p_3^k} = \tilde{P}_j^k(z^k) \equiv \frac{\partial \tilde{U}^k / \partial z_j^k}{\partial \tilde{U}^k / \partial z_3^k} \equiv \frac{\tilde{U}_j^k}{\tilde{U}_3^k} \quad (j, k = 1, 2)$$

where p_j^k denotes the price of commodity j in country k denominated in the latter's currency. The *inverse (indirect) trade-demand functions* for country $k = 1, 2$ are defined by

$$(4) \quad \begin{aligned} \hat{P}_j^k(z_1^k, z_2^k) &= \tilde{P}_j^k(z_1^k, z_2^k, 0) \quad \text{for } j = 1, 2; \\ \hat{P}_3^k(z_1^k, z_2^k) &= z_1^k \hat{P}_1^k(z_1^k, z_2^k) + z_2^k \hat{P}_2^k(z_1^k, z_2^k). \end{aligned}$$

The first two equations of (4) may be identified with Bickerdike's inverse supply and demand functions. Following Samuelson (1950, p. 377), the otherwise wasted symbol \hat{P}_3^k is used in the third equation of (4) to denote country k 's trade deficit—denominated in the internal prices of its tradable goods relative to the price of its nontradable—as a function of its trades.

Let e denote the exchange rate, defined as the price of country 2's currency in units of country 1's; and let $T_2^2 = 1 + \tau_2^2$ be the tariff factor corresponding to an *ad valorem* tariff rate of τ_2^2 imposed by country 1 on its import of commodity 2, expressed as a proportion of the country-2 price after it has been converted to country-1 currency units.⁶ Assuming the tariff to be nonprohibitive, the following relations must hold in equilibrium:

$$(5) \quad p_1^1 = ep_1^2; \quad p_2^1 = T_2^2 ep_2^2.$$

Country 1's inverse trade-demand functions must satisfy a budget constraint yielding balanced trade in terms of country-2 prices:

$$(6) \quad z_1^1 \hat{P}_1^1(z_1^1, z_2^1) + z_2^1 \hat{P}_2^1(z_1^1, z_2^1) / T_2^2 = 0.$$

Likewise for country 2:

$$(7) \quad z_1^2 \hat{P}_1^2(z_1^2, z_2^2) + z_2^2 \hat{P}_2^2(z_1^2, z_2^2) = 0.$$

These two equations define the Marshallian offer curves (tariff-adjusted for country 1) for countries 1 and 2 respectively. From (3), the first two equations of (4), and (5), we also

⁶To explain the notation: the subscript j in τ_j^k refers to the commodity being taxed while the superscript k refers to the country whose price forms the basis of the tax. Thus, an *ad valorem* tax on country 1's export of commodity 1 based on the country-1 price is denoted τ_1^1 , and the prices of commodity 1 in the two countries are related by $p_1^1 + \tau_1^1 p_1^1 = ep_1^2$, or $p_1^2 = (1 + \tau_1^1)p_1^1/e$; consequently the relevant tax factor is $T_1^1 = 1 + \tau_1^1$. On the other hand, if the tax is instead reckoned on the country-2 price of commodity 1 as a base, it is denoted τ_1^2 and the prices in the two countries are related by $p_1^1 + \tau_1^2 ep_1^2 = ep_1^2$, or $p_1^1 = (1 - \tau_1^2)ep_1^2$; in this case the relevant tax factor is $T_1^2 = 1 - \tau_1^2$. In the case of a tariff on country 1's import of commodity 2, which is normally reckoned on the external country-2 price p_2^2 , the two countries' prices are related by $p_2^1 = (1 + \tau_2^2)ep_2^2$ and the relevant tariff factor is $T_2^2 = 1 + \tau_2^2$. Should a tariff τ_2^1 be expressed as a proportion of the domestic (country-1) price of commodity 2, however, the two countries' prices would have to be related by $p_2^1 = ep_2^2 + \tau_2^1 p_2^1$, or $p_2^1 = (1 - \tau_2^1)p_2^1/e$, and the relevant tariff factor would be $T_2^1 = 1 - \tau_2^1$. In general, then, we may write $T_j^k = 1 - (\text{sign } z_j^k)\tau_j^k$; from the definitions these satisfy $T_j^1 T_j^2 = 1$ for $j = 1, 2$.

have immediately

$$(8) \quad \frac{\hat{P}_2^1(z_1^1, z_2^1)}{T_2^2 \hat{P}_2^2(z_1^2, z_2^2)} = \frac{\hat{P}_1^1(z_1^1, z_2^1)}{\hat{P}_1^2(z_1^2, z_2^2)} = \frac{e}{\mu} \quad \text{where} \quad \mu \equiv \frac{p_3^1}{p_3^2}.$$

The common value e/μ of the first two expressions of (8) was referred to by Bickerdike (1907a, p. 100n) as the “ratio of exchange of moneys” (of this, more below).

Note that equations (3) and (5) are invariant with respect to multiplication of all six prices by a positive scalar, hence one of them may be chosen as numéraire. The logical choice in Bickerdike’s scheme is one of the prices of nontradables, say p_3^1 . Accordingly we may impose the two restrictions

$$(9) \quad p_3^1 = \bar{p}_3^1; \quad p_3^2 = p_3^1.$$

Thus, $\mu = 1$ in (8). Finally, given our choice of notation we must specify the material-balance condition

$$(10) \quad z_j^1 + z_j^2 = 0 \quad (j = 1, 2).$$

Given (10), any two of the equations (6), (7), and the first equation of (8) imply the third, as is easily verified. Bickerdike (1907, p. 100n) worked with (6) and the first equation of (8), but it is easier to work with (6) and (7). The four equations $p_j^k/p_3^k = \hat{P}_j^k(z_1^k, z_2^k)$ from (3) and (4) ($j, k = 1, 2$); any two out of equations (6), (7), and the first equation of (8); the second equation of (8); the two equations of (9); and the two equations of (10), constitute eleven equations in the eleven unknowns $p_1^1, p_2^1, p_3^1, p_1^2, p_2^2, p_3^2, e, z_1^1, z_2^1, z_1^2, z_2^2$. Fixing the prices p_3^1 and p_3^2 of the two nontradables in Bickerdike’s model according to (9) by means of appropriate monetary policies in the two countries amounts to defining the “constant value of money” in each country by the fixed amount of money that can be obtained for a unit of the nontradable good. This, then, satisfies Marshall’s requirement that “the pure theory must be capable of exact interpretation.”

3 Incipient and optimal tariffs

From (6) and (7), together with (10), we may solve for the functions $z_1^2 = \bar{z}_1^2(T_2^2)$ and $z_2^1 = \bar{z}_2^1(T_2^2)$. Defining, following Bronfenbrenner (1942),⁷ the *flexibilities* as the elasticities of the indirect (inverse) trade-demand functions \hat{P}_i^k of (4) $i = 1, 2$) with respect to the trades z_j^k ($j = 1, 2$):

$$(11) \quad \pi_{ij}^k = \frac{z_j^k}{\hat{P}_i^k} \frac{\partial \hat{P}_i^k}{\partial z_j^k} \quad (i, j, k = 1, 2),$$

we find readily that

$$(12) \quad \frac{T_2^2}{z_1^2} \frac{d\bar{z}_1^2}{dT_2^2} = -\frac{1 - \pi_{12}^2 + \pi_{22}^2}{\Delta}, \quad \frac{T_2^2}{z_2^1} \frac{d\bar{z}_2^1}{dT_2^2} = -\frac{1 + \pi_{11}^2 - \pi_{21}^2}{\Delta}$$

⁷The terminology “flexibility” goes back to Frisch (1932).

where

$$(13) \quad \Delta = \begin{vmatrix} -1 - \pi_{11}^1 + \pi_{21}^1 & 1 - \pi_{12}^1 + \pi_{22}^1 \\ 1 + \pi_{11}^2 - \pi_{21}^2 & -1 + \pi_{12}^2 - \pi_{22}^2 \end{vmatrix}.$$

Now country 1's advantage from the tariff may be defined as

$$(14) \quad \bar{U}^1(T_2^2) = \hat{U}^1(-\bar{z}_1^2(T_2^2), \bar{z}_2^1(T_2^2)).$$

Differentiating (14) and taking account of (4) and (12) we obtain

$$(15) \quad \frac{d\bar{U}^1}{dT_2^2} = \frac{\partial \tilde{U}^1}{\partial z_3^1} \frac{z_1^2 \hat{P}_1^1}{T_2^2} \frac{1 - \pi_{12}^2 + \pi_{22}^2 - T_2^2(1 + \pi_{11}^2 - \pi_{21}^2)}{\Delta}.$$

This is a generalization of the expression obtained by Bickerdike (1907a, p. 100n). Its relation to Bickerdike's formula will be discussed in the next section.

Bickerdike derived two results: (1) that a sufficiently small ("incipient") tariff would necessarily improve a country's welfare, and (2) that a particular rate of tariff (the optimal tariff) would maximize the country's welfare.

To establish Bickerdike's first result for this more general model it is necessary to show that the expression

$$(16) \quad \frac{1 - \pi_{12}^2 + \pi_{22}^2 - T_2^2(1 + \pi_{11}^2 - \pi_{21}^2)}{\Delta}$$

is positive for $T_2^2 = 1$; then, starting from a zero tariff rate ($T_2^2 = 1$), the country will gain from a small or "incipient" tariff. To establish Bickerdike's second result (Bickerdike 1907a, p. 101n), conditions are needed to show that expression (16) can vanish for $T_2^2 > 1$; then

$$(17) \quad T_2^2 = \frac{1 - \pi_{12}^2 + \pi_{22}^2}{1 + \pi_{11}^2 - \pi_{21}^2}$$

is the optimal tariff factor.

Edgeworth (1908, p. 544n)—who went along with Bickerdike's separability assumption $\pi_{ij}^k = 0$ for $i \neq j$ —saw the need to invoke dynamic stability conditions in order to ensure $\Delta > 0$ in (16). The simplest way to proceed is to postulate the dynamic-adjustment process

$$(18) \quad \begin{aligned} \dot{z}_1^2 \propto \hat{D}^1(z_1^2, z_2^1, T_2^2) &\equiv -z_1^2 \hat{P}_1^1(-z_1^2, z_2^1) + z_2^1 \hat{P}_2^1(-z_1^2, z_2^1)/T_2^2 \\ \dot{z}_2^1 \propto \hat{D}^2(z_1^2, z_2^1) &\equiv z_1^2 \hat{P}_1^2(z_1^2, -z_2^1) - z_2^1 \hat{P}_2^2(z_1^2, -z_2^1) \end{aligned}$$

where \hat{D}^k is the deficit in country k 's balance of trade, denominated in its external (tariff-exclusive) own-currency prices and measured relatively to the price of its nontradable, and expressed as a function of the trades in accordance with the third equation of (4). Assuming that points (z_1^2, z_2^1) (Marshall's "exchange index"⁸) to the left and right of country 1's offer curve, and below and above country 2's offer curve, correspond to deficits and surpluses of

⁸Marshall (1879, p. 9; 1923, p. 340).

their trade balances respectively, (18) is a possible rendition of Marshall's (1879a, p. 18) adjustment process.⁹

Defining the Jacobian matrix $J = \partial(D^1, D^2) / \partial(z_1^2, z_2^1)$ we see that

$$(19) \quad \begin{bmatrix} \frac{1}{z_1^2 P_1^1} & 0 \\ 0 & \frac{1}{z_1^2 P_1^2} \end{bmatrix} J \begin{bmatrix} z_1^2 & 0 \\ 0 & z_2^1 \end{bmatrix} = \begin{bmatrix} -1 - \pi_{11}^1 + \pi_{21}^1 & 1 - \pi_{12}^1 + \pi_{22}^1 \\ 1 + \pi_{11}^2 - \pi_{21}^2 & -1 + \pi_{12}^2 - \pi_{22}^2 \end{bmatrix}.$$

From the results of Metzler (1945) and Arrow (1974), for the system (18) to be stable independently of the speeds of adjustment, the principal minors of (19) must be alternately nonpositive and nonnegative. For an equilibrium solution of (18) (a solution of (6), (7), and (10)) to be isolated, the determinant Δ of (19) must be nonzero, hence positive.¹⁰ From the argument in footnote 9 it follows that in accordance with Marshall's dynamic-adjustment process, the diagonal elements of (19) must be strictly negative.

To establish Bickerdike's first result it is necessary to obtain conditions for the numerator of (16) to be positive when $T_2^2 = 1$, i.e., for

$$(20) \quad \pi_{22}^2 - \pi_{12}^2 + \pi_{21}^2 - \pi_{11}^2 > 0.$$

⁹Marshall argued (1879, p. 18; 1923, p. 341) that if exchange took place at a point to the left of country 1's offer curve, say at a point $(z_1^2(t), z_2^1(0))$ to the left of the point $(z_1^2(0), z_2^1(0))$ on its offer curve (where $z_1^2(t) < z_1^2(0)$), this would mean that country 1 was exporting only $z_1^2(t)$ of commodity 1 in exchange for imports of $z_2^1(0)$ of commodity 2, when it was capable of exporting the larger quantity $z_1^2(0)$ in a competitive equilibrium with zero profits. This would imply, according to Marshall, that industry 1 "must be a trade which affords abnormally high profits"; accordingly, exports of commodity 1 will increase. This argument is not entirely convincing, however, since if industry 1 is making a profit in the sense that it is earning more than it is spending, and other industries are breaking even, then the country as a whole must be experiencing a trade surplus. But on the contrary, the first equation of (18) implies that if $z_1^2(t)$ is below its offered value $z_1^2(0)$ when $z_2^1(t) = z_2^1(0)$ (i.e., is to the left of country 1's offer curve), it must rise if stability is to hold (i.e., move to the right, back towards the offer curve); this means that \dot{z}_1^2 must be positive, i.e., we must have $\hat{D}^1(z_1^2(t), z_2^1(0), T_2^2) > 0$ for $z_1^2(t) < z_1^2(0)$ (so that $\partial \hat{D}^1(z_1^2(0), z_2^1(0), T_2^2) / \partial z_1^2 < 0$). Therefore a position to the left of country 1's offer curve must represent a deficit in country 1's balance of trade, rather than a surplus as would be implied by Marshall's argument. The difficulty is that Marshall's adjustment process provides no information about price movements, only quantity movements; a more satisfactory procedure would be to allow for simultaneous price and quantity adjustments as has been done, e.g., by Beckmann and Ryder 1969. Note that Marshall's subsequent description (1923, p. 341) of his adjustment process is somewhat confused, since Figure 10 on p. 340 should correspond to Figure 7 of the 1879 version, but does not.

An alternative representation of Marshall's adjustment process was suggested by Samuelson (1947, pp. 266–7) and has been further analyzed in Chipman 1987, pp. 934–7. This consists in defining the offer functions F^k implicitly by $\hat{D}^1(F^1(z_2^1, T_2^2), z_2^1, T_2^2) = 0$ and $\hat{D}^2(z_1^2, F^2(z_1^2)) = 0$ and setting \dot{z}_1^2 and \dot{z}_2^1 proportional to $F^1(z_2^1, T_2^2) - z_2^1$ and $F^2(z_1^2) - z_1^2$ respectively. This leads to a simpler Jacobian matrix in place of (19) below, with diagonal elements -1 and off-diagonal elements equal to what Alexander (1951) called "elasticities of trade"

$$\alpha^1 = \frac{z_2^1}{F^1} \frac{\partial F^1}{\partial z_2^1} = \frac{1 - \pi_{12}^1 + \pi_{22}^1}{1 + \pi_{11}^1 - \pi_{21}^1} \quad \text{and} \quad \alpha^2 = \frac{z_1^2}{F^2} \frac{\partial F^2}{\partial z_1^2} = \frac{1 - \pi_{21}^2 + \pi_{11}^2}{1 + \pi_{22}^2 - \pi_{12}^2}.$$

¹⁰This result may also be obtained on the basis of an alternative dynamic-adjustment system analyzed in Chipman (1978, pp. 58–9). The 3×3 matrix A of that system must have a negative determinant. It is easily seen by elementary row operations that this determinant is the negative of Δ ; and one of the second-order stability conditions is equivalent to the condition on the second diagonal element of Δ .

Denoting $\tilde{U}_j^k = \partial \tilde{U}^k / \partial z_j^k$ and $\tilde{U}_{ij}^k = \partial^2 \tilde{U}^k / \partial z_i^k \partial z_j^k$ we see from the definitions (11) and (3) that

$$(21) \quad \pi_{ij}^2 = \frac{-z_j^2}{\hat{P}_i^2 [\tilde{U}_3^2]^2} \begin{vmatrix} \tilde{U}_i^2 & \tilde{U}_{ij}^2 \\ \tilde{U}_3^2 & \tilde{U}_{3j}^2 \end{vmatrix}$$

hence

$$(22) \quad \pi_{2j}^2 - \pi_{1j}^2 = \frac{z_j^2}{\tilde{U}_1^2 \tilde{U}_2^2} \begin{vmatrix} \tilde{U}_1^2 & \tilde{U}_{1j}^2 \\ \tilde{U}_2^2 & \tilde{U}_{2j}^2 \end{vmatrix} \quad \text{for } j = 1, 2;$$

thus their sum is

$$(23) \quad \pi_{22}^2 - \pi_{12}^2 + \pi_{21}^2 - \pi_{11}^2 = \frac{1}{\tilde{U}_1^2 \tilde{U}_2^2} \left\{ z_2^2 \begin{vmatrix} \tilde{U}_1^2 & \tilde{U}_{12}^2 \\ \tilde{U}_2^2 & \tilde{U}_{22}^2 \end{vmatrix} + z_1^2 \begin{vmatrix} \tilde{U}_1^2 & \tilde{U}_{11}^2 \\ \tilde{U}_2^2 & \tilde{U}_{21}^2 \end{vmatrix} \right\}.$$

Now from (3) and country 2's trade-balance constraint (7) we have

$$(24) \quad \frac{z_2^2}{z_1^2} = -\frac{\hat{P}_1^2}{\hat{P}_2^2} = -\frac{\tilde{U}_1^2}{\tilde{U}_2^2}$$

so that (23) implies the proportionality relation

$$(25) \quad \pi_{22}^2 - \pi_{12}^2 + \pi_{21}^2 - \pi_{11}^2 \propto -\tilde{U}_1^2 \begin{vmatrix} \tilde{U}_1^2 & \tilde{U}_{12}^2 \\ \tilde{U}_2^2 & \tilde{U}_{22}^2 \end{vmatrix} + \tilde{U}_2^2 \begin{vmatrix} \tilde{U}_1^2 & \tilde{U}_{11}^2 \\ \tilde{U}_2^2 & \tilde{U}_{21}^2 \end{vmatrix} = \begin{vmatrix} 0 & \tilde{U}_1^2 & \tilde{U}_2^2 \\ \tilde{U}_1^2 & \tilde{U}_{11}^2 & \tilde{U}_{12}^2 \\ \tilde{U}_2^2 & \tilde{U}_{21}^2 & \tilde{U}_{22}^2 \end{vmatrix}.$$

But this is positive by the assumed strong quasi-concavity of each country's utility function (see footnote 3).

In order to establish conditions for the possibility of an optimal tariff $T_2^2 > 1$ given by (17), it suffices to note that the expression $\alpha^2 = (1 + \pi_{11}^2 - \pi_{21}^2) / (1 - \pi_{12}^2 + \pi_{22}^2)$ is related to the elasticity ω^2 of the Marshallian offer curve for country 2 by the formula $\omega^2 = 1 / (1 - \alpha^2)$ (cf. Chipman 1978, p. 67). Thus, $T_2^2 = 1 / \alpha^2 > 1$ if and only if $\omega^2 > 1$, i.e., if and only if the foreign offer curve is elastic (in the relevant range). In fact, from this formula we immediately obtain the formula

$$(26) \quad \tau_2^2 = \frac{1}{\omega^2 - 1}$$

for the optimal tariff rate first obtained by Johnson (1950, p. 29).

4 Edgeworth's approach

In his elegant interpretation, Edgeworth (1908, p. 550n) showed how the optimal-tariff formula (17) could be reached directly by an argument mirroring Bickerdike's (1906, p. 532) geometric treatment in terms of Marshallian offer curves and Edgeworth's indifference curves. Edgeworth's approach was to maximize country 1's trade-utility function $\hat{U}^1(z_1^1, z_2^1) = \tilde{U}^1(z_1^1, z_2^1, 0)$ subject to country 2's offer curve (7) (as well as (10)), and then to use (4) to obtain (but under Bickerdike's and his special assumption $\partial \hat{P}_1^2 / \partial z_2^2 = \partial \hat{P}_2^2 / \partial z_1^2 = 0$)

$$(27) \quad \frac{\hat{P}_2^1}{\hat{P}_1^1} = \frac{\hat{P}_2^2 + z_1^2 \frac{\partial \hat{P}_2^2}{\partial z_2^2} + z_2^2 \frac{\partial \hat{P}_2^2}{\partial z_2^2}}{\hat{P}_1^2 + z_1^2 \frac{\partial \hat{P}_1^2}{\partial z_1^2} + z_2^2 \frac{\partial \hat{P}_1^2}{\partial z_1^2}}.$$

The term on the left is the slope of country 1's trade-indifference curve, and that on the right is the slope of country 2's offer curve. Now substituting the first equation of (8) into the left side of (27) (the passage in Edgeworth's footnote states that (6) is employed—but this must be a misprint) and making use of (7) and (10), we immediately obtain (17) once again.

5 The separable case

The main limiting assumption in Bickerdike's theory is the implicit one of additive separability leading to the special form (2) of the net-utility function. In Graaff's (1949, p. 53) terminology, this is the case in which “‘cross-elasticities’ vanish identically.” Since the net-utility function involves production in addition to consumption, the meaning of this assumption is not at all apparent; since conditions for this are fairly complicated, discussion of this question is taken up in the Appendix.

Given the form (2), since the functions (3) will always be evaluated at $z_3^k = 0$ as in (4), the latter may be written in the form¹¹

$$(28) \quad \hat{P}_j^k(z_j^k) = \frac{\varphi_j^{k'}(z_j^k)}{\varphi_3^{k'}(0)} \quad (j \neq k); \quad \hat{P}_k^k(z_k^k) = \frac{\varphi_k^{k'}(-z_k^k)}{\varphi_3^{k'}(0)}.$$

Thus, $\pi_{ij}^k = 0$ for $i \neq j$. Bickerdike defined the two countries' elasticities of supply of exports and demand for imports as the reciprocals of the corresponding flexibilities (elasticities of the inverse functions):

$$(29) \quad e_s = -\frac{1}{\pi_{11}^1}, \quad e_d = -\frac{1}{\pi_{22}^2}; \quad \eta_\sigma = -\frac{1}{\pi_{22}^2}, \quad \eta_\delta = -\frac{1}{\pi_{11}^1}.$$

His measure of the “net advantage of trade” was given by the consumers'-surplus integral

$$(30) \quad \hat{u}^1(z_1^2, z_2^1) = \int_0^{z_2^1} \hat{P}_2^1(\zeta_2^1) d\zeta_2^1 - \int_0^{z_1^2} \hat{P}_1^1(-\zeta_1^2) d\zeta_1^2.$$

Given (2) and the definitions (28), this is just the same as $\hat{U}^1(-z_1^2, z_2^1)/\varphi_3^{1'}(0) = [\varphi_2^1(z_2^1) - \varphi_1^1(z_1^2)]/\varphi_3^{1'}(0)$. Thus, the “marginal utility of money” (in this case the marginal net-utility of the numéraire) is constant since the net trade in commodity 3 is zero. This is a case in which consumers'-surplus analysis is fully in accord with modern utility theory.¹²

¹¹For readers of Bickerdike (1907a, p. 100n) the following correspondence may be found helpful: $\hat{P}_2^1(z_2^1) = f_1(\alpha_1)$, $\hat{P}_1^1(-z_1^2) = F_2(\alpha_2)$, $\hat{P}_2^2(-z_2^2) = F_1(\alpha_1)$, and $\hat{P}_1^2(z_1^2) = f_2(\alpha_2)$.

¹²Since \hat{P}_2^1 in (30) is country 1's “demand price” for its import and \hat{P}_1^1 its “supply price” for its export (terms that were subsequently introduced by Pigou 1910), Pigou (1907, pp. 290–1) in his reply to Bickerdike interpreted the two integrals of (30) as consumers' and producers' surplus respectively, and questioned the legitimacy of using the latter as a welfare measure since, according to Marshall (1898, p. 521n), this would require the function $\hat{P}_1^1(z_2^1)$ to be a “particular expenses curve.” In short, according to him, internal and external economies would have to be absent (or equally balanced). This involved both a quibble and a misunderstanding on Pigou's part, the misunderstanding being a confusion between a country's reciprocal supply and its factor supply. Pigou's discussion also went on at length concerning the indirect effects on country 2—which appear to be irrelevant since these are incorporated in the dependence of z_1^2 and z_2^1 on T_2^2 from the solution of (31) and (32). In his rejoinder, Bickerdike (1907b, p. 585) pointed out that both integrals in (30) involve supply as well as demand. In his use of (30) Bickerdike was defended by Edgeworth (1908, p. 551).

Bickerdike (1907a, p. 100n) proceeded to take the derivative of this expression with respect to T_2^2 (strictly speaking, the negative of the derivative with respect to $r = 1/T_2^2$), presumably by solving

$$(31) \quad z_2^1 \hat{P}_2^1(z_2^1) = T_2^2 z_1^2 \hat{P}_1^1(-z_1^2)$$

and the first equation of

$$(32) \quad \frac{\hat{P}_2^1(z_2^1)}{T_2^2 \hat{P}_2^2(-z_2^1)} = \frac{\hat{P}_1^1(-z_1^2)}{\hat{P}_1^2(z_1^2)} = e$$

(the counterparts of (6) and (8) above) for the functions $z_1^2 = \bar{z}_1^2(T_2^2)$ and $z_2^1 = \bar{z}_2^1(T_2^2)$ to obtain the composed function $\bar{u}^1(T_2^2) = \hat{u}^1(\bar{z}_1^2(T_2^2), \bar{z}_2^1(T_2^2))$, as in (14) above. He was thus led to

$$(33) \quad \frac{d\bar{u}^1}{dT_2^2} = \frac{1 - 1/\eta_\sigma - T_2^2(1 - 1/\eta_\delta)}{(1 - 1/\eta_\sigma)(1 - 1/e_s) - (1 - 1/\eta_\delta)(1 - 1/e_d)}$$

—a formula he displayed only for $T_2^2 = 1$ —and to the formula for the optimal tariff factor

$$(34) \quad T_2^2 = \frac{1 - 1/\eta_\sigma}{1 - 1/\eta_\delta}$$

—a formula he displayed in its reciprocal form (p. 101n).¹³ The expression (33) corresponds precisely to (16) when the “cross-elasticities” vanish and the expressions (29) are substituted. Thus, Bickerdike neglected the term $z_1^2 \hat{P}_1^1$ appearing in (15)—a minor blemish.

Bickerdike pointed out (1907a, pp. 100–1) that “exactly the same expression is derived in the case of export tax.” There is no reason to doubt that he performed the computation (or, indeed, that he saw that formulas (31) and (32) would be unaffected). Thus, he completely anticipated Lerner’s (1936) “symmetry theorem.”¹⁴

¹³From (34) we readily obtain Kahn’s (1947, p. 16) formula

$$(\dagger) \quad \tau_2^2 = \frac{1/\eta + 1/\varepsilon}{1 - 1/\varepsilon}$$

for the optimal tariff rate, where $\eta = -\eta_\sigma$ and $\varepsilon = \eta_\delta$.

¹⁴If country 1 imposes a tax of τ_1^1 on its export good (as a proportion of the domestic value), in addition to a tariff τ_2^2 on its import good, the first equation of (5) is modified to $T_1^1 p_1^1 = e p_1^2$, where $T_1^1 = 1 + \tau_1^1$, and in the ensuing formulas (6) and (8), as well as (31)–(34), T_2^2 is replaced by the product $T_1^1 T_2^2 = (1 + \tau_1^1)(1 + \tau_2^2)$. In essence, this is Lerner’s symmetry theorem; the export-tax-cum-import-tariff combination $(\tau_1^1, \tau_2^2) = (\tau, 0)$ is equivalent to the combination $(0, \tau)$. In 1944 Lerner defined (implicitly) an export tax as an *ad valorem* tax τ_1^2 expressed as a proportion of the foreign (country-2) price (Lerner 1944, p. 383); the domestic price of the export good would then have to be less than the foreign price by the factor $T_1^2 = 1 - \tau_1^2 = 1/(1 + \tau_1^1) = 1/T_1^1$ (see footnote 6 above). Thus T_2^2 in (34) is replaced by T_2^2/T_1^2 . Formula (34) may then be written (in Kahn’s notation—see footnote 13)

$$(\ddagger) \quad \frac{T_2^2}{T_1^2} = \frac{1 + \tau_2^2}{1 - \tau_1^2} = \frac{1 + 1/\eta}{1 - 1/\varepsilon}.$$

This led Lerner (1944, pp. 382–3) to the elegant formula $(\tau_1^2, \tau_2^2) = (1/\varepsilon, 1/\eta)$ for “the” optimal export-tax-cum-import-tariff combination. According to Lerner’s own (1936) symmetry theorem, however, equivalent combinations would be $\tau_1^2 = 0, \tau_2^2 = (1/\eta + 1/\varepsilon)/(1 - 1/\varepsilon)$ and $\tau_1^2 = (1/\eta + 1/\varepsilon)/(1 + 1/\eta), \tau_2^2 = 0$. In fact it is obvious from (\ddagger) that there is an entire continuum of optimal combinations. See also the discussion in Graaff 1949, p. 53.

It was left to Edgeworth (1908, p. 544n) to show that the positive sign of the denominator in (33) followed from dynamic stability—probably the first use of what Samuelson (1947) was later to call the “correspondence principle.” It is these stability conditions of Edgeworth’s rather than Bickerdike’s (1920) subsequent development of them, that most probably influenced Robinson¹⁵ (1937, p. 194n) and Lerner (1944, p. 378). As for the numerator of (33), if, as apparently Bickerdike tacitly assumed, the original utility function $U^k(x^k)$ is also additively separable, so that $\tilde{U}_{ij}^2 = 0$ for $i \neq j$ and $\tilde{U}_{ii}^2 < 0$, then it is clear from (21) that $\pi_{11}^2 < 0$ and $\pi_{22}^2 > 0$ (and of course that the bordered determinant of (25) is positive), hence $\eta_\delta > 0$ and $\eta_\sigma < 0$ from (38). Thus, the numerator of (33) is unambiguously positive for $T_2^2 = 1$.

It is of some interest to inquire into the meaning of these assumptions if the original utility function is not necessarily separable. In terms of the more general formulation of Section 3 this amounts to assuming that $\pi_{22}^2 - \pi_{12}^2$ and $\pi_{21}^2 - \pi_{11}^2$ are both nonnegative and not both zero, rather than assuming that their sum is positive. Denoting by Υ the 4×4 bordered matrix of footnote 3 and by μ the marginal utility of income, and letting c_j and s_{ij} respectively denote the marginal income effects and Slutsky terms of the consumer demand function, we have from (22) (dropping country superscripts and tildes for convenience)

$$\pi_{22} - \pi_{12} \propto - \begin{vmatrix} U_1 & U_{12} \\ U_2 & U_{22} \end{vmatrix} = \frac{|\Upsilon|}{\mu^2} \begin{vmatrix} c_1 & c_3 \\ s_{13} & s_{33} \end{vmatrix} \geq 0 \iff \begin{vmatrix} c_1 & c_3 \\ s_{13} & s_{33} \end{vmatrix} \leq 0$$

and

$$\pi_{21} - \pi_{11} \propto + \begin{vmatrix} U_1 & U_{11} \\ U_2 & U_{21} \end{vmatrix} = \frac{|\Upsilon|}{\mu^2} \begin{vmatrix} c_2 & c_3 \\ s_{23} & s_{33} \end{vmatrix} \geq 0 \iff \begin{vmatrix} c_2 & c_3 \\ s_{23} & s_{33} \end{vmatrix} \leq 0,$$

by Jacobi’s theorem (cf. Chipman 1974, pp. 42–3). If all three goods are strongly superior (i.e., $c_i > 0$ for $i = 1, 2, 3$) then a sufficient condition for the above inequalities is $s_{13} \geq 0$ and $s_{23} \geq 0$, i.e., that the tradable goods be weak Hicksian substitutes of the nontradable good. Since the Slutsky terms must satisfy $p_1 s_{13} + p_2 s_{23} + p_3 s_{33} = 0$, s_{13} and s_{23} , and hence $\pi_{22} - \pi_{12}$ and $\pi_{21} - \pi_{11}$, cannot both be zero, hence the sum of the latter must be positive. But of course these conditions are much stronger than necessary for Bickerdike’s first theorem.

Thus we see that Bickerdike’s results can be given a rigorous interpretation in terms of modern general-equilibrium theory—provided that the separability assumption can be accepted. The latter assumption is fairly restrictive, as is evident from the discussion in the Appendix. However, as we saw in Section 3, the objection to the separability assumption in no way invalidates Bickerdike’s approach, which is easily extended to the general case.

6 The question of dynamic stability

The discussion of Bickerdike’s model would not be complete without consideration of Bickerdike’s own and subsequent approach to the stability problem (1920). He retained the

¹⁵The first mention in the literature of which I am aware of Bickerdike’s 1920 article was that of Metzler 1948. It is thus not at all certain that this paper was known to Robinson (1937). On the other hand, Edgeworth’s 1908 paper was cited by Robinson (1947, p. 107n).

three equations (31) and (32) above (with $T_2^2 = 1$), and added the equation of country 2's offer curve, modified to allow for a payment (such as a loan or repayment) L^2 from country 1 to country 2, denominated in country 2's currency:

$$(35) \quad z_1^2 \hat{P}_1^2(z_2^2) - z_2^1 \hat{P}_2^2(-z_2^1) = L^2$$

(Bickerdike used the symbol Z for L^2). Strictly speaking, L^2 must be expressed not only in country 2's currency but relatively to the nominal price of the nontradable in country 2; but we may assume that units are chosen so that $p_3^1 = p_3^2 = 1$ in (3).

Bickerdike (1920, p. 120) appears to have overlooked the fact that the four equations (31), (32), and (35) are not independent, so that (31) should be replaced by

$$(36) \quad z_2^1 \hat{P}_2^1(z_1^1) - z_1^2 \hat{P}_1^1(-z_1^2) = -eL^2.$$

However, he did not use (31) in his subsequent calculations so the results were unaffected.

Bickerdike posed the problem of "stability of foreign exchange" not in expressly dynamic terms, but in terms of a problem in comparative statics. Equations (32) (with $T_2^2 = 1$) may be used to solve for country 1's imports z_2^1 and exports z_1^2 as functions $\tilde{z}_2^1(e)$ and $\tilde{z}_1^2(e)$ of the exchange rate e (in this paper Bickerdike defined the exchange rate as $1/e$). When these are inserted in (35) we obtain a relationship between the exchange rate and the payment, which may be denoted $e = \bar{e}(L^2)$. Bickerdike's criterion for stability was $d\bar{e}/dL^2 > 0$, i.e., a capital movement from country 1 to country 2 should raise the value of country 2's currency. In terms of Bickerdike's revised 1920 notation, in which (29) becomes

$$(37) \quad \pi_{11}^1 = -\frac{1}{e_s} = s, \quad \pi_{22}^1 = -\frac{1}{e_d} = -d, \quad \pi_{22}^2 = -\frac{1}{\eta_\sigma} = \sigma, \quad \pi_{11}^2 = -\frac{1}{\eta_\delta} = -\delta,$$

one obtains, evaluated at $L^2 = 0$,¹⁶

$$(38) \quad \frac{d(1/\bar{e})}{dL^2} = \frac{1}{z_2^1 \hat{P}_2^1} \frac{-1}{\frac{1+\sigma}{d+\sigma} + \frac{1-\delta}{s+\delta}}.$$

Accordingly, Bickerdike's "stability condition" is

$$(39) \quad \frac{1+\sigma}{d+\sigma} + \frac{1-\delta}{s+\delta} > 0.$$

Rewriting this as

$$(40) \quad \frac{(1+\sigma)(s+\delta) + (1-\delta)(d+\sigma)}{(d+\sigma)(s+\delta)} = \frac{(1+\sigma)(1+s) - (1-\delta)(1-d)}{(d+\sigma)(s+\delta)} > 0$$

we see that it is equivalent to Edgeworth's stability condition (since the numerator of (40) is the same as the denominator of (33)) provided the denominator of (40) is positive (each term of which was assumed positive by Bickerdike).¹⁷

¹⁶Note the misprint "when $Z = 1$ " for "when $Z = 0$ " in Bickerdike 1920, p. 120. Bickerdike also assumed that units were chosen so that $z_2^1 \hat{P}_2^1 = 1$. Note also the misprint in formula (5.21) of Chipman (1978, p. 60) where the numerator of the fraction should be -1 rather than 1 .

¹⁷For the relation of these formulas to those of Robinson (1937) and Metzler (1948) see Chipman 1978, p. 61. Bickerdike noted (1920, p. 121) that for $s = \sigma = 0$ ("infinite elasticities of supply of exports") (39) reduces to $1/d + 1/\delta - 1 > 0$, which is equivalent to Lerner's (1944, p. 378) famous formula $e_d + \eta_\delta - 1 > 0$.

It is of some interest to inquire whether condition (39) can be obtained as the stability condition for a true dynamic process. The following process fills the bill:

$$(41) \quad \dot{e} \propto B^2(e, L^2) \equiv \tilde{z}_2^1(e) \hat{P}_2^2(-\tilde{z}_2^1(e)) - \tilde{z}_1^2(e) \hat{P}_1^2(\tilde{z}_1^2(e)) - L^2.$$

This expresses, in positive rather than negative form, Robinson's (1937, p. 195) postulate: "If, at a given exchange rate, the balance of trade falls short of the balance of lending, the exchange depreciates."¹⁸ From standard theory of differential equations, asymptotic stability requires that $\partial B^2/\partial e < 0$, i.e., that a depreciation of country 2's currency should improve its *ex ante* (scheduled) balance of payments.¹⁹

7 Concluding remarks

The study of past economic doctrines has much in common with archeology and paleontology. One is presented with a fragment from which one must try to reconstruct the whole. In those disciplines, the whole once existed, and the task is to reconstitute it. In the analysis of past doctrines, the whole may never have existed except possibly in the subconscious mind of the author, or within the context of a verbal tradition that has since been forgotten. The task of interpretation consists of asking, not "what did the author mean?", but rather, "what must the author have meant if the whole is to make sense?" Of course, not all past (or present) writers have developed doctrines that could make sense. The trick is to sense, from the flashes of originality and insight, whether there is something there behind it all, and to give the author the benefit of the doubt; the correct interpretation of a doubtful passage, then, is the one that makes sense in the wider context.

The distinction between tradable and nontradable goods, as an analytic device in the theory of international trade and payments, was not known to Bickerdike in 1907. It was introduced by Taussig (1917), and later developed by Graham (1925), Hawtrey (1928, pp. 67–88), and Ohlin (1928). The concept took a very long time to become incorporated in formal trade models. But it presumably would have been congenial to Bickerdike who, in discussing the purchasing-power parity doctrine, stated (1922, p. 36):

For problems such as those under discussion it is not admissible to abstract all transport charges. That is to rule out essential and fundamental conditions . . . when we are concerned with paper money and the connection between foreign exchanges and index-numbers of prices.

¹⁸In (41), the burden of adjustment is placed entirely on country 2. A more symmetric procedure would be to replace $\hat{P}_2^2(-\tilde{z}_2^1(e))$ by $\hat{P}_2^1(\tilde{z}_2^1(e))/e$; but this would not lead to Bickerdike's condition.

¹⁹It is a common fallacy to conclude from this that the condition is equivalent to the criterion for a "successful devaluation," i.e., to conclude that in a regime of pegged, adjustable exchange rates a devaluation will improve a country's *ex post* (equilibrium) balance of payments. The source of the fallacy is very simple: a Bretton-Woods type of regime would violate Bickerdike's basic assumption (1906, p. 533) that "money can be regarded as a constant measure" in each country, i.e. (in the interpretation presented in this paper), that the nominal prices p_3^1 and p_3^2 of the two countries' nontradables are kept fixed by the respective countries' monetary authorities. Bickerdike cannot be accused of having committed this fallacy. For detailed discussion of this point see Chipman 1989a, 1989b.

We have seen that with this analytic device, the Bickerdike model becomes a consistent general-equilibrium model of international trade and payments, and Bickerdike's main propositions and formulas—whose validity have been questioned because of their alleged partial-equilibrium nature—all fall neatly into place. For this reason his work has lasting value.

Appendix: Conditions for vanishing cross-flexibilities of indirect net-demand functions

This appendix, stimulated by Tower's (1993) note, is addressed to the problem of obtaining necessary and sufficient conditions for the vanishing of the cross-flexibilities π_{ij}^k of (11) when $i \neq j$, $i = 1, 2$. To simplify notation I shall deal just with country 1, and I shall drop the country superscripts. It will be assumed throughout that the domain of variation of the world prices is limited to a manifold such that the country exports commodity 1 and imports commodity 2. Consumer preferences are assumed to be strictly monotone and convex, and the country's production-possibility set is assumed to be convex.

To obtain a necessary condition on consumer preferences it suffices to select a particular convex production-possibility set. Let it be Haberler's (1950) box-shaped set defined by

$$(A1) \quad 0 \leq y_j \leq \eta_j \quad (j = 1, 2, 3).$$

The net-utility function is obtained by maximizing $U(y_1 + z_1, y_2 + z_2, y_3 + z_3)$ subject to (A1), and since the utility function $U(x_1, x_2, x_3)$ is assumed increasing in its three arguments, the net-utility function (1) is given simply by

$$(A2) \quad \tilde{U}(z_1, z_2, z_3) = U(\eta_1 + z_1, \eta_2 + z_2, \eta_3 + z_3).$$

From (3) and (11), vanishing cross-flexibilities yield

$$(A3) \quad \begin{vmatrix} U_1 & U_{12} \\ U_3 & U_{32} \end{vmatrix} = \begin{vmatrix} U_2 & U_{21} \\ U_3 & U_{31} \end{vmatrix} = 0, \quad \text{hence} \quad \begin{vmatrix} U_1 & U_{31} \\ U_2 & U_{32} \end{vmatrix} = 0.$$

From Leontief's (1947) results these three equalities imply that U satisfies

$$(A4) \quad U(x_1, x_2, x_3) = F(x_1, f(x_2, x_3)) = G(x_2, g(x_1, x_3)) = H(x_3, h(x_1, x_2)).$$

From a result of Aczél (1966, p. 312, Corollary 1 of Theorem 1) the functional equation $F(x_1, f(x_2, x_3)) = H(x_3, h(x_1, x_2))$ from (A4) implies in turn that U is of the additively separable form

$$(A5) \quad U(x_1, x_2, x_3) = \Phi(\varphi_1(x_1) + \varphi_2(x_2) + \varphi_3(x_3)),$$

where the φ_i are increasing and concave and Φ is an increasing function.

A further restriction on the form of U is obtained by selecting a production-possibility frontier of the form

$$(A6) \quad y_3 = \eta_3 - \psi_1(y_1)$$

where ψ_1 is an increasing convex function. (A6) implies that country 1 specializes on commodities 1 and 3 and relies entirely on imports for its supplies of commodity 2. Without loss of generality we may choose the additive representation of (A5),

$$(A7) \quad U(x_1, x_2, x_3) = \sum_{i=1}^3 \varphi_i(x_i),$$

and maximize (A7) subject to (A6). Setting up the Lagrangean function

$$(A8) \quad L(y, z, \lambda) = \varphi_1(y_1 + z_1) + \varphi_2(z_2) + \varphi_3(y_3 + z_3) - \lambda[\psi_1(y_1) + y_3 - \eta_3]$$

and differentiating with respect to y_1 and y_3 and using (A6), we obtain the condition

$$(A9) \quad F(z_1, z_3, y_1) \equiv \varphi_1'(y_1 + z_1) - \psi_1'(y_1)\varphi_3'(\eta_3 + z_3 - \psi_1(y_1)) = 0.$$

Since $\partial F/\partial y_1 < 0$ (from the concavity of φ_i and convexity of ψ_1), (A9) implicitly defines the function $y_1 = F_1(z_1, z_3)$. The net-utility function is then

$$(A10) \quad \tilde{U}(z_1, z_2, z_3) = \varphi_1(F_1(z_1, z_3) + z_1) + \varphi_2(z_2) + \varphi_3(\eta_3 + z_3 - \psi_1(F_1(z_1, z_3))).$$

Denoting $\tilde{U}_{ij} = \partial \tilde{U} / \partial z_i \partial z_j$, clearly $\tilde{U}_{12} = \tilde{U}_{32} = 0$ hence from (3) and (11) it follows that $\pi_{12} = 0$. Now $\pi_{21} = 0$ if and only if

$$\begin{vmatrix} \tilde{U}_2 & \tilde{U}_{21} \\ \tilde{U}_3 & \tilde{U}_{31} \end{vmatrix} = \begin{vmatrix} \tilde{U}_2 & 0 \\ \tilde{U}_3 & \tilde{U}_{31} \end{vmatrix} = \tilde{U}_2 \tilde{U}_{31} = 0,$$

i.e., if and only if $\tilde{U}_{31} = 0$, and after some computation one finds that

$$(A11) \quad \tilde{U}_{31} = \frac{\varphi_1'' \varphi_3'' \psi_1'}{\varphi_1'' - \varphi_3' \psi_1'' + \varphi_3'' \psi_1'^2}$$

which vanishes if and only if the numerator does, i.e., any one of the terms φ_1'' , φ_3'' , ψ_1' vanishes.

If $\psi_1' = 0$, we are back to the case of fixed production. Assuming $\psi_1' > 0$, we obtain the following

Proposition 1. *A necessary condition for the cross-flexibilities π_{ij} ($i = 1, 2$, $i \neq j$) of a country's indirect net-demand functions to vanish, for any convex production-possibility set, is that the consumption-utility function be additively separable (of the form (A5)), with additive representation (A7) which is linear in either the amount of the exportable, x_1 , or the amount of the nontradable, x_3 . If the country specializes in the production of the exportable and the nontradable, this condition is also sufficient.*

Using this result, we may consider the more general case in which the country produces all three commodities. Without loss of generality its production-possibility frontier may be defined by

$$(A12) \quad \Psi(y_1, y_2) + y_3 = \eta_3$$

where Ψ is an increasing convex function. I will confine attention to the case in which the additive representation of the consumption-utility function (A7) is linear in x_3 , i.e.,

$$(A13) \quad U(x_1, x_2, x_3) = \varphi_1(x_1) + \varphi_2(x_2) + x_3.$$

We thus set up the Lagrangean function

$$(A14) \quad L(y, z, \lambda) = \varphi_1(y_1 + z_1) + \varphi_2(y_2 + z_2) + y_3 + z_3 - \lambda[\Psi(y_1, y_2) + y_3 - \eta_3].$$

The necessary conditions for a constrained maximum yield the transformation $(z_1, z_2) = \Upsilon(y_1, y_2)$ defined by

$$(A15) \quad \begin{aligned} z_1 &= \Upsilon_1(y_1, y_2) \equiv (\varphi_1')^{-1} \left(\frac{\partial}{\partial y_1} \Psi(y_1, y_2) \right) - y_1 \\ z_2 &= \Upsilon_2(y_1, y_2) \equiv (\varphi_2')^{-1} \left(\frac{\partial}{\partial y_2} \Psi(y_1, y_2) \right) - y_2 \end{aligned}$$

whose Jacobian

$$(A16) \quad \begin{bmatrix} (\varphi_1'')^{-1} & 0 \\ 0 & (\varphi_2'')^{-1} \end{bmatrix} \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is negative definite. Thus, the mapping $(y_1, y_2) = \Upsilon^{-1}(z_1, z_2)$ is well defined. The net-utility function is then given by

$$(A17) \quad \tilde{U}(z) = \varphi_1 \left(\Upsilon_1^{-1}(z_1, z_2) + z_1 \right) + \varphi_2 \left(\Upsilon_2^{-1}(z_1, z_2) + z_2 \right) + \eta_3 - \Psi \left(\Upsilon^{-1}(z_1, z_2) \right) + z_3.$$

For the cross-flexibilities to vanish, this function must be additively separable, and for this it is clearly necessary and sufficient that the function Ψ in turn be additively separable, i.e., of the form

$$(A18) \quad \Psi(y_1, y_2) = \psi_1(y_1) + \psi_2(y_2).$$

Consequently Υ_i depends only on y_i ($i = 1, 2$) and (A17) becomes

$$(A19) \quad \tilde{U}(z) = \varphi_1 \left(\Upsilon_1^{-1}(z_1) + z_1 \right) + \varphi_2 \left(\Upsilon_2^{-1}(z_2) + z_2 \right) + \eta_3 - \psi_1 \left(\Upsilon_1^{-1}(z_1) \right) - \psi_2 \left(\Upsilon_2^{-1}(z_2) \right) + z_3.$$

We therefore have

Proposition 2. *In order for the cross-flexibilities of the indirect net-demand functions to vanish when the production-possibility frontier is concave to the origin, it is necessary and sufficient that*

- (a) *consumer preferences be representable by a utility function of the form (A7);*
- (b) *the production-possibility function be representable in the form*

$$(A20) \quad \psi_1(y_1) + \psi_2(y_2) + y_3 = \eta_3$$

where ψ_1 and ψ_2 are increasing convex functions; and

- (c) *either*
 - (c1) φ_1 or φ_3 is linear, or
 - (c2) φ_1, φ_2 , or φ_3 is linear and the country produces all three commodities.

It remains finally to consider the implications of condition (b) of the above proposition in the case in which the production-possibility frontier is derived from a model with three commodities and two or three factors in fixed total supply and perfectly mobile among industries, which operate under constant returns to scale and concave single-output production functions.

The Two-Factor Case

If there are three commodities and two factors, the production-possibility frontier is a ruled surface; characterizations of this surface have been developed by Melvin (1968), Inoue (1984), and Inoue and Wegge (1986). Inoue and Wegge showed that this surface would be the intersection of the nonnegative octant with either (a) a cone, (b) a cylinder, or (c) a tangent-developable surface. I will consider just (a) and (b).

As shown by Inoue (1984), a cone with vertex $\eta = (\eta_1, \eta_2, \eta_3)$ is representable by the implicit function

$$(A21) \quad F\left(\frac{y_1 - \eta_1}{y_3 - \eta_3}, \frac{y_2 - \eta_2}{y_3 - \eta_3}\right) = 0.$$

For this to be consistent with (A20) one would have to have

$$\frac{\psi_1(y_1)}{y_3 - \eta_3} + \frac{\psi_2(y_2)}{y_3 - \eta_3} + 1 = 0$$

for $\psi_i(y_i) = c_i(y_i - \eta_i)$ and some constants c_i , $i = 1, 2$; but this would imply

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = \eta_1 + \eta_2 + \eta_3 = \text{constant},$$

i.e., that the cone degenerates to a plane. This case must therefore be ruled out.

As shown in Inoue (1984), a cylinder is representable by the implicit function

$$(A22) \quad F(y_1 + c_1 y_3, y_2 + c_2 y_3) = 0$$

which together with (A20) implies that either

$$(A22.1) \quad y_1 + c_1 y_3 = G(y_2 + c_2 y_3) = c_1[\eta_3 - \psi_2(y_2)], \quad \text{hence } c_1 \neq 0, c_2 = 0,$$

or

$$(A22.2) \quad y_2 + c_2 y_3 = H(y_1 + c_1 y_3) = c_2[\eta_3 - \psi_1(y_1)], \quad \text{hence } c_1 = 0, c_2 \neq 0,$$

i.e., either ψ_1 or ψ_2 is linear in (A20) (but not both).

If ψ_1 is linear (case (A22.1)) then by the assumption that all three commodities are produced, the relative price p_1/p_3 is constant (equal to $1/c_1$); but this can happen only if industries 1 and 3 have identical technologies, i.e., $y_1 = f_1(v_{11}, v_{21}) = c_1 f_3(v_{11}, v_{21})$, where f_j denotes the production function for commodity j and v_{ij} denotes the amount of factor i employed in the production of commodity j ; for, the prices are then related to each other via the minimum-unit-cost functions by $p_1 = g_1(w_1, w_2) = c_1^{-1} g_3(w_1, w_2) = p_3$, where the w_i are the factor rentals. This is a case of “infinite elasticity of supply of exports.”

A special case of (A22.1) is that in which industries 1 and 3 use only one factor of production, say factor 1. Then industry 2 uses up the entire endowment of factor 2. Since by constant returns to scale we have $y_j = \mu_j v_{1j}$ for $j = 1, 3$, where v_{ij} is the amount of factor i allocated to industry j , the equation of the production-possibility frontier is given by

$$(A23) \quad y_2 = f_2\left(l_1 - \frac{y_1}{\mu_1} - \frac{y_3}{\mu_3}, l_2\right),$$

where $f_2(v_{12}, v_{22})$ is the production function for commodity 2 and l_i is the country's endowment in factor i . Defining the function $v_{12} = f_2^{-1}(y_2, v_{22})$ implicitly by

$$f_2(f_2^{-1}(y_2, v_{22}), v_{22}) = y_2$$

(which gives the isoquant for the production function f_2 for each output level y_2), we may express (A23) in the equivalent form

$$\frac{\mu_3}{\mu_1}y_1 + \mu_3 f_2^{-1}(y_2, l_2) + y_3 = \mu_3 l_1,$$

which is of the form (A20) or (A22.1) for

$$\psi_1(y_1) = \frac{y_1}{c_1} = \left(\frac{\mu_3}{\mu_1}\right)y_1, \quad \psi_2(y_2) = \mu_3 f_2^{-1}(y_2, l_2),$$

and $\eta_3 = \mu_3 l_1$. The function Υ_1 of (A19) becomes

$$z_1 = \Upsilon_1(y_1) = (\varphi'_1)^{-1} \left(\frac{\mu_3}{\mu_1} \right) - y_1,$$

hence the argument of the first term on the right in (A19) becomes

$$\Upsilon_1^{-1}(z_1) + z_1 = (\varphi'_1)^{-1} \left(\frac{\mu_3}{\mu_1} \right),$$

which is constant. The fourth term on the right in (A19) becomes

$$\psi_1(\Upsilon_1^{-1}(z_1)) = \frac{\mu_3}{\mu_1} \left[(\varphi'_1)^{-1} \left(\frac{\mu_3}{\mu_1} \right) - z_1 \right],$$

which is linear. Thus, $\tilde{U}(z)$ is linear in z_1 (as well as in z_3) and therefore the flexibilities satisfy $\pi_{11} = 0$ in addition to the separability conditions $\pi_{12} = \pi_{21} = 0$.

If instead ψ_2 is linear (case (A22.2)), then $p_2/p_3 = 1/c_2$, which can happen only if industries 2 and 3 have identical technologies, i.e., $y_2 = f_2(v_{12}, v_{22}) = c_2 f_3(v_{12}, v_{22})$. This is a case of "infinite elasticity of demand for imports."

A special case of (A22.2) is that in which industries 2 and 3 use a single factor of production (say factor 2) and industry 1 uses all of the other factor. In this case $y_j = \mu_j v_{2j}$ for $j = 2, 3$ and the equation of the production-possibility frontier is

$$(A24) \quad y_1 = f_1 \left(l_1, l_2 - \frac{y_2}{\mu_2} - \frac{y_3}{\mu_3} \right).$$

Defining the function $v_{21} = f_1^{-1}(y_1, v_{11})$ implicitly by $f_1(v_{21}, f_1^{-1}(y_1, v_{11})) = y_1$ (which gives the isoquant for the production function f_1 for each output level y_1), we may express (A24) in the equivalent form

$$\mu_3 f_1^{-1}(y_1, l_1) + \frac{\mu_3}{\mu_2} y_2 + y_3 = \mu_3 l_2,$$

which is of the form (A20) or (A22.2) for $\psi_1(y_1) = \mu_3 f_1^{-1}(y_1, l_1)$, $\psi_2(y_2) = y_2/c_2 = (\mu_3/\mu_2)y_2$ and $\eta_3 = \mu_3 l_2$. An argument similar to the above shows that the net-utility function $\tilde{U}(z)$ of (A19) is linear in z_2 as well as additively separable, hence $\pi_{22} = 0$ in addition to $\pi_{12} = \pi_{21} = 0$.

The Three-Factor Case

For strict concavity to the origin of the production-possibility frontier, at least three factors are required, and this is also sufficient except for singular cases (cf., e.g., Chipman 1987, p. 930).

The Rybczynski (supply) functions are obtained by maximizing the value of the national product, $\sum_{j=1}^3 p_j y_j$, subject to (A20), yielding the conditions

$$(A25) \quad p_j = p_3 \psi'_j(y_j) \quad (j = 1, 2).$$

The Rybczynski functions for commodities 1 and 2 are therefore given by

$$(A26) \quad \hat{y}_j(p_1, p_2, p_3) = (\psi'_j)^{-1}(p_j/p_3) \quad (j = 1, 2).$$

Thus, $\partial \hat{y}_1 / \partial p_2 = \partial \hat{y}_2 / \partial p_1 = 0$, i.e., the supply cross-elasticities must be zero. On the other hand we know that the Rybczynski functions, when all three commodities are produced, have the form (cf., e.g., Chipman 1987, p. 932)

$$(A27) \quad \hat{y}_i(p_1, p_2, p_3) = \sum_{j=1}^3 b^{ij}(p_1, p_2, p_3) l_j$$

where l_j is the amount of the j th factor endowment and $B^{-1} = [b^{ij}]$ is the inverse of the matrix $B = [b_{ij}]$ of factor-output coefficients $b_{ij}(w_1, w_2, w_3)$, where the w_i are the factor rentals and the b^{ij} are evaluated at $w = \hat{w}(p)$, $\hat{w}(p)$ being the Stolper-Samuelson mapping inverse to the system of minimum-unit-cost functions $p = g(w)$. For the supply cross-elasticities to vanish independently of the factor endowments, we must have

$$(A28) \quad \partial b^{ij} / \partial p_k = \partial^2 \hat{w}_j / \partial p_i \partial p_k = 0 \quad (i \neq k = 1, 2; j = 1, 2, 3).$$

Our task is to translate this into conditions on the technology.

I shall limit myself here to the special Cobb-Douglas case²⁰ in which

$$(A29) \quad y_j = \mu_j \prod_{i=1}^3 v_{ij}^{\beta_{ij}} \left(\beta_{ij} > 0, \sum_{i=1}^3 \beta_{ij} = 1, j = 1, 2, 3 \right), \quad \sum_{j=1}^3 v_{ij} = l_i \quad (i = 1, 2, 3),$$

where v_{ij} is the amount of factor i used in industry j . Then as is well known, the cost-minimizing factor-output ratios are given by

$$(A30) \quad \frac{v_{ij}}{y_j} = b_{ij}(w) = \frac{\partial g_j(w)}{\partial w_i} = \frac{p_j \beta_{ij}}{w_i}$$

and likewise

$$(A31) \quad b^{ij}(p) = w_j \beta^{ij} / p_i = \prod_{k=1}^3 (p_k / \nu_k)^{\beta^{kj}} \beta^{ij} / p_i \quad (i, j = 1, 2, 3)$$

²⁰Similar results hold in the case in which all three industries operate under production functions with constant and equal elasticities of substitution (cf., e.g., Chipman 1966, pp. 57ff.).

where $[\beta^{ij}] = [\beta_{ij}]^{-1}$ and $\nu_j = 1/\mu_j \prod_{i=1}^3 \beta_{ij}^{\beta_{ij}}$. Condition (A28) then implies that $\beta^{ij}\beta^{kj} = 0$ for $i \neq k = 1, 2$ and $j = 1, 2, 3$. This states that for each column of $[\beta_{ij}]^{-1}$, the element in either the first or second row must be zero. Now I shall assume that the conditions of Gale and Nikaido (1965) hold, namely that the matrix $B = [b_{ij}]$ has positive principal minors, from which it follows (cf. Chipman 1969) that $B^{-1} = [b^{ij}]$ also has positive principal minors, as of course then do $[\beta_{ij}]$ and $[\beta^{ij}]$. From this assumption and the condition $\beta^{ij}\beta^{kj} = 0$ just obtained it follows that $\beta^{21} = \beta^{12} = 0$ and *either* $\beta^{23} = 0$ *or* $\beta^{13} = 0$.

If $\beta^{23} = 0$ then from $\beta^{21} = \beta^{23} = 0$ and $[\beta_{ij}][\beta^{ij}] = I$ we have

$$(\beta_{21}, \beta_{23}) \begin{bmatrix} \beta^{11} & \beta^{13} \\ \beta^{31} & \beta^{33} \end{bmatrix} = (0, 0), \quad \text{hence } (\beta_{21}, \beta_{23}) = w_2(b_{21}, b_{23}) \begin{bmatrix} 1/p_2 & 0 \\ 0 & 1/p_3 \end{bmatrix} = (0, 0);$$

and since $\beta^{12} = 0$ we have

$$\begin{vmatrix} \beta_{12} & \beta_{13} \\ \beta_{32} & \beta_{33} \end{vmatrix} = \frac{w_1 w_3}{p_2 p_3} \begin{vmatrix} b_{12} & b_{13} \\ b_{32} & b_{33} \end{vmatrix} = 0.$$

These conditions state that (1) factor 2 is not used in either industry 1 or 3, and (2) industries 2 and 3 always use factors 1 and 3 in the same proportions.

If $\beta^{13} = 0$ then since $\beta^{12} = \beta^{13} = 0$, $[\beta_{ij}][\beta^{ij}] = I$ gives

$$(\beta_{12}, \beta_{13}) \begin{bmatrix} \beta^{22} & \beta^{23} \\ \beta^{32} & \beta^{33} \end{bmatrix} = (0, 0), \quad \text{hence } (\beta_{12}, \beta_{13}) = w_1(b_{12}, b_{13}) \begin{bmatrix} 1/p_2 & 0 \\ 0 & 1/p_3 \end{bmatrix} = (0, 0),$$

and since $\beta^{21} = 0$ we have

$$\begin{vmatrix} \beta_{21} & \beta_{23} \\ \beta_{31} & \beta_{33} \end{vmatrix} = \frac{w_2 w_3}{p_1 p_3} \begin{vmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{vmatrix} = 0.$$

These conditions state that (1) factor 1 is not used in industries 2 and 3, and (2) industries 1 and 3 always use factors 2 and 3 in the same proportions.

Examples of matrices $[\beta_{ij}]$ satisfying the above conditions are

$$(A32) \quad [\beta_{ij}] = \begin{bmatrix} 13/15 & 1/30 & 1/5 \\ 0 & 5/6 & 0 \\ 2/15 & 2/15 & 4/5 \end{bmatrix} \quad \text{with} \quad [\beta_{ij}]^{-1} = \begin{bmatrix} 1.2 & 0.0 & -0.3 \\ 0.0 & 1.2 & 0.0 \\ -0.2 & -0.2 & 1.3 \end{bmatrix}$$

and

$$(A33) \quad [\beta_{ij}] = \begin{bmatrix} 5/6 & 0 & 0 \\ 1/32 & 13/16 & 3/16 \\ 13/96 & 3/16 & 13/16 \end{bmatrix} \quad \text{with} \quad [\beta_{ij}]^{-1} = \begin{bmatrix} 1.2 & 0.0 & 0.0 \\ 0.0 & 1.3 & -0.3 \\ -0.2 & -0.3 & 1.3 \end{bmatrix}.$$

Thus we have found an exact interpretation of Bickerdike's assumption of vanishing cross-elasticities.

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