A General-Equilibrium Framework for Analyzing the Responses of Imports and Exports to External Price Changes: An Aggregation Theorem

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Abstract

Assuming preferences to be generated by an aggregate utility function, an aggregate demand function for imports and exports is defined as a function of import and export prices and the deficit in the balance of payments on current account, in a model with nontradable as well as traded goods. It is shown that this function is integrable, hence can be generated by maximizing a utility function (with import and export quantities as arguments) subject to a balance-of-payments constraint.

1 Introduction

Empirical work on the measurement of responses of imports and exports to external price changes is typically based on the specification of an aggregate demand function for imports and exports with prices of imports and exports, as well as national income, as arguments (cf. Leamer & Stern, 1970; Stern, Francis & Schumacher, 1976). In this paper an aggregate demand function for imports and exports is defined in a model in which there are nontradable as well as tradable goods, and in which an aggregate consumer demand function for final products is assumed to exist (generable by an aggregate utility function). However, factor rentals and hence consumer incomes are determined in this model on the basis of external prices and any exogenously determined deficit or surplus in the balance of payments on current account. National income cannot therefore be legitimately treated as an exogenous variable. The aggregate net-import demand function therefore has as arguments the import and export prices and the deficit in the balance of payments on current account.

It is shown that this aggregate net-import demand function is itself generable by an aggregate "trade-utility function" with quantities of imports and exports as arguments. No nontradables appear as arguments in this function. The result is thus a generalization of Meade's (1952) concept of "trade-indifference curves" to a model with nontraded goods. The result also furnishes a theoretical foundation for the parametric specification of net-import demand functions that can be used in econometric applications.

*Research supported by NSF grant SES–8008047 and a fellowship from the John Simon Guggenheim Memorial Foundation.

2 The Basic Model

Following Chipman (1978, 1980), it will be assumed that there are four categories of commodities: n_1 tradable commodities produced at home, n_2 importable commodities not produced at home, n_3 nontradable goods and services produced at home, and $n_4 \equiv m$ primary factors of production; and that commodities do not switch categories during the period studied. Commodities in the first three categories are assumed to be producible according to single-valued, homogeneous-of-degree-one and concave production functions

$$y_j^k = f_j^k(v_j^k) \quad (j = 1, 2, \ldots, n_k; k = 1, 2, 3) \tag{1}$$

where

$$v_j^k = (v_{1j}^k, v_{2j}^k, \ldots, v_{mj}^k)' \tag{2}$$

is a vector of inputs of m primary factors of production. Dual to (1) are the Samuelson (1953)–Shephard (1953) minimum-unit-cost functions

$$g_j^k(w) \quad (j = 1, 2, \ldots, n_k; k = 1, 2, 3) \tag{3}$$

which are also homogeneous of degree one and concave, where

$$w = (w_1, w_2, \ldots w_m)' \tag{4}$$

is the vector of factor rentals, identical among industries owing to the assumption of perfect factor mobility. Denoting by p_j^k the price of the jth commodity in category k, we have

$$g_j^k(w) = p_j^k \quad (j = 1, 2, \ldots, n_k; k = 1, 3) \tag{5}$$

in the producing industries (owing to the assumption of perfect competition) and

$$g_j^2(w) > p_j^2 \quad (j = 1, 2, \ldots, n_2) \tag{6}$$

in the nonproducing industries, owing to the assumption that these are nonproducing and continue to be so after small (upward) perturbations in the prices.

By Shephard's (1953) duality theorem, the factor-output coefficient for the ith factor in the jth industry in category k is given by

$$v_{ij}^k / y_j^k = b_{ij}^k(w) = \partial g_j^k(w)/\partial w_i \quad (j = 1, 2, \ldots, n_k; k = 1, 3). \tag{7}$$

Denoting by ℓ_i the country's endowment of factor i, the assumptions of full employment of all factors and of their perfect mobility among industries entail

$$\sum_{j=1}^{n_1} b_{ij}^1(w) y_j^1 + \sum_{j=1}^{n_3} b_{ij}^3(w) y_j^3 = \ell_i \quad (i = 1, 2, \ldots, m). \tag{8}$$

To complete the model we introduce the aggregate demand functions

$$x_j^k = h_j^k(p^1, p^2, p^3, Y) \quad (j = 1, 2, \ldots, n_k; k = 1, 2, 3) \tag{9}$$

which are assumed to be generated by maximization of an aggregate utility function $U(x^1, x^2, x^3)$ subject to a budget constraint

$$\sum_{k=1}^{3} \sum_{j=1}^{n_k} p_j^k x_j^k \leq Y, \tag{10}$$

where Y is disposable national income, or "absorption" in the terminology of balance-of-payments theory.[1] This is defined as

$$Y = p^{1\prime}y^1 + p^{3\prime}y^3 + D \tag{11}$$

where D is the deficit in the balance of payments on current account.[2] A positive deficit permits the country to "live beyond its means" and spend more than the value of its output.[3] By definition of category 3, demand for nontradables is equal to the supply, i.e., $x_j^3 = y_j^3$, so that in equilibrium we have, from (9) and (11),

$$y_j^3 = h_j^3(p^1, p^2, p^3, p^{1\prime}y^1 + p^{3\prime}y^3 + D) \quad (j = 1, 2, \ldots, n_3). \tag{12}$$

Our four basic sets of equations (12), (8) and (5) may be written in matrix notation as

$$\begin{aligned} h^3(p^1, p^2, p^3, p^{1\prime}y^1 + p^{3\prime}y^3 + D) &= y^3 \\ B^1(w)y^1 + B^3(w)y^3 &= \ell \\ g^1(w) &= p^1 \\ g^3(w) &= p^3 \end{aligned} \tag{13}$$

where unprimed vectors and vector-valued functions denote column vectors and the $B^k(w)$'s are the $m \times n_k$ factor-output matrices

$$B^k(w) = [b_{ij}^k(w)]. \tag{14}$$

Equations (13) are $n_3 + m + n_1 + n_3$ in number, and can be solved for the $n_3 + m + n_1 + n_3$ endogenous variables p^3, w, y^1 and y^3 as functions of the $n_1 + n_2 + 1 + m$ exogenous variables p^1, p^2, D and ℓ:

$$\begin{aligned} p^3 = \tilde{p}^3(p^1, p^2, D, \ell); \quad y^1 = \tilde{y}^1(p^1, p^2, D, \ell); \\ w = \tilde{w}(p^1, p^2, D, \ell); \quad y^3 = \tilde{y}^3(p^1, p^2, D, \ell). \end{aligned} \tag{15}$$

The functions (15) will be referred to as the "reduced form" of the system (13).

We define the net import of commodity j in category k as

$$z_j^k = x_{ij}^k - y_j^k \quad (j = 1, 2, \ldots, n_k; k = 1, 2). \tag{16}$$

This is positive in the case of imports and negative in the case of exports. Of course, $z_j^2 = x_j^2$. The *net-import demand functions* are defined for $k = 1, 2$ by

$$\begin{aligned} &\tilde{z}_j^k(p^1, p^2, D, \ell) = \\ &h_j^k[p^1, p^2, D, \ell, p^{1\prime}\tilde{y}^1(p^1, p^2, D, \ell) + \tilde{p}^3(p^1, p^2, D, \ell)'\tilde{y}^3(p^1, p^2, D, \ell)' + D] \\ &\quad - \tilde{y}_j^k(p^1, p^2, D, \ell), \end{aligned} \tag{17}$$

[1] For conditions under which an aggregate demand function can be defined, see Chipman (1974).

[2] Note that this formulation assumes that all domestically-situated capital is owned by domestic residents, and conversely that domestic residents own only domestic capital; it likewise assumes no repatriation of earnings by guest workers, etc., except to the extent that such transfers are subsumed under D. Obvious adjustments should be made to (11) in a more general formulation.

[3] When a version of this paper was first presented at the second Latin American Regional Congress of the Econometric Society in Rio de Janeiro, 17 July 1981, my discussant Yair Mundlak made the interesting suggestion that the "absorption" term Y in (9) be replaced by an indicator of "permanent income" or wealth. Such a procedure would allow payment deficits to be explained endogenously as the consequence of expenditure or expected ("permanent") income exceeding actual income (absorption). I hope to take this suggestion up in a later formulation.

where of course

$$\tilde{y}_j^2(p^1,p^2,D,\ell) \equiv 0 \quad (j=1,2,\ldots,n_2). \tag{18}$$

The main proposition to be proved in this paper is that the net-import demand functions (17) may be considered as being generated by maximization of an aggregative trade-utility function $\tilde{U}(z^1,z^2)$ subject to the budget constraint

$$\sum_{k=1}^{2}\sum_{j=1}^{n_k} p_j^k z_j^k \leq D. \tag{19}$$

This generalizes the concept of Meade's (1952) "trade-indifference curves", or the general concept of a "trade-utility function" (Chipman, 1979), to the case of a model of international trade in the presence of nontraded goods. It is important to stress that only the imports and exports of tradables enter as arguments of this trade-utility function.

There are three possible approaches to establishing this proposition. One is a direct set-theoretic approach such as that adopted in Chipman (1979), in which a trade-utility function is defined by maximization of the utility function over an appropriate shifted production-possibility set. Another is to start with an indirect utility function $V(p^1,p^2,p^3,Y)$ associated with (9) and then define an indirect trade-utility function $\tilde{V}(p^1,p^2,D,\ell)$ by substituting the reduced-form equations in (11) and thence in the arguments of V, and showing that the net-import demand functions (17) satisfy the Antonelli-Allen-Roy partial differential equations

$$\frac{\partial \tilde{V}(p^1,p^2,D,\ell)}{\partial p_j^k} = -\frac{\partial \tilde{V}(p^1,p^2,D,\ell)}{\partial D}\tilde{z}_j^k(p^1,p^2,D,\ell) \tag{20}$$

(cf. Chipman & Moore 1976, p. 74). A third method, which is the one I shall adopt here, is to compute the Slutsky matrix of the net-import demand function (17) and verify that it is symmetric and negative semi-definite, and then appeal to the results of Hurwicz & Uzawa (1972). I present this third approach here since it is the way in which I actually stumbled across the result. The expressions obtained for the Slutsky matrix are also of independent interest. However, I shall be able within the confines of this paper to present the computations only for two of the three cases

$$\text{(i)}\;\; m \leq n_1; \;\;\text{(ii)}\; n_1 < m < n_1+n_3; \;\;\text{(iii)}\;\; m \geq n_1+n_3. \tag{21}$$

Case (iii) is the most straightforward, since it is the case in which the country's production-possibility frontier is strictly concave to the origin; it will be dealt with in Section 3. As for Case (i), it reduces to an equality $m = n_1$ when one takes account of the fact that the third set of equations of (13) require the external prices p^1 to be in the manifold of price vectors swept out by $g^1(w)$ as w varies, and this manifold has dimension at most m. If the external prices are to be regarded as truly exogenous, we must have $n_1 \leq m$. The case $m = n_1$ is dealt with in Section 4. Treatment of Case (ii) requires elaborate matrix computations of which only a general indication is given in Section 5.

3 The Case of Single-Valued Rybczynski Functions

In this section I take up Case (iii) of (21), in which the number of primary factors of production exceeds or equals the number of commodities (tradable and nontradable) produced at home, so that (except for singular cases which we can ignore here) the production-possibility frontier is strictly concave to the origin and thus there exist single-valued Rybczynski (supply) functions $\hat{y}^k_j(p^1, p^2, p^3; \ell)$ satisfying

$$\hat{y}^k_j(p^1, p^2, p^3; \ell) = \partial\Pi(p^1, p^2, p^3; \ell)/\partial p^k_j, \tag{22}$$

where Π is the domestic-product function, defined as the maximum value of output $\sum_{k=1}^3 p^k \cdot y^k$ over the production-possibility set (cf. Samuelson, 1953; Chipman, 1972). Since, by hypothesis, commodities in category 2 are not produced, in what follows the argument p^2 will be dropped from the Rybczynski and domestic-product functions $\hat{y}^k_j$ and Π.

In this case we need only consider the first set of equations of (13), and we may define the function $\tilde{p}^3(p^1, p^2, D, \ell)$ implicitly by

$$\begin{aligned} &h^3[p^1, p^2, \tilde{p}^3(p^1, p^2, D, \ell), \Pi(p^1, \tilde{p}^3(p^1, p^2, D, \ell); \ell) + D] \\ &\qquad = \hat{y}^3[p^1, \tilde{p}^3(p^1, p^2, D, \ell); \ell]. \end{aligned} \tag{23}$$

The functions $\tilde{y}^k$ of (15) are then given by

$$\tilde{y}^k(p^1, p^2, D, \ell) = \hat{y}^k[p^1, \tilde{p}^3(p^1, p^2, D, \ell); \ell], \tag{24}$$

and the net-import demand functions (17) are given by

$$\begin{aligned} &\tilde{z}^r(p^1, p^2, D, \ell) = \\ &h^r[p^1, p^2, \tilde{p}^3(p^1, p^2, D, \ell), \Pi(p^1, \tilde{p}^3(p^1, p^2, D, \ell); \ell) + D] - \hat{y}^r[p^1, \tilde{p}^3(p^1, p^2, D, \ell); \ell] \end{aligned} \tag{25}$$

for $r = 1, 2$ (note that $\hat{y}^2 \equiv 0$).

We define the $n_r \times n_k$ Slutsky submatrix

$$\tilde{S}^{rk} = \frac{\partial \tilde{z}^r}{\partial p^k} + \frac{\partial \tilde{z}^r}{\partial D} \tilde{z}^{k\prime} \quad (r, k = 1, 2) \tag{26}$$

where $\partial \tilde{z}^r / \partial p^k = [\partial \tilde{z}^r_i / \partial p^k_j]$. The Slutsky matrix $\tilde{S}$ of the net-import demand function $\tilde{z}(\tilde{z}^1, \tilde{z}^2)'$ is then the $(n_1 + n_2) \times (n_1 + n_2)$ matrix consisting of the 2×2 block matrix $[\tilde{S}^{rk}]$, $r, k = 1, 2$; I shall call this the "trade-Slutsky matrix". Analogously we define the partial $n_r \times n_k$ Slutsky matrix of the demand function h by

$$S^{rk} = \frac{\partial h^r}{\partial p^k} + \frac{\partial h^r}{\partial Y} h^k, \tag{27}$$

and the $n_r \times n_k$ transformation matrix associated with the Rybczynski function $\hat{y}^r$ by

$$T^{rk} = \frac{\partial \hat{y}^r}{\partial p^k}. \tag{28}$$

The full Slutsky and transformation matrices S and T are therefore the $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ matrices formed by the 3×3 block matrices $[S^{rk}]$ and $[T^{rk}]$ for $r, k = 1, 2, 3$. I assume that the submatrices S^{33} and T^{33} have full rank n_3.[4]

From (23) we find that

$$\partial \tilde{p}^3 / \partial p^k = (S^{33} - T^{33})^{-1} [c^3 z^{k\prime} - (S^{3k} - T^{3k})] \tag{29}$$

[4]This is a regularity condition which is assured if the full matrices S and T have rank $n_1 + n_2 + n_3 - 1$, and this condition in turn can be interpreted as a smoothness condition on indifference surfaces and on the production-possibility frontier.

and, as in Chipman (1980),

$$\partial\tilde{p}^3/\partial D = -(S^{33} - T^{33})^{-1}c^3, \tag{30}$$

where we define the column vector

$$c^k = \partial h^k/\partial Y. \tag{31}$$

Now, differentiating (25) with respect to p^k and D we obtain, respectively,

$$\partial\tilde{z}^r/\partial p^k = S^{rk} - T^{rk} + (S^{r3} - T^{r3})\partial\tilde{p}^3/\partial p^k - c^r z^{k\prime} \tag{32}$$

and (keeping in mind that $x^3 = y^3$)

$$\partial\tilde{z}^r/\partial D = (S^{r3} - T^{r3})\partial\tilde{p}^3/\partial D + c^r. \tag{33}$$

Substituting (29) in (32) and (30) in (33) we obtain

$$\begin{aligned}\partial\tilde{z}^r/\partial p^k &= [(S^{rk} - T^{rk}) - (S^{r3} - T^{r3})(S^{33} - T^{33})^{-1}(S^{3k} - T^{3k})] \\ &\quad -[c^r - (S^{r3} - T^{r3})(S^{33} - T^{33})^{-1}c^3]z^{k\prime},\end{aligned} \tag{34}$$

and

$$\partial\tilde{z}^r/\partial D = c^r - (S^{r3} - T^{r3})(S^{33} - T^{33})^{-1}c^3, \tag{35}$$

so that the partial trade-Slutsky matrix (26) is precisely

$$\tilde{S}^{rk} = (S^{rk} - T^{rk}) - (S^{r3} - T^{r3})(S^{33} - T^{33})^{-1}(S^{3k} - T^{3k}). \tag{36}$$

The full trade-Slutsky matrix may therefore be written

$$\begin{aligned}\tilde{S} &= \begin{bmatrix} \tilde{S}^{11} & \tilde{S}^{12} \\ \tilde{S}^{21} & \tilde{S}^{22} \end{bmatrix} = \begin{bmatrix} S^{11} - T^{11} & S^{12} - T^{12} \\ S^{21} - T^{21} & S^{22} - T^{22} \end{bmatrix} \\ &\quad - \begin{bmatrix} S^{13} - T^{13} \\ S^{23} - T^{23} \end{bmatrix} (S^{33} - T^{33})^{-1} [\; S^{31} - T^{31} \quad S^{32} - T^{32} \;].\end{aligned} \tag{37}$$

It is clearly symmetric. Since the net Slutsky matrix $S - T$ is symmetric and nonpositive definite, we can find a $\rho \times (n_1 + n_2 + n_3)$ matrix $R = [R_1, R_2, R_3]$ (where $\rho = \text{rank}(S - T) \leq n_1 + n_2 + n_3 - 1$) such that

$$-(S - T) = R'R, \tag{38}$$

so that (37) may be written

$$-\tilde{S} = [R_1, R_2]'[I - R_3(R_3'R_3)^{-1}R_3'][R_1, R_2]. \tag{39}$$

Since the matrix $I - R_3(R_3'R_3)^{-1}R_3'$ is symmetric and idempotent (of rank $\rho - n_3$) this proves that $-\tilde{S}$ is nonnegative definite, hence $\tilde{S}$ is nonpositive definite.[5]

[5] The proof just given is simply a proof of the matrix Schwarz inequality. Cf. Chipman (1976).

4 The Case of Equal Numbers of Factors and Produced Tradables

I come now to Case (i) of (21) where $m = n_1$. This is the case in which the reasoning underlying Samuelson's (1953) factor-price-equalization theorem can be applied. The third set of equations of (13) is in this case a system of $n_1 = m$ equations in an equal number of unknowns, the factor rentals. These can then be solved for as functions of the prices p^1, and the solutions can be substituted into the fourth set of equations of (13) to solve for the domestic prices p^3. We may denote these reduced-form functional relationships by

$$w = \hat{w}(p^1) \equiv (g^1)^{-1}(p^1); \quad p^3 = \hat{p}^3(p^1) = g^3 \circ \hat{w}(p^1). \tag{40}$$

The Jacobian matrices of these mappings are, by virtue of (7) and (14),

$$\partial\hat{w}/\partial p^1 = (B^{1\prime})^{-1}; \quad \partial\hat{p}^3/\partial p^1 = B^{3\prime}(B^{1\prime})^{-1}. \tag{41}$$

Since the minimum-unit-cost functions $g_i^k(w)$ are homogeneous of degree 1, we have by Euler's theorem

$$g^k(w) = B^k(w)'w \tag{42}$$

(a relationship which of course is very well known), so that equations (40) may be written

$$w = \{B^1[\hat{w}(p^1)]'\}^{-1}p^1, \quad p^3 = B^3[\hat{w}(p^1)]'\{B^1[\hat{w}(p^1)]'\}^{-1}p^1 \tag{43}$$

or more simply (suppressing arguments of the functions)

$$w' = p^{1\prime}(B^1)^{-1}, \quad p^{3\prime} = p^{1\prime}(B^1)^{-1}B^3. \tag{44}$$

The reduced-form functions $\hat{w}$ and $\hat{p}^3$ of (40) may now be substituted into the first two sets of equations of (13) to obtain

$$\begin{aligned} \phi[\hat{w}(p^1), y^1, y^3] \equiv B^1[\hat{w}(p^1)]y^1 + B^3[\hat{w}(p^1)]y^3 &= \ell \\ h^3[p^1, p^2, \hat{p}^3(p^1), p^1 \cdot y^1 + \hat{p}^3(p^1) \cdot y^3 + D] &= y^3, \end{aligned} \tag{45}$$

where the first equality defines the factor-demand function $\phi(w, y^1, y^3)$. These are $m + n_3$ equations in the $n_1 + n_3$ unknowns y^1 and y^3, which may be solved since $m = n_1$. Differentiating, we obtain

$$\begin{aligned} \begin{bmatrix} B^1 & B^3 \\ -c^3p^{1\prime} & I - c^3p^{3\prime} \end{bmatrix} \begin{bmatrix} dy^1 \\ dy^3 \end{bmatrix} &= \begin{bmatrix} -\Phi \cdot (B^{1\prime})^{-1} \\ S^{31} + S^{33}B^{3\prime}(B^{1\prime})^{-1} - c^3z^{1\prime} \end{bmatrix} dp^1 \\ &+ \begin{bmatrix} 0 \\ \partial h^3/\partial p^2 \end{bmatrix} dp^2 + \begin{bmatrix} 0 \\ c^3 \end{bmatrix} dD + \begin{bmatrix} I_m \\ 0 \end{bmatrix} d\ell, \end{aligned} \tag{46}$$

where we define the $m \times m$ matrix

$$\Phi(w) = [\partial\phi_i(w, y^1, y^3)/\partial w_j]. \tag{47}$$

Using Schur's formula for the inverse of a partitioned matrix (cf. Chipman, 1980), and keeping in mind the relation (44), we find that

$$\begin{aligned} &\begin{bmatrix} B^1 & B^3 \\ -c^3p^{1\prime} & I - c^3p^{3\prime} \end{bmatrix}^{-1} = \\ &\begin{bmatrix} (B^1)^{-1} & 0 \\ 0 & I_{n_3} \end{bmatrix} \begin{bmatrix} B^1 - B^3c^3p^{1\prime} & -B^3 \\ c^3p^{1\prime} & I_{n_3} \end{bmatrix} \begin{bmatrix} (B^1)^{-1} & 0 \\ 0 & I_{n_3} \end{bmatrix}, \end{aligned} \tag{48}$$

from which the matrices of partial derivatives of the reduced-form functions $\tilde{y}^1(p^1, p^2, D, \ell)$ and $\tilde{y}^3(p^1, p^2, D, \ell)$ may readily be derived. For later use we note in particular, again using (44), that (48) implies

$$(49) \qquad (p^{1\prime}, p^{3\prime}) \begin{bmatrix} B^1 & B^3 \\ -c^3 p^{1\prime} & I - c^3 p^{3\prime} \end{bmatrix}^{-1} = (w', 0).$$

Noting further that the functions ϕ_i of (45) are homogeneous of degree 0 in the factor rentals w, so that by Euler's theorem the matrix-valued function (47) satisfies $\Phi(w)w = 0$, we see from (49) and (46) that the following envelope condition is satisfied:

$$(50) \qquad p^{1\prime}\frac{\partial \tilde{y}^1}{\partial p^1} + p^{3\prime}\frac{\partial \tilde{y}^3}{\partial p^1} = -w'\Phi(w)(B^{1\prime})^{-1} = 0.$$

With this preparation we may now proceed to derive the expression for the trade-Slutsky matrix. The net-import demand functions (17) are now defined by

$$(51) \qquad \tilde{z}^r(p^1, p^2, D, \ell) = \tilde{x}^r(p^1, p^2, D, \ell) - \tilde{y}^r(p^1, p^2, D, \ell)$$

where

$$(52) \qquad \tilde{x}^r(p^1, p^2, D, \ell) = \\ h^r[p^1, p^2, \hat{p}^3(p^1), p^{1\prime}\tilde{y}^1(p^1, p^2, D, \ell) + \hat{p}^3(p^1)'\tilde{y}^3(p^1, p^2, D, \ell) + D]$$

and $\tilde{y}^r$ is defined implicitly by (45). We verify from (52), using (50) and (41), that

$$(53) \qquad \partial \tilde{x}^r / \partial p^1 = S^{r1} + S^{r3}B^{3\prime}(B^{1\prime})^{-1} - c^r z^{1\prime}$$

and

$$(54) \qquad \partial \tilde{x}^r / \partial p^2 = \partial h^r / \partial p^2 = S^{r2} - c^r z^{2\prime}.$$

Likewise we have

$$(55) \qquad \partial \tilde{x}^r / \partial D = c^r.$$

From (46) and (48) we have, making use of (50),

$$(56) \qquad \partial \tilde{y}^1 / \partial p^1 = -(B^1)^{-1}\Phi(B^{1\prime})^{-1} - (B^1)^{-1}B^3[S^{31} + S^{33}B^{3\prime}(B^{1\prime})^{-1} - c^3 z^{1\prime}]$$

(where the last bracketed expression is the expression for $\partial \tilde{y}^3/\partial p^1 = \partial \tilde{x}^3/\partial p^1$), and

$$(57) \qquad \partial \tilde{y}^1 / \partial p^2 = -(B^1)^{-1}B^3(\partial h^3/\partial p^2) = -(B^1)^{-1}B^3(S^{32} - c^3 z^{2\prime}),$$

as well as

$$(58) \qquad \partial \tilde{y}^1 / \partial D = -(B^1)^{-1}B^3 c^3.$$

From (53), (57), (55), (58), and (26) we then obtain

$$(59) \qquad \tilde{S}^{11} = (B^1)^{-1}\left\{\begin{bmatrix} B^1 & B^3 \end{bmatrix}\begin{bmatrix} S^{11} & S^{13} \\ S^{31} & S^{33} \end{bmatrix}\begin{bmatrix} B^{1\prime} \\ B^{3\prime} \end{bmatrix} + \Phi\right\}(B^{1\prime})^{-1},$$

which is certainly symmetric and negative semi-definite. Continuing our computations we obtain from (54), (57), (55), (58) and (26) the expression

$$(60) \qquad \tilde{S}^{12} = S^{12} + (B^1)^{-1}B^3 S^{32},$$

and from (53), (55), and (26) (recalling that $\tilde{y}^2 = 0$) the expression

$$\tilde{S}^{21} = S^{21} + S^{23}B^{3\prime}(B^{1\prime})^{-1}, \tag{61}$$

which is the transpose of (60). Finally, from (53), (55) and (26) we obtain

$$\tilde{S}^{22} = S^{22}. \tag{62}$$

Putting together the expressions (59), (60), (61) and (62) it is not hard to see that we finally obtain for the full trade-Slutsky matrix $\tilde{S}$ the matrix

$$\begin{bmatrix} \tilde{S}^{11} & \tilde{S}^{12} \\ \tilde{S}^{21} & \tilde{S}^{22} \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & (B^1)^{-1}B^3 \\ 0 & I_{n_2} & 0 \end{bmatrix} \begin{bmatrix} S^{11} & S^{12} & S^{13} \\ S^{21} & S^{22} & S^{23} \\ S^{31} & S^{32} & S^{33} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \\ B^{3\prime}(B^{1\prime})^{-1} & 0 \end{bmatrix} + \begin{bmatrix} (B^1)^{-1}\Phi(B^{1\prime})^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{63}$$

which is obviously symmetric and negative semi-definite.

5 The Intermediate Case

There seems to be no way of handling Case (ii) of (21) except by brute force. It was shown in Chipman (1980) that the differentials of the reduced-form functions (15) of the system (13) are given by

$$\begin{bmatrix} d\tilde{p}^3 \\ d\tilde{w} \\ d\tilde{y}^1 \\ d\tilde{y}^3 \end{bmatrix} = \begin{bmatrix} S^{33} & 0 & c^3p^{1\prime} & c^3p^{3\prime} - I \\ 0 & \Phi & B^1 & B^3 \\ 0 & B^{1\prime} & 0 & 0 \\ -I & B^{3\prime} & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -(S^{31} - c^3z^{1\prime})dp^1 - \dfrac{\partial h^3}{\partial p^2}dp^2 - c^3dD \\ d\ell \\ dp^1 \\ 0 \end{bmatrix}. \tag{64}$$

Expressions were obtained there for some but not all of the submatrices of the inverse of the above submatrix. In a future paper I plan to complete the computations required to extend the proposition established here to the general case $m \geq n_1$.

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