

Intra-Industry Trade in a Loglinear Model*

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Summary. Within the framework of an n -commodity, n -factor, K -country loglinear model with identical Cobb–Douglas production coefficients and identical consumer-expenditure shares across countries, balanced trade, and a pattern of world endowments permitting positive outputs of all commodities in each country, it is shown that, given any mode of aggregating the n commodities into at most $n - 1$ industrial categories, there exists an allocation of the world factor endowments among the K countries such that each country engages in trade and 100% of each country’s trade is intra-industry trade (i.e., the values of imports and exports balance each other in each aggregate category). It is also shown that in the special case $n = m = K = 2$, a movement of either production function in the direction of the other causes a greater intensity of trade between the two countries as measured by their export-output ratios.

1 Introduction

As any international economist knows who has attempted to analyze statistics of international trade flows, at even the finest subdivisions one will find imports to and exports from the same country in virtually every commodity category. This phenomenon has been described by Grubel & Lloyd [17] as “intra-industry trade”.

These authors start out from the hypothesis, as stated in Grubel [16, p. 35], that “the principle of aggregation used in the compilation of international trade statistics is the proximity of products’ substitutability in consumption and/or the similarity of input requirements.” It is their contention that while some forms of intra-industry trade are compatible with the “traditional Heckscher–Ohlin model”, others are not. Included in the former are trade in goods—such as wooden and metal tables—which are highly substitutable in consumption but whose production processes differ; “for this group of goods the intra-industry trade phenomenon is simply the result of statistical aggregation” (p. 87). However, they argue differently in the case of goods—such as steel sheets and bars—which are not substitutable in consumption but employ very similar production techniques. Since their conclusions have been widely accepted by the profession it is worth subjecting their argument to a brief examination.

Arguing that in many industries (as defined by the standard classification systems), “input requirements . . . [are] so similar that they may be considered identical” (p. 89) they consider

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a model of two countries producing two commodities with two factors of production, with identical technologies for the two commodities within and between the countries. They also assume that the actual input proportions (which depend upon the input prices) are the same in the two countries; this entails the implicit assumption that the countries' factor endowment ratios are also the same, since the assumed coincidence of the two technologies implies that the cone of diversification in factor space collapses to a ray; and if both commodities are produced then each country's factor-endowment vector must lie on this ray. It follows from these assumptions that the two countries will have linear and parallel production-possibility frontiers. From this, Grubel and Lloyd conclude:

As a result the exchange of these commodities with identical input requirements is not profitable, because profits arise from the exploitation of differences in relative prices among countries. Yet we observe the exchange of such products. The inconsistency between the theory and reality can be explained by relaxing either the assumption that production functions are identical across countries or the assumption that there are no economies of scale.

However, this contains an analytic error. Since prices are always equal across countries in competitive equilibrium, the same argument could be used to prove that there can never be any trade.¹ Under the given assumptions, the correct conclusion is that the amount of trade is indeterminate; it could be zero, and it could be very large. The case at hand is one of locally nonunique or "neutral" equilibrium. If we abandon the approximation of similarity by identity, we can say that when production functions are very similar, each country's production-possibility frontier will be very flat (as opposed to curved); and if the countries' endowment ratios are very close, the shapes of the two countries' (very flat) production-possibility frontiers will be slightly different. Under these circumstances there can be a considerable amount of trade.² In fact, under some precise assumptions it is shown in Section 4 below that the more similar the production functions for the two commodities, the larger is the volume of trade between the two countries as measured by their export-output ratios.

Grubel and Lloyd's conclusions appears to have been widely accepted. Thus, Lancaster [24, p. 151] states: "Intra-industry trade on a large scale, an undeniable fact of trade between modern industrial economies, is simply not a prediction of traditional trade theory." Likewise we find the following fairly typical statement by Helpman & Krugman [20, p. 2]:

actual trade patterns seem to include substantial two-way trade in goods of similar factor intensity. This "intra-industry" trade seems both pointless and hard to explain from the point of view of conventional trade analysis.

¹In fact, we find a similar argument in Cassel [4, p. 8]:

To say, in an exact sense, that two currencies have the same purchasing power is possible only when the prices of all goods are precisely the same in both countries. . . . But then no international trade could take place. The fundamental condition of international trade is that relative prices in the countries are different.

²This shows, incidentally, the danger in making assumptions such as the above-quoted one that "input requirements . . . [are] so similar that they may be considered identical." As input requirements become more and more similar, the sequence of equilibria corresponding to different input requirements does not converge to the set of equilibria corresponding to identical input requirements. This lack of continuity means precisely that it is illegitimate to treat very similar input requirements as if they were identical.

No independent reasoning is supplied as to why such trade should be hard to explain in terms of the conventional Heckscher-Ohlin-Lerner-Samuelson (HOLS) model.

In this paper I start with the thesis, which I believe can command fairly wide agreement, that the great bulk of intra-industry trade consists of trade in economically distinct commodities. To be sure, there are documented cases of simultaneous cross-haulage of homogeneous standardized products such as steel and cement which result from cartel arrangements such as the basing-point system.³ But I do not believe that many trade economists would attribute to such arrangements a substantial proportion of intra-industry trade. As long as arbitrage is allowed in tradable goods, simultaneous cross-haulage of identical homogeneous goods is as pointless when they are produced under increasing as when they are produced under constant returns to scale. And aggregates of distinct brands of differentiated products do not come under the heading of identical homogeneous goods. As Drèze [14, p. 12] has stated, “American cigarettes, English cigarettes, French cigarettes, Egyptian cigarettes are four quite different things: if you doubt this, try offering a Gauloise to an American.” That being the case, intra-industry trade may be considered to be for the most part a “statistical phenomenon.”

This, of course, does not mean that the HOLLS model is better or worse equipped to explain the phenomenon than alternative models. But it does mean that whatever model is used should take explicit account of the aggregation process. That is, it should recognize that the data provided by international-trade statistics are aggregates of the variables of the pure theory—whether this theory assumes constant or increasing returns to scale, perfect or monopolistic competition.

In this paper I show that the HOLLS model, specialized to the case of loglinear production and utility functions, is capable of explaining any amount of intra-industry trade. This of course does not imply that it is necessarily the most appropriate model, i.e., the model that best explains the phenomenon; other models may succeed in explaining it better. But I submit that no model, whether conventional or otherwise, can do so if it ignores the nature of the aggregation process itself. The HOLLS model has the advantage of being a consistent general-equilibrium model; and it has been disfavored for wrong reasons, such as the argument rebutted above as well as the fact that it has been represented by a caricature—rather than the rich general version formulated by Samuelson [35].

For example, the so-called “Heckscher–Ohlin theorem” (cf. Jones [21])—which states that a country will export the commodity that uses relatively intensively the factor in which it is relatively well endowed—is valid only under very stringent conditions: free and costless trade between two countries in two commodities using two factors of production under conditions of perfectly competitive markets and constant returns to scale, absence of reversal of factor intensities, identical production functions and identical homothetic preferences between countries, and balanced trade. Given the stringency of these conditions it is somewhat surprising that in most of the literature it is only to the absence of perfect competition and constant returns to scale that the discrepancy between the “factor-proportions theory” and observed trade patterns is attributed, although it is obvious that in the real world most of the other assumptions—in particular that there are only two commodities, factors, and countries, and that trade is balanced—are clearly violated. The reason for this may be the general lack of awareness in the profession of how stringent these conditions are, given the scarcity of

³Cf. Machlup [29]. Obviously such cartel-induced cross-haulage is not limited to internal trade. For this observation see Brander [2].

correct statements to be found in the literature.⁴ Moreover it does not seem to be generally recognized that the Heckscher–Ohlin theorem is concerned only with the *direction* of trade; it says nothing about the *volume* of trade.

In this paper I show that the HOLS model, in its multi-commodity, multi-factor, multi-country version (cf. Samuelson [35]), is compatible with any degree of intra-industry trade; in particular, it is compatible with a situation in which all countries trade with each other and 100% of their trade is intra-industry trade. I show this for a special case of the HOLS model—that of identical Cobb–Douglas technologies and tastes, in which the number of commodities is equal to the number of factors, and in which trade is balanced. It goes without saying that under less stringent assumptions it would be even easier for the model to be compatible with intra-industry trade. The loglinear model—developed in Section 2—permits a simple algebraic solution to the world equilibrium problem, avoiding the complications of fixed-point theorems.⁵ Thus it provides explicit closed-form expressions for equilibrium trades and prices, making it possible to obtain answers to questions that would otherwise be difficult to obtain. The model of Section 2 also allows for interindustrial relationships and trade in intermediate products.

The main result, Theorem 1 of Section 3, takes as given arbitrary production and consumption coefficients, as well as any arbitrary mode of aggregating commodities into a smaller number of industrial groups. It assumes that the world factor endowments are such as to make possible, when appropriately allocated among countries, positive production of all n commodities in all countries and equalization of the n factor rentals among them. The theorem states that it is possible to find allocations of the world factor endowments among countries (in fact, many such allocations) such that all countries engage in trade and 100% of their trade is intra-industry trade.

The idea of the proof of Theorem 1 can be easily described. First it is assumed that the world factor endowments are initially distributed proportionately among countries; assuming nondegenerate technology (i.e., assuming the diversification cone to have a nonempty interior), this implies that there is no trade. Now a displacement vector is constructed—whose components are positive and negative increments in factor endowments—which has the property that adding any scalar multiple of this vector to a country’s factor-endowment vector (and subtracting a corresponding amount from those of the other countries) causes the country to start trading with its neighbors in such a way that the values of imports and exports remain the same in each aggregated industry, i.e., 100% of trade is intra-industry trade. For a sufficiently small scalar multiple this is always possible without violating any boundary constraints.

While examples may be found in which countries’ factor-endowment vectors depart substantially from proportionality and yet 100% of their trade remains intra-industry, nevertheless it is of interest that the theorem is in accord with the widespread observation that one finds large amounts of intra-industry trade between countries with similar factor endowments.

The final result is Theorem 2 in Section 4, which takes as its starting point the argument of Grubel and Lloyd criticized above. In the special case of two commodities, factors, and

⁴To my knowledge, the first correct statement and proof of the Heckscher–Ohlin theorem in the literature was that of Riezman [33], and the next that of Leamer [26]. Most statements to be found in the literature omit at least two crucial conditions. For a discussion see Chipman [11, pp. 937–8].

⁵For previous developments of models assuming Cobb–Douglas technology and/or preferences, see Laursen [25], Radner [32], Leamer [26], Chipman [9], [10], Hartigan & Tower [19], Krelle [23, pp. 634ff], Leamer [27]. Undoubtedly there have been others.

countries, an initial equilibrium is assumed with positive production of both commodities in both countries and equalization of factor rentals between them. It is then assumed that the production functions for the two commodities (which are identical between countries) become closer, one to the other (it does not matter which). It is shown, under the assumption of identical loglinear technology and preferences between countries, that this results in increased trade as measured by each country's export-output ratio.⁶ Since, according to Grubel and Lloyd, greater similarity of input requirements implies greater likelihood that the corresponding production processes will be classified as belonging to the same industry,⁷ this suggests that one may be more likely to observe intra- than inter-industry trade.

2 A Simple Loglinear Model of the World Economy

Let production functions in all countries be of the identical Cobb–Douglas type

$$(2.1) \quad q_j^k = f_j(u_{.j}^k, v_{.j}^k) = \mu_j \prod_{i=1}^n (u_{ij}^k)^{\alpha_{ij}} \prod_{i=1}^m (v_{ij}^k)^{\beta_{ij}}$$

$$(\alpha_{ij} \geq 0, \beta_{ij} \geq 0, \sum_{i=1}^n \alpha_{ij} < 1, \sum_{i=1}^n \alpha_{ij} + \sum_{i=1}^m \beta_{ij} = 1),$$

where q_j^k is the gross output of commodity j in country k and u_{ij}^k, v_{ij}^k are respectively the amounts of the i th intermediate and the i th primary input into the production of commodity j , and $u_{.j}^k, v_{.j}^k$ are vectors of these amounts. The minimum-unit-cost function dual to (2.1) is

$$(2.2) \quad g_j(p, w^k) = \nu_j \prod_{i=1}^n p_i^{\alpha_{ij}} \prod_{i=1}^m (w_i^k)^{\beta_{ij}} \quad (\nu_j = 1/f_j(\alpha_{.j}, \beta_{.j})),$$

where p_i and w_i^k are the i th commodity price and factor rental in country k respectively, and p, w^k denote the corresponding $n \times 1$ and $m \times 1$ vectors. Thus, the exponents α_{ij}, β_{ij} may be interpreted both as elasticities of gross outputs with respect to intermediate and primary inputs, and as elasticities of minimum unit costs with respect to prices of intermediate and primary inputs respectively. The cost-minimizing input-output and factor-output coefficients are given by $a_{ij} = \partial g_j / \partial p_i = p_j \alpha_{ij} / p_i$ and $b_{ij}^k = \partial g_j / \partial w_i^k = p_j \beta_{ij} / w_i^k$ respectively (cf. Shephard [36]). Denoting the $n \times n$ input-output and $m \times n$ factor-output matrices $A = [a_{ij}]$ and $B^k = [b_{ij}^k]$ respectively, as well as the corresponding elasticity matrices $A = [\alpha_{ij}]$ and $B = [\beta_{ij}]$ (which are distinguished from A and B by being set in roman rather than italic⁸), these relationships may be indicated in matrix notation by

$$(2.3) \quad A = P^{-1}AP, \quad B^k = (W^k)^{-1}BP$$

where P and W^k are diagonal matrices $P = \text{diag } p$ and $W^k = \text{diag } w^k$ of commodity prices and factor rentals respectively.

⁶Under the same set of assumptions it is also possible to derive a monotone relationship between the countries' export-output ratios and the the degree of similarity between their relative factor endowments—extending the Heckscher–Ohlin theorem from a proposition about directions of trade to one about the volume of trade. But such monotonicity cannot be established in general.

⁷This proposition has been challenged by Finger [15], but I do not try to argue this point here.

⁸This follows the notational convention introduced by Koopmans (1950, p. xiii).

In accordance with the well-known Leontief [28] theory, the $n \times 1$ vector of net outputs is defined by

$$(2.4) \quad y^k = (I - A)q^k, \quad \text{or} \quad Py^k = (I - A)Pq^k,$$

where q^k is the vector of gross outputs. The resource-allocation constraint (assuming full employment) is given by $\sum_{j=1}^n v_{ij}^k = \sum_{j=1}^n b_{ij}^k y_j^k = l_i^k$, yielding

$$(2.5) \quad B^k q^k = l^k, \quad \text{or} \quad BPq^k = W^k l^k,$$

where l^k is the $m \times 1$ vector of country k 's factor endowments. Note that in this loglinear model, linear relations between vectors of quantities defined by the input-output and factor-output matrices carry over into corresponding linear relations between vectors of *values* of these quantities defined by the respective matrices of input-output and factor-output elasticities.

It will now be assumed that all commodities are produced in each country, so that prices are equal to minimum unit costs for all n commodities. Since from (2.1) it follows that $I - A$ has a positive dominant diagonal (cf. McKenzie, [30]), its inverse exists and it satisfies the Hawkins–Simon [18] conditions, hence it satisfies $(I - A)^{-1} \geq I$ (cf. Chipman [5]). From (2.2) we have

$$(2.6) \quad \log p = [I - A']^{-1} \log \nu + [I - A']^{-1} B' \log w^k.$$

This defines a system of consolidated cost functions $p = \psi(w^k)$. Denoting

$$(2.7) \quad \Gamma = B(I - A)^{-1}, \quad C^k = B^k(I - A)^{-1},$$

where

$$C^k = (W^k)^{-1} \Gamma P,$$

clearly Γ has all its elements nonnegative. We verify that it has unit column sums, hence is column-stochastic: since $[A', B']$ has unit row sums from (2.1), the matrix $[I - A', -B']$ has zero row sums hence, letting ι denote a column vector of ones (of appropriate dimension), we have

$$\iota' \begin{bmatrix} I \\ -\Gamma \end{bmatrix} = \iota' \begin{bmatrix} I \\ -B(I - A)^{-1} \end{bmatrix} = \iota' \begin{bmatrix} I - A \\ -B \end{bmatrix} (I - A)^{-1} = 0$$

hence $\iota' \Gamma = \iota'$.

In order for (2.6) to provide a unique solution w^k for given p , it is necessary and sufficient that $\text{rank}(B) = m$,⁹ which implies in particular that $m \leq n$. This will be assumed in the rest

⁹Let B have rank $r \leq \min(m, n)$. From Penrose's [31] theory (see, e.g., Chipman [8]), (2.6) has a solution, $\log w^k$, if and only if $\log p$ is confined to an r -dimensional affine subspace of n -dimensional space defined by the projection

$$(I - A') \log p - \log \nu = B' B^{-'} [(I - A') \log p - \log \nu],$$

where B^{-} is a generalized inverse of B , i.e., any matrix satisfying $B B^{-} B = B$. (If $r = n$ this entails no restriction; otherwise it reflects the fact that country k 's production–possibility frontier is a ruled surface, and the budget hyperplane defined by the world prices must be tangential to this ruled surface.) Under these conditions the general solution of (2.6) is

$$\log w^k = B^{-'} [(I - A') \log p - \log \nu] + (I_m - B^{-'} B') \zeta$$

where ζ is an arbitrary $m \times 1$ vector. This is unique if and only if $B B^{-} = I_m$, i.e., if and only if $\text{rank}(B) = m$. This is equivalent to the condition that the diversification cone have a nonempty interior.

of the paper. From this assumption it follows that factor rentals are equalized throughout the world, and the k superscript may be dropped from w , b_{ij} , B , and C . From (2.4) and (2.5) we obtain the relationship between net outputs and factor endowments:

$$(2.8) \quad Cy^k = l^k, \text{ or } \Gamma Py^k = Wl^k.$$

If $n > m$, the solutions of these equations are not unique, reflecting the fact that country k 's production-possibility frontier is a ruled surface. In this case there is a basic indeterminacy in trade patterns, and to proceed in our analysis we would have to deal with sets of possible equilibrium trades. In order to obtain definite results, I shall therefore assume henceforth that $\text{rank}(B)$ (hence $\text{rank}(\Gamma)$) = n , hence $m = n$, which is equivalent to assuming that the production-possibility frontier is strictly concave to the origin (cf. Chipman [11]). Denoting $L^k = \text{diag } l^k$, we may write the solution of the second equation of (2.8) as

$$(2.9) \quad Py^k = \Gamma^{-1}Wl^k = \Gamma^{-1}L^k w.$$

To complete the model, let us now assume that preferences in each country are generated by the same Mill-Cobb-Douglas utility function

$$(2.10) \quad U(x^k) = \prod_{j=1}^n (x_j^k)^{\theta_j} \quad (\theta_j > 0, \quad \sum_{j=1}^n \theta_j = 1),$$

where x_j^k is the consumption of commodity j in country k . This implies that in each country, consumers spend a constant proportion θ_j of the disposable income Y^k on commodity j , i.e.,

$$(2.11) \quad Px^k = \theta Y^k, \quad \text{where } Y^k = \iota' Py^k + D^k,$$

where D^k is the deficit in country k 's balance of payments on current account.

World equilibrium may now be solved for explicitly as follows. Let there be K countries, and denote the world consumption, net output, and factor-endowment vectors and world income respectively by

$$(2.12) \quad x = \sum_{k=1}^K x^k, \quad y = \sum_{k=1}^K y^k, \quad l = \sum_{k=1}^K l^k, \quad Y = \sum_{k=1}^K Y^k.$$

World equilibrium is defined by the condition $x = y$. Aggregating (2.9) and (2.11) over countries and taking account of the fact that $\sum_{k=1}^K D^k = 0$ we obtain

$$(2.13) \quad Py = \Gamma^{-1}Wl$$

and

$$(2.14) \quad Px = \theta Y, \quad \text{where } Y = \iota' Px = \iota' Py = \iota' \Gamma^{-1}Wl.$$

Denoting $L = \text{diag } l$, upon equating (2.13) and (2.14) we obtain

$$(2.15) \quad w = L^{-1}\Gamma\theta Y.$$

The solution (2.15) is unique up to a proportionality factor. One could set $w_1 = 1$, or more symmetrically, $Y = 1$. The latter normalization will be used in Section 3.

Country k 's output-value vector is determined from (2.9) and (2.15) by

$$(2.16) \quad Py^k = \Gamma^{-1}Wl^k = \Gamma^{-1}L^k w = \Gamma^{-1}L^k L^{-1}\Gamma\theta Y,$$

and its consumption-expenditure vector is determined from (2.11) and (2.16) by

$$(2.17) \quad Px^k = \theta Y^k = \theta(\iota' Py^k + D^k) = \theta[\iota'\Gamma^{-1}L^k L^{-1}\Gamma\theta Y + D^k];$$

thus the values of its net imports $z^k = x^k - y^k$ are determined from

$$(2.18) \quad Pz^k = P(x^k - y^k) = \theta D^k - (I - \theta\iota')\Gamma^{-1}L^k L^{-1}\Gamma\theta Y.$$

Note that these vectors of values of production, consumption, and net imports (relative to world income, Y) depend only on the technology, preferences, and endowments, and may be obtained without solving for the equilibrium world prices. To obtain these prices one first solves (2.15) for the factor rentals and then substitutes these into the consolidated cost function (2.6) to obtain

$$(2.19) \quad \log p = \log \psi(w) = \log \nu^* + \Gamma' \log w = \log \nu^* + \Gamma' \log(L^{-1}\Gamma\theta Y),$$

where $\log \nu^* = [I - A']^{-1} \log \nu$.

3 The Main Theorem

Assuming that n traded commodities are produced in each of K countries with the aid of n factors of production, let

$$(3.1) \quad t^k = Pz^k$$

denote the vector of country k 's *trades*, defined as the values of its net imports—by (2.18). Country k 's gross imports and gross exports of commodity j may be defined by

$$(3.2) \quad z_j^k(+)=\max(z_j^k,0), \quad z_j^k(-)=-\min(z_j^k,0)$$

respectively, yielding the corresponding vectors

$$(3.3) \quad z^k(+)= (z_1^k(+), \dots, z_n^k(+))', \quad z^k(-)= (z_1^k(-), \dots, z_n^k(-))'$$

In value terms we have the vectors of gross imports and exports

$$(3.4) \quad t^k(+)= Pz^k(+), \quad t^k(-)= Pz^k(-),$$

which have the property that net imports and total trades (in absolute value) are given respectively by

$$(3.5) \quad t^k = t^k(+)-t^k(-) \quad \text{and} \quad |t^k| = t^k(+)+t^k(-).$$

International trade statistics provide data not on the net and absolute trades (3.5), but on aggregates of these. Let G denote an $\bar{n} \times n$ *grouping matrix*, defined as a matrix with zeros and ones with exactly one unit element in each column (cf. Chipman [6], [7], [8]), where $\bar{n} < n$. Trade statistics provide data on

$$(3.6) \quad \bar{t}^k(+)= Gt^k(+), \quad \bar{t}^k(-)= Gt^k(-),$$

i.e., on total gross imports and gross exports of all commodities in each category, and therefore also of net imports

$$(3.7) \quad \bar{t}^k(+)-\bar{t}^k(-)=G[t^k(+)-t^k(-)]=Gt^k$$

and of total trade

$$(3.8) \quad \bar{t}^k(+)+\bar{t}^k(-)=G[t^k(+)+t^k(-)]=G|t^k|$$

in each aggregated category.¹⁰

By their definitions, the disaggregated vectors of imports and exports $t^k(+)$ and $t^k(-)$ are orthogonal to one another; there cannot be both imports and exports in the same disaggregated category. In general this is not true of the aggregated vectors $\bar{t}^k(+)$ and $\bar{t}^k(-)$. If it were true of the aggregate vectors, for all aggregated categories, then we would have $|Gt^k|=G|t^k|$, and we would say that there is no intra-industry trade. At the other extreme, if import values exactly balance export values in each aggregated category, then $Gt^k=0$. The *Grubel-Lloyd index of intra-industry trade* (cf. Grubel & Lloyd [17, p. 22]) is defined (for $t^k \neq 0$) by

$$(3.9) \quad Q_{GL}=1-\frac{\iota'|Gt^k|}{\iota'|G|t^k|}=1-\frac{\iota'|Gt^k|}{\iota'|t^k|},$$

(the last equality following from the fact that $\iota'G=\iota'$, by the definition of G). The Grubel-Lloyd index takes on the values 0 and 1 in the two extreme cases just considered.

The second of these extreme cases is the major object of interest in this section. Is it possible, and if so under what conditions, for 100% of trade to be intra-industry trade?

From the form of (3.9), it is clear that $Q_{GL}=1$ if and only if $Gt^k=0$. The Grubel-Lloyd index is an appropriate measure only if trade is balanced,¹¹ but this assumption will be adhered to here. Accordingly, from (3.1) and (2.18) we see that $Q_{GL}=1$ if and only if

$$(3.10) \quad Gt^k=GPz^k=-G(I-\theta\iota')\Gamma^{-1}L^kL^{-1}\Gamma\theta Y=0.$$

This formula may be simplified in the following way. Define the vector

$$(3.11) \quad \lambda^k=\Lambda^k\iota, \quad \text{where } \Lambda^k=L^kL^{-1}.$$

Its components are country k 's endowments relative to the world's. Recalling that L^k and L are diagonal matrices (see (2.9) and (2.15)), the product of the diagonal matrix Λ^k and the

¹⁰The process of grouping imports and exports according to (3.2) is of course itself a form of aggregation. We may suppose the grouping matrix G to be decomposed into a product $G=H^*G^*$ of $\bar{n}\times 2\bar{n}$ and $2\bar{n}\times n$ grouping matrices H^* and G^* , where G^* is partitioned into two $\bar{n}\times n$ import and export grouping submatrices G_+ and G_- . These grouping submatrices distinguish and group together the import (resp. export) subindustries within standard industries, so that

$$G_+t^k=Gt^k(+) \quad \text{and} \quad G_-t^k=-Gt^k(-).$$

The matrix H^* aggregates these import and export subindustries together into standard industries.

A problem facing the econometrician is that while there are data on trades $G_\sigma t^k$ as well as on import and export price indices of the form $G_\sigma^{\dagger'}p=(G_\sigma DG'_\sigma)^{-1}G_\sigma Dp$ (where D is a diagonal matrix of weights—the absolute quantities of imports and exports in a base year), there are no corresponding data on $G_\sigma x^k$, $G_\sigma y^k$, $G_\sigma q^k$ ($\sigma=+,-$). For a solution see Chipman [12].

¹¹A more general measure which reduces to the Grubel-Lloyd index when trade is balanced was provided by Aquino [1]. See also Chipman [11].

vector $\Gamma\theta$ may be written as the product of the diagonal matrix $\text{diag}(\Gamma\theta)$ and the vector λ^k , i.e.,

$$(3.12) \quad \Lambda^k \Gamma \theta = \text{diag}(\Gamma \theta) \lambda^k,$$

hence (3.10) yields the condition

$$(3.13) \quad G(I - \theta \iota') \Gamma^{-1} \text{diag}(\Gamma \theta) \lambda^k = 0.$$

For given Γ and θ , (3.13) expresses a condition on the vector λ^k of country k 's endowments relative to world endowments.

Three more conditions must be satisfied by λ^k . First, it must be consistent with positive gross outputs. From (2.4), (2.3), and (2.16), and using the normalization $Y = 1$, we then have the condition

$$(3.14) \quad Pq^k = (I - A)^{-1} \Gamma^{-1} \text{diag}(\Gamma \theta) \lambda^k > 0 \quad (k = 1, 2, \dots, K).$$

A second condition is required in order to rule out trivialities, namely that each country be engaged in some trade, i.e., $t^k \neq 0$, or $\iota' |t^k| > 0$; this condition is also needed for the Grubel–Lloyd index to be well defined. From (3.1) and (2.18), and given our assumption of balanced trade and the normalization $Y = 1$, this condition reduces to

$$(3.15) \quad t^k = Pz^k = -(I - \theta \iota') \Gamma^{-1} \text{diag}(\Gamma \theta) \lambda^k \neq 0 \quad (k = 1, 2, \dots, K).$$

Finally we of course require that the countries' relative factor endowments be positive and sum to unity, i.e.,

$$(3.16) \quad \lambda^k > 0 \quad (k = 1, 2, \dots, K), \quad \sum_{k=1}^K \lambda^k = \iota.$$

Note from (2.1), (2.5), and (3.11) that the inequalities (3.16) are redundant, being implied by (3.14).¹²

To summarize, our problem is to show that there exists a set of endowment-ratio vectors λ^k ($k = 1, 2, \dots, K$) satisfying (3.13), (3.14), (3.15), and (3.16).

Theorem 1. *There exists a solution λ^k ($k = 1, 2, \dots, K$) to (3.13), (3.14), (3.15), and (3.16); i.e., given any technology matrix Γ and any positive vector θ of consumer-expenditure shares, as well as any vector l of world factor endowments permitting positive production of all commodities in all countries, and given any way of grouping commodities into a smaller number of industries (i.e., given any $\bar{n} \times n$ grouping matrix G with $\bar{n} < n$), there exists an allocation of the world factor endowments among countries such that each country engages in trade and 100% of each country's trade is intra-industry trade.*

Proof. Defining

$$(3.17) \quad M = (I - \theta \iota') \Gamma^{-1} \text{diag}(\Gamma \theta),$$

we start by showing that there exists a vector ρ such that

$$(3.18) \quad M\rho \neq 0 \quad \text{and} \quad GM\rho = 0.$$

¹²In fact, zero outputs will be reached when a factor–endowment ray hits the edge of a diversification cone, long before it hits one of the axes.

Observe that

$$(3.19) \quad M\iota = (I - \theta\iota')\Gamma^{-1}\Gamma\theta = 0.$$

Since $I - \theta\iota'$ is idempotent of rank $n - 1$, M has rank $n - 1$ and nullity 1; therefore from (3.19) it follows that the null space of M is exactly the space

$$(3.20) \quad \mathcal{N}(M) = \{\rho \in E^n : \rho = \omega\iota \text{ for some } \omega, \quad -\infty < \omega < \infty\}.$$

Hence ρ satisfies the first condition of (3.18) if and only if it is not proportional to ι , i.e., if and only if its components are not all equal to one another.

Now G is $\bar{n} \times n$ of rank \bar{n} . Since it satisfies $\iota'G = \iota'$, it follows that

$$\iota'G(I - \theta\iota') = \iota'(I - \theta\iota') = 0,$$

i.e., that the rows of $G(I - \theta\iota')$ (hence those of GM) are linearly dependent. Therefore GM has rank $\bar{n} - 1$ and nullity $n - (\bar{n} - 1) = n - \bar{n} + 1 \geq 2$. The null space of GM contains that of M , and $n - \bar{n}$ dimensions in addition; it follows that it contains vectors ρ whose elements are not all equal to one another, i.e., that (3.18) is satisfied.

Now define, for such a ρ satisfying (3.18),

$$(3.21) \quad \lambda^k = \omega_k\iota + \varepsilon_k\rho \quad (\omega_k > 0, \quad \varepsilon_k \neq 0)$$

where

$$(3.22) \quad \sum_{k=1}^K \omega_k = 1, \quad \sum_{k=1}^K \varepsilon_k = 0.$$

Scaling down the ε_k s by a common factor (if necessary), it is clear that they can be chosen so that (3.16) is satisfied; similarly, for sufficiently small ε_k s we have

$$(3.23) \quad \begin{aligned} (I - A)^{-1}\Gamma^{-1}\text{diag}(\Gamma\theta)\lambda^k = \\ (I - A)^{-1}\{\omega_k\theta + \varepsilon_k\Gamma^{-1}\text{diag}(\Gamma\theta)\rho\} > 0 \quad (k = 1, 2, \dots, K) \end{aligned}$$

from (3.21) and the facts that $\theta > 0$ and $(I - A)^{-1} \geq I$, hence (3.14) is satisfied. Finally we verify from (3.17), (3.21), and (3.18) that

$$(3.24) \quad M\lambda^k = \varepsilon_k M\rho \neq 0 \quad \text{and} \quad GM\lambda^k = \varepsilon_k GM\rho = 0,$$

hence (3.15) and (3.13) are satisfied. Q.E.D.

In interpreting this theorem the following points should be kept in mind. (a) If $\bar{n} = 1$ (which in particular will be the case if $n = 2$), the conclusion of the theorem is of course trivial, since if trade is balanced and all goods are grouped into one industry, obviously all trade is intra-industry trade. (b) If $\bar{n} > n/2$, then necessarily at least one of the n commodities will be left unaggregated. The theorem therefore implies that this commodity is not traded. If the theorem is to be consistent with the existence of trade in every category, obviously one must have $\bar{n} \leq n/2$. From (a) and (b) it follows that if the theorem is to be interesting one must have $2 \leq \bar{n} \leq n/2$. Since in the real world n is of the order of tens of thousands, and even at the crudest levels of aggregation (say, 1-digit SITC) $\bar{n} \geq 10$, the result can claim to be of practical interest. (c) Note from (3.23) that allowance for trade in intermediate products widens the possibilities for intra-industry trade.

There is also an interesting interpretation in terms of the criterion of similarity of factor endowments. If $\lambda^k = \omega_k \ell$ for $0 < \omega_k < 1$, then from (2.16) and (3.12) it follows that $y^k = \omega_k P^{-1} \theta Y$ (net output equals consumption) and country k will not trade. The form (3.21) chosen for λ^k ensures that this will not be the case; but for small ε_k it may be interpreted as saying that countries' factor endowments are "similar." As the ε_k s increase in absolute value, sooner or later inequality (3.14) will be violated; world equilibrium would have to be computed in a different way, but it is safe to say that there would be no reason to assume that all trade would continue to be intra-industry trade. In this sense it may be said that a strong relationship appears to hold between similarity of factor endowments and the phenomenon of intra-industry trade.

4 Trade and Similarity of Production Functions —the $2 \times 2 \times 2$ Case

In this section I consider the specialization of the model of Section 2 to the textbook case $n = m = K = 2$. Starting from a situation in which each country produces positive amounts of both commodities, it will be shown that a movement of either production coefficient in the direction of the other will increase the proportion exported of each country's export good. Thus we may say that greater similarity in production functions leads to more trade.

Let us assume that, in terms of the integrated technology defined by the matrix $\Gamma = [\gamma_{ij}]$ of (2.7), commodity j uses factor j relatively intensively for $j = 1, 2$, so that $\gamma_{11} > \gamma_{21}$ and $\gamma_{22} > \gamma_{12}$; this is equivalent to the condition that $\gamma_{21} < \frac{1}{2}$ and $\gamma_{12} < \frac{1}{2}$. Let us also assume that country 1 is relatively well endowed in factor 1, i.e., $l_1^1/l_2^1 > l_1^2/l_2^2$; this is equivalent to the condition $l_1^1/l_2^1 > l_1/l_2$ (and $l_2^2/l_1^2 > l_2/l_1$). The final assumption to be used is that of balanced trade. As a normalization I take $Y = 1$.

As a measure of intensity of trade I choose for each country $k = 1, 2$ its export-output ratio

$$(4.1) \quad -\frac{z_k^k}{y_k^k} = \frac{p_k y_k^k - p_k x_k^k}{p_k y_k^k} = 1 - \frac{\theta(p_j y_j^k + p_k y_k^k)}{p_k y_k^k} \\ = 1 - \theta_k - \theta_k \frac{p_j y_j^k}{p_k y_k^k} \quad (j \neq k),$$

using (2.11) with the balanced-trade condition $D^k = 0$. An alternative and simpler measure to work with is therefore $r_k = p_k y_k^k / p_j y_j^k$ —the ratio of net values of output of the exportable and importable ($j \neq k$). From (2.16) it is clear that this ratio is independent of the prices. Denoting $\gamma = \Gamma \theta$ and $\lambda_i^k = l_i^k / l_i$, we have from (2.16)

$$(4.2) \quad r_k = \frac{(1 - \gamma_{kj}) \gamma_k \lambda_k^k - \gamma_{kj} \gamma_j \lambda_j^k}{(1 - \gamma_{jk}) \gamma_j \lambda_j^k - \gamma_{jk} \gamma_k \lambda_k^k} = \frac{\eta_k}{\eta_j} \quad (j \neq k).$$

The numerator and denominator of (4.2) define the expressions η_k and η_j .

Theorem 2. *In the loglinear model of the world economy, when there are two commodities, factors, and countries with identical production functions across countries in each industry and equal factor rentals across countries, if the integrated factor–output ratios are*

such that commodity 1 uses a higher ratio of factor 1 to factor 2 than commodity 2 (i.e., $\gamma_{11} > \gamma_{12}$), and if country 1 has a higher relative endowment in factor 1 compared with factor 2 (i.e., $l_1^1/l_2^1 > l_1^2/l_2^2$), then a movement of either production coefficient γ_{12} , γ_{21} in the direction of the other will increase both countries' export–output ratios.

Proof. Since country k 's export–output ratio (4.1) is a monotone increasing function of the variable r_k of (4.2), it suffices to show that under the given assumptions $\partial r_k/\partial \gamma_{kj}$ and $\partial r_k/\partial \gamma_{jk}$ are both positive.

We find after some tedious calculations that $\partial r_k/\partial \gamma_{kj}$ is proportional to

$$(4.3) \quad \eta_j \frac{\partial \eta_k}{\partial \gamma_{kj}} - \eta_k \frac{\partial \eta_j}{\partial \gamma_{kj}} = (\lambda_j^k)^2 (\lambda_k - 1) [\gamma_{jk} (\gamma_k)^2 \lambda_k + \gamma_{kk} (\gamma_j)^2],$$

where

$$(4.4) \quad \lambda_k = \frac{\lambda_k^k}{\lambda_j^k} = \frac{l_k^k/l_k}{l_j^k/l_j}.$$

Thus, $\partial r_k/\partial \gamma_{kj} > 0$ as long as $\lambda_k > 1$, i.e., as long as country k is relatively well endowed in factor k , as has been assumed.

Likewise we find that $\partial r_k/\partial \gamma_{jk}$ is proportional to

$$(4.5) \quad \eta_j \frac{\partial \eta_k}{\partial \gamma_{jk}} - \eta_k \frac{\partial \eta_j}{\partial \gamma_{jk}} = (\lambda_j^k)^2 (\lambda_k - 1) [\gamma_{jj} (\gamma_k)^2 \lambda_k + \gamma_{kj} (\gamma_j)^2],$$

where λ_k is defined by (4.4), hence $\partial r_k/\partial \gamma_{jk} > 0$ as long as $\lambda_k > 1$, as assumed. Q.E.D.

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