Product Diversification, Equalization of Factor Rentals, and Consumer Preferences

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1 Introduction

As formulated by Samuelson (1953), the theorem that rentals of factors of production in two countries are equalized through trade between them contains as one of its premises the assumption that both countries produce positive amounts of both commodities. However, from the point of view of general-equilibrium theory, the question of whether or not this product-diversification condition is satisfied is an *outcome* of the process. As originally formulated by Samuelson, therefore, the theorem may be best thought of as part of a taxonomy: some outcomes will involve specialization and some diversification, and some of the latter (in some circumstances, all) will be characterized by factor-rental equalization. The main difficulty with this approach is that it does not provide insight as to what basic conditions will give use to the famous result.

In the present paper I derive, for the simple two-country, two-commodity, two-factor case, conditions on relative factor endowments that will make it possible for both countries to produce both commodities efficiently and in positive amounts (Section 2). Then in the latter case, it is shown how the question of whether such an outcome will indeed occur depends on consumer preferences (Section 3). To help understand the result, an explicit computation is carried out in Section 4.

Use is made of the concept of a world production-possibility frontier developed by Lerner (1932) and further by Sohmen (1969), as well as of Rybczynski functions and Stolper-Samuelson functions developed in Chipman (1972). For general background the reader may be referred to Chipman (1966).

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2 Conditions for Specialization and Diversification

Consider two countries producing (or capable of producing) two commodities with the aid of two factors of production by means of the production functions

$$\begin{aligned} y_1 &= f_1(v_{11}, v_{21}) \\ y_2 &= f_2(v_{12}, v_{22}), \end{aligned} \tag{1}$$

assumed homogeneous of degree 1, concave, and strictly quasi-concave. The outputs are y_1 and y_2, and v_{ij} is the input of factor i into industry j. The inputs satisfy

$$\begin{aligned} v_{11} + v_{12} &= l_1 \\ v_{21} + v_{22} &= l_2, \end{aligned} \tag{2}$$

where l_i is the fixed endowment of the country in factor i. When necessary for clarity, a superscript k $(= 1, 2)$ will be used to distinguish the country.

Let the dual minimum-unit-cost functions be denoted by $g_j(w_1, w_2)$, $j = 1, 2$, where w_i is the rental of factor i. We know that the factor-product ratios are given by $v_{ij}/y_j = b_{ij}(w_1, w_2)$ where, by Shephard's (1953) duality theorem,

$$\begin{bmatrix} \dfrac{\partial g_1}{\partial w_1} & \dfrac{\partial g_1}{\partial w_2} \\ \dfrac{\partial g_2}{\partial w_1} & \dfrac{\partial g_2}{\partial w_2} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} = B'. \tag{3}$$

It will be assumed that the functions $b_{ij}(w_1, w_2)$ satisfy

$$|B| = \begin{vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{21}b_{12} = b_{11}b_{12}\left[\frac{b_{22}}{b_{12}} - \frac{b_{21}}{b_{11}}\right] > 0 \tag{4}$$

for all $w_1, w_2 > 0$, i.e., commodity 1 uses factor 1 (say labor) relatively intensively, and commodity 2 uses factor 2 (say capital) relatively intensively, for all wage rates and rentals. We also assume that $b_{ij} > 0$, so in conjunction with the above it follows that the mapping

$$\begin{aligned} g_1(w_1, w_2) &= p_1 \\ g_2(w_1, w_2) &= p_2 \end{aligned} \tag{5}$$

is one-to-one. The inverse mapping is denoted

$$\begin{aligned} g_1^{-1}(p_1, p_2) &= w_1 \\ g_2^{-1}(p_1, p_2) &= w_2 \end{aligned} \tag{6}$$

and its Jacobian matrix is denoted

$$\begin{bmatrix} \partial g_1^{-1}/\partial p_1 & \partial g_1^{-1}/\partial p_2 \\ \partial g_2^{-1}/\partial p_1 & \partial g_2^{-1}/\partial p_2 \end{bmatrix} = \begin{bmatrix} b^{11} & b^{21} \\ b^{12} & b^{22} \end{bmatrix}. \tag{7}$$

This is the inverse of the Jacobian matrix (3). Thus we have

$$\begin{aligned} &\begin{bmatrix} b^{11}(p_1, p_2) & b^{21}(p_1, p_2) \\ b^{12}(p_1, p_2) & b^{22}(p_1, p_2) \end{bmatrix} \\ &= \begin{bmatrix} b_{11}(g_1^{-1}(p_1, p_2), g_2^{-1}(p_1, p_2)) & b_{21}(g_1^{-1}(p_1, p_2), g_2^{-1}(p_1, p_2)) \\ b_{12}(g_1^{-1}(p_1, p_2), g_2^{-1}(p_1, p_2)) & b_{22}(g_1^{-1}(p_1, p_2), g_2^{-1}(p_1, p_2)) \end{bmatrix}^{-1} \end{aligned} \tag{8a}$$

and

$$\text{(8b)}\quad \begin{aligned}&\begin{bmatrix} b_{11}(w_1,w_2) & b_{21}(w_1,w_2) \\ b_{12}(w_1,w_2) & b_{22}(w_1,w_2) \end{bmatrix}\\ &= \begin{bmatrix} b^{11}(g_1(w_1,w_2),g_2(w_1,w_2)) & b^{21}(g_1(w_1,w_2),g_2(w_1,w_2)) \\ b^{12}(g_1(w_1,w_2),g_2(w_1,w_2)) & b^{22}(g_1(w_1,w_2),g_2(w_1,w_2)) \end{bmatrix}^{-1}\end{aligned}$$

Let us find the conditions for specialization and diversification. If both commodities are produced, we have

$$\text{(9)}\quad \begin{aligned} b_{11}(w_1,w_2)y_1 + b_{12}(w_1,w_2)y_2 &= l_1, \\ b_{21}(w_1,w_2)y_1 + b_{22}(w_1,w_2)y_2 &= l_2, \end{aligned}$$

where w_1 and w_2 are determined by (5). Inverting (9) and using (8) we obtain the solution

$$\text{(10)}\quad \begin{aligned} y_1 &= b^{11}(p_1,p_2)l_1 + b^{12}(p_1,p_2)l_2, \\ y_2 &= b^{21}(p_1,p_2)l_1 + b^{22}(p_1,p_2)l_2. \end{aligned}$$

These are the segments of the Rybczynski functions within the zone of diversification. If $y_1 = 0$ we have from the first equation of (10)

$$\text{(11a)}\quad \frac{l_2}{l_1} = -\frac{b^{11}(p_1,p_2)}{b^{12}(p_1,p_2)},$$

and if $y_2 = 0$ we obtain from the second equation of (10)

$$\text{(11b)}\quad \frac{l_2}{l_1} = -\frac{b^{21}(p_1,p_2)}{b^{22}(p_1,p_2)}.$$

From (3) and (7) and the formula for an inverse matrix we have, using (4),

$$\text{(12)}\quad B^{-1} = \begin{bmatrix} b^{11} & b^{12} \\ b^{21} & b^{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^{-1} = \frac{1}{|B|}\begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix},$$

so that the limits of (11a) and (11b) may be also written as

$$\text{(13a)}\quad \frac{l_2}{l_1} = \frac{b_{22}(g_1^{-1}(p_1,p_2), g_2^{-1}(p_1,p_2))}{b_{12}(g_1^{-1}(p_1,p_2), g_2^{-1}(p_1,p_2))}$$

and

$$\text{(13b)}\quad \frac{l_2}{l_1} = \frac{b_{21}(g_1^{-1}(p_1,p_2), g_2^{-1}(p_1,p_2))}{b_{11}(g_1^{-1}(p_1,p_2), g_2^{-1}(p_1,p_2))}$$

respectively.

Since $|B| > 0$ from (4), in order for (9) to have a nonnegative solution $y_1 \geqq 0$, $y_2 \geqq 0$ we must have

$$\text{(14)}\quad \frac{b_{21}}{b_{11}} \leqq \frac{l_2}{l_1} \leqq \frac{b_{22}}{b_{12}},$$

or equivalently,

$$\text{(15)}\quad -\frac{b^{21}(p_1,p_2)}{b^{22}(p_1,p_2)} \leqq \frac{l_2}{l_1} \leqq -\frac{b^{11}(p_1,p_2)}{b^{12}(p_1,p_2)}.$$

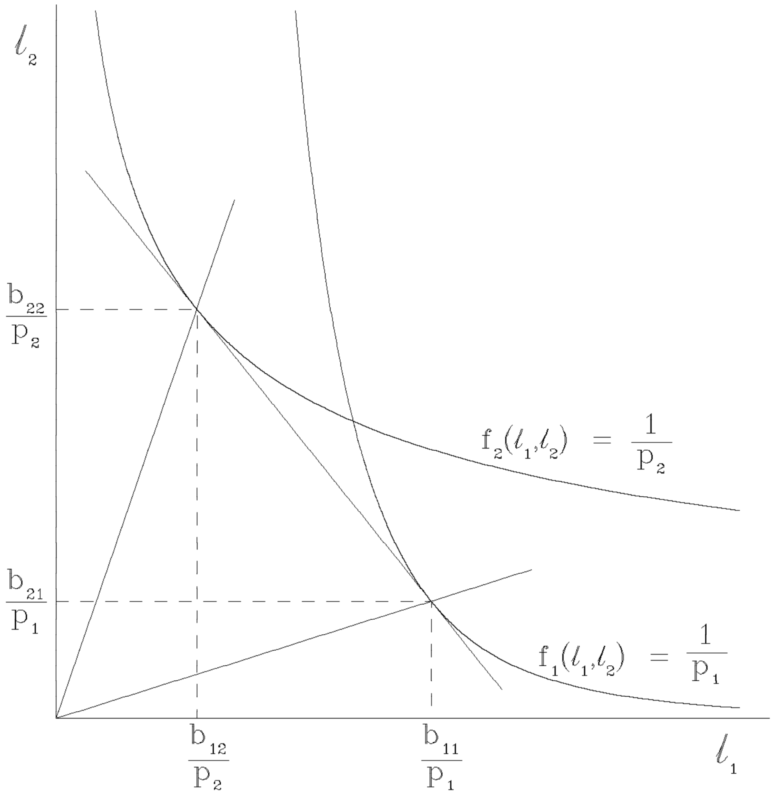

Figure 1

Geometrically, this states that the factor-endowment ratio must lie within the diversification cone (see Figure 1).

Now, just as the functions $b_{ij}(w_1, w_2)$ are homogeneous of degree zero in w_1, w_2, so are the functions $b^{ij}(p_1, p_2)$ homogeneous of degree zero in p_1, p_2. Thus, they may be expressed as functions of the terms of trade p_1/p_2. So we may rewrite (15) in the form

(16) $$L\left(\frac{p_1}{p_2}\right) \leqq \frac{l_2}{l_1} \leqq U\left(\frac{p_1}{p_2}\right).$$

Thus, by definition,

(16a) $$L\left(\frac{p_1}{p_2}\right) = -\frac{b^{21}\left(\frac{p_1}{p_2}, 1\right)}{b^{22}\left(\frac{p_1}{p_2}, 1\right)} = \frac{b_{21}\left(g_1^{-1}\left(\frac{p_1}{p_2}, 1\right), g_2^{-1}\left(\frac{p_1}{p_2}, 1\right)\right)}{b_{11}\left(g_1^{-1}\left(\frac{p_1}{p_2}, 1\right), g_2^{-1}\left(\frac{p_1}{p_2}, 1\right)\right)}$$

and

(16b) $$U\left(\frac{p_1}{p_2}\right) = -\frac{b^{11}\left(\frac{p_1}{p_2}, 1\right)}{b^{12}\left(\frac{p_1}{p_2}, 1\right)} = \frac{b_{22}\left(g_1^{-1}\left(\frac{p_1}{p_2}, 1\right), g_2^{-1}\left(\frac{p_1}{p_2}, 1\right)\right)}{b_{12}\left(g_1^{-1}\left(\frac{p_1}{p_2}, 1\right), g_2^{-1}\left(\frac{p_1}{p_2}, 1\right)\right)}.$$

We shall show presently that the functions $L(p_1/p_2)$ and $U(p_1/p_2)$ are monotone increasing. The geometric meaning of this is shown in Figure 2, where p_1 has been decreased to p_1', resulting in an outward shift in the isoquant for the production of a dollar's worth of commodity 1. This results in a rightward and downward movement of the points $(b_{11}/p_1, b_{21}/p_1)$ and $(b_{12}/p_2, b_{22}/p_2)$ at which the isoquants $f_1(v_{11}, v_{21}) = 1/p_1$ and $f_2(v_{12}, v_{22}) = 1/p_2$ have a common tangent.

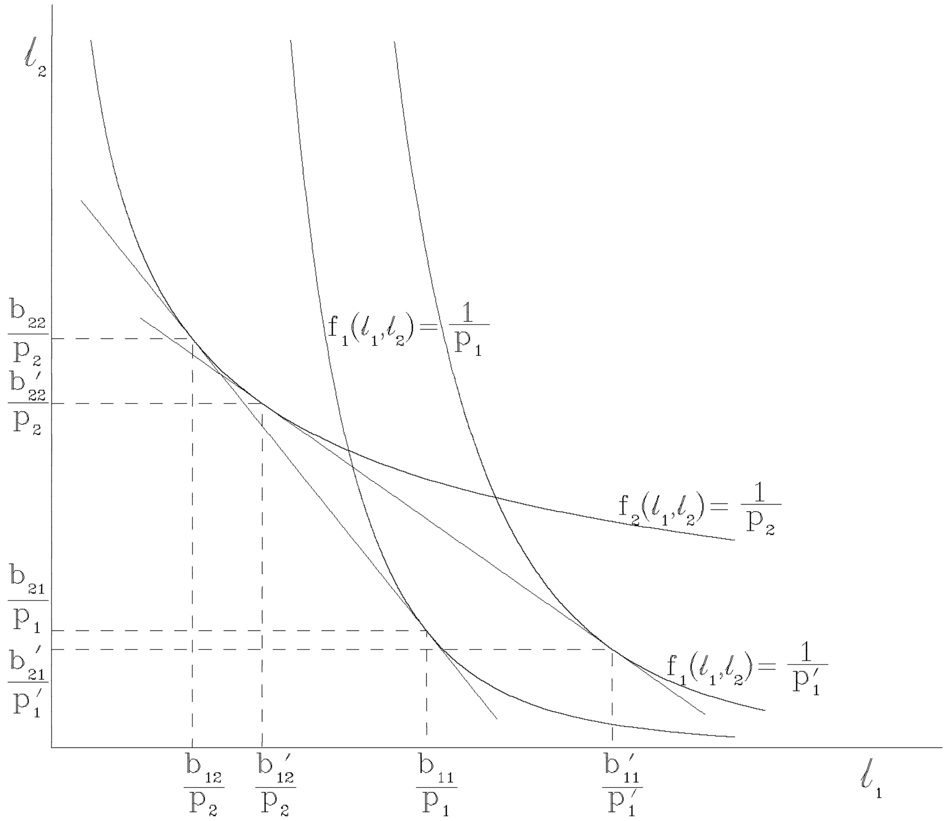

Figure 2

Before verifying that $L'(p_1/p_2) > 0$ and $U'(p_1/p_2) > 0$ (the prime denotes differentiation), let us see the implications for the inequality (16). Let us now consider l_1/l_2 as fixed and p_1/p_2 as variable. Since L and U are monotone, they are invertible hence we may write (16) in the form

$$\pi_2 \equiv U^{-1}\Big(\frac{l_2}{l_1}\Big) \leqq \frac{p_1}{p_2} \leqq L^{-1}\Big(\frac{l_2}{l_1}\Big) \equiv \pi_1. \tag{17}$$

This is the formula for the slope of the country's production-possibility frontier at the points of complete specialization, as shown in Figure 3. The relation between inequalities (16) and (17) is evident from Figure 4.

Let us now consider two countries 1 and 2, whose factor endowment ratios are given by

$$\frac{l_2^1}{l_1^1} < \frac{l_2^2}{l_1^2}, \tag{18}$$

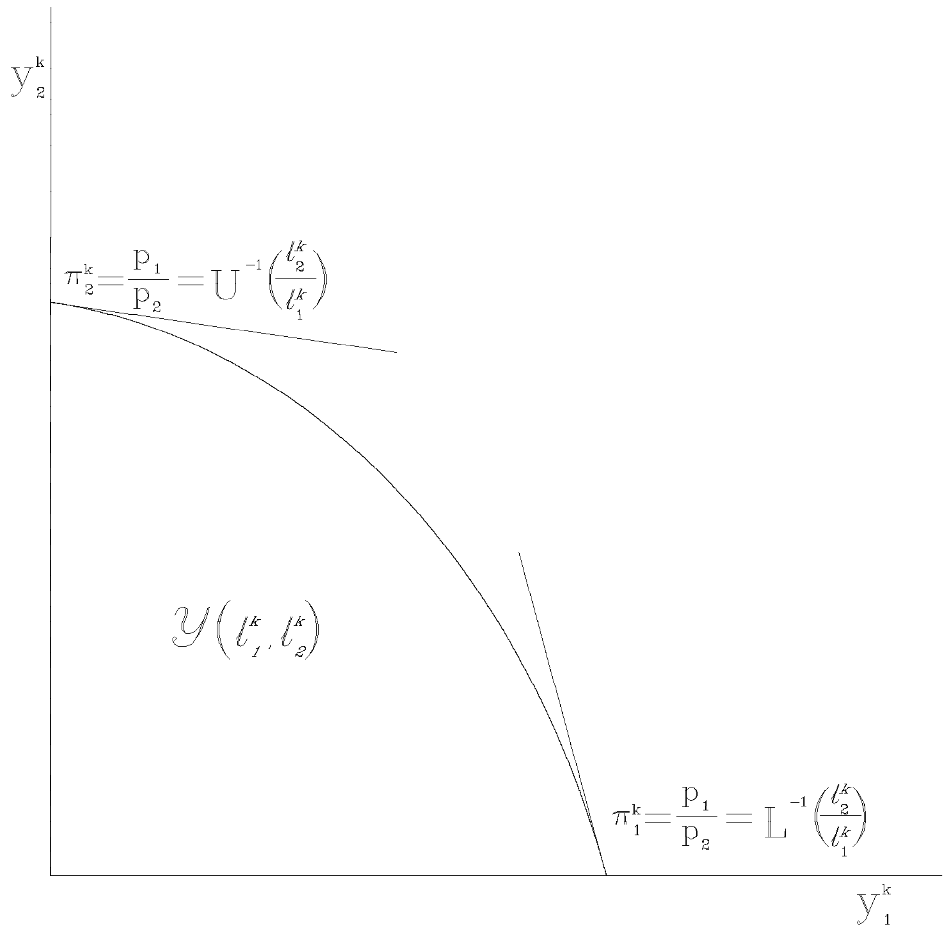

Figure 3

i.e., country 1 is relatively well endowed in factor 1 and country 2 is relatively well endowed in factor 2. Since identical production functions have been assumed, the same U and L functions apply to both. In order therefore that there exist a (nondegenerate) segment on the world production-possibility frontier for which there is diversification in both countries, it is necessary and sufficient that

$$U^{-1}\left(\frac{l_2^2}{l_1^2}\right) < L^{-1}\left(\frac{l_2^1}{l_1^1}\right). \tag{19}$$

To establish the monotonicity of L and U we may proceed as follows. First of all we have from (6), (7), and (12)

$$\begin{aligned}\frac{\partial g_1^{-1}}{\partial p_1} &= b^{11} = \frac{b_{22}}{|B|} > 0\\ \frac{\partial g_2^{-1}}{\partial p_1} &= b^{12} = -\frac{b_{12}}{|B|} < 0\end{aligned}$$

whence, defining the function ψ by

$$\frac{w_1}{w_2} = \frac{g_1^{-1}(p_1/p_2, 1)}{g_2^{-1}(p_1/p_2, 1)} = \psi\left(\frac{p_1}{p_2}\right),$$

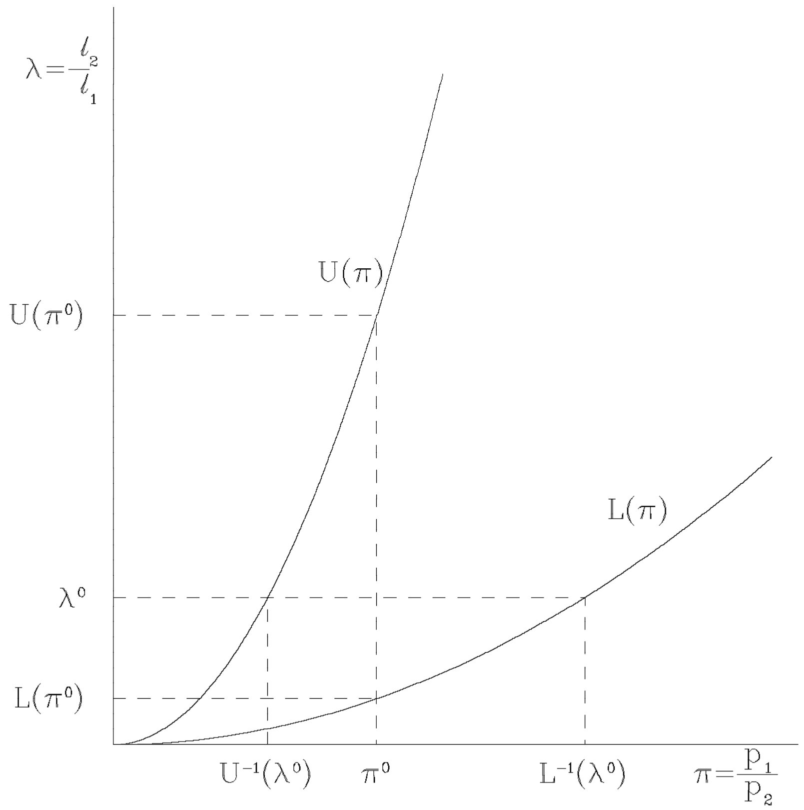

Figure 4

we have

$$\frac{d\psi}{d(p_1/p_2)} = \frac{g_2^{-1}\Big(\frac{p_1}{p_2},1\Big)\frac{\partial g_1^{-1}(p_1/p_2,1)}{\partial p_1} - g_1^{-1}\Big(\frac{p_1}{p_2},1\Big)\frac{\partial g_2^{-1}(p_1/p_2,1)}{\partial p_1}}{g_2^{-1}\Big(\frac{p_1}{p_2},1\Big)^2}$$

$$= \frac{1}{|B|}\cdot\frac{w_2 b_{22} + w_1 b_{12}}{(w_2)^2} = \frac{1}{|B|}\frac{p_2}{(w_2)^2} > 0.$$

Now define the functions

$$L^*\Big(\frac{w_1}{w_2}\Big) = \frac{b_{21}(w_1/w_2,1)}{b_{11}(w_1/w_2,1)}, \quad U^*\Big(\frac{w_1}{w_2}\Big) = \frac{b_{22}(w_1/w_2,1)}{b_{12}(w_1/w_2,1)}.$$

Then

$$L^*\Big(\psi\Big(\frac{p_1}{p_2}\Big)\Big) = L\Big(\frac{p_1}{p_2}\Big), \quad U^*\Big(\psi\Big(\frac{p_1}{p_2}\Big)\Big) = U\Big(\frac{p_1}{p_2}\Big),$$

so it is sufficient to prove that L^* and U^* are monotone increasing functions. We have

$$\frac{dL^*}{d(w_1/w_2)} = \frac{b_{11}\Big(\frac{w_1}{w_2},1\Big)\frac{\partial b_{21}(w_1/w_2,1)}{\partial w_1} - b_{21}\Big(\frac{w_1}{w_2},1\Big)\frac{\partial b_{11}(w_1/w_2,1)}{\partial w_1}}{b_{11}\Big(\frac{w_1}{w_2},1\Big)^2}$$

$$
(20) \qquad = \frac{b_{11}\left(\frac{w_1}{w_2},1\right)\frac{\partial^2 g_1(w_1/w_2,1)}{\partial w_2 \partial w_1} - b_{21}\left(\frac{w_1}{w_2},1\right)\frac{\partial^2 g_1(w_1/w_2,1)}{\partial w_1^2}}{b_{11}\left(\frac{w_1}{w_2},1\right)^2}.
$$

This has to be positive for the following reason. The matrix

$$
\begin{bmatrix} \frac{\partial^2 g_1}{\partial w_1^2} & \frac{\partial^2 g_1}{\partial w_1 \partial w_2} \\ \frac{\partial^2 g_1}{\partial w_2 \partial w_1} & \frac{\partial^2 g_1}{\partial w_2^2} \end{bmatrix}
$$

is the Hessian of g_1 (which is concave) hence its principal minors oscillate in sign. Since g_1 is homogeneous of degree one, $\partial g_1/\partial w_1$ and $\partial g_1/\partial w_2$ are homogeneous of degree zero, hence the Hessian matrix is singular, i.e.,

$$
\begin{bmatrix} \frac{\partial^2 g_1}{\partial w_1^2} & \frac{\partial^2 g_1}{\partial w_1 \partial w_2} \\ \frac{\partial^2 g_1}{\partial w_2 \partial w_1} & \frac{\partial^2 g_1}{\partial w_2^2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

The diagonal elements are negative, hence the off-diagonal terms are positive. Thus both terms in the numerator of (20) are positive. This shows that $dL^*/d(w_1/w_2) > 0$, and a similar argument shows that $dU^*/d(w_1/w_2) > 0$.

3 Endowment Ratios, Preferences, and Equalization of Factor Rentals

Let us introduce the notation π_j^k to signify the critical price ratio p_1/p_2 at which country k will specialize in commodity j. From (17) these critical price ratios are given by

$$
(21) \qquad \pi_1^k = L^{-1}\left(\frac{l_2^k}{l_1^k}\right) \quad \text{and} \quad \pi_1^k = U^{-1}\left(\frac{l_2^k}{l_1^k}\right)
$$

(see Figure 3). From the strict concavity to the origin of the production-possibility frontier, for each country one necessarily has

$$
(22) \qquad \pi_2^k < \pi_1^k \quad (k = 1, 2).
$$

From the monotonicity of the functions L and U and the assumption (18) that country 1 is relatively well endowed in factor 1 and country 2 in factor 2, it follows that

$$
(23) \qquad \pi_1^1 < \pi_1^2 \quad \text{and} \quad \pi_2^1 < \pi_2^2.
$$

From (22) and (23) it follows that only the following three configurations are possible:

$$
(24a) \qquad \pi_2^1 < \pi_1^1 < \pi_2^2 < \pi_1^2;
$$

$$
(24b) \qquad \pi_2^1 < \pi_1^1 = \pi_2^2 < \pi_1^2;
$$

(24c) $$\pi_2^1 < \pi_2^2 < \pi_1^1 < \pi_1^2.$$

In configuration (24a) the intervals of price ratios $[\pi_2^1, \pi_1^1], [\pi_2^2, \pi_1^2]$ at which the respective countries produce both commodities do not overlap, hence there is no possibility of equalization of factor rentals, regardless of the nature of consumer preferences. The world production-possibility frontier is composed of only two segments: one along which country 1 will produce both commodities and country 2 only commodity 2, and the other along which country 2 will produce both commoditites and country 1 only commodity 1. These segments will be separated by a kink, at which both countries specialize.

In configuration (24b) there is exactly one point on the world production-possibility frontier separating the above two segments, at which each country is specializing yet is on the verge of diversifying. This is the case in which one country's factor-endowment vector is at one edge of the diversification cone, and the other country's factor-endowment vector at the other. Thus, factor rentals will be equalized even though both countries specialize. For this to be a world equilibrium, a world indifference curve would have to be tangential to the world production-possibility frontier precisely at the intersection-point of the two segments. Since configuration (24b) is already exceptional, this would be doubly exceptional. It is an interesting freak case.

Configuration (24c) is the only one that allows for both countries to produce positive amounts of both commodities simultaneously and efficiently. It gives rise to a third, intermediate segment on the world production-possibility frontier. Equalization of factor rentals takes place if and only if consumer preferences are such that world equilibrium occurs on this segment (including the end-points). If world preferences can be aggregated, then the dependence of factor-rental equalization on preferences can be depicted in a particularly simple way. This will now be done.

Let x_j^k denote the consumption of commodity j in country k, and let world consumption of commodity j be denoted

(25) $$x_j = x_j^1 + x_j^2 \quad (j = 1, 2).$$

Assuming world preferences to be aggregable, world equilibrium will take place where a world utility function $U(x_1, x_2)$ is maximized subject to (x_1, x_2) belonging to the world production-possibility set. Now let us assume that configuration (24c) holds, and let us assume that the maximum occurs on the intermediate segment on the world production-possibility frontier where both countries produce positive amounts of both commodities. Then the equilibrium prices satisfy

(26) $$\pi_2^2 = U^{-1}\left(\frac{l_2^2}{l_1^2}\right) \leqq \frac{p_1}{p_2} \leqq L^{-1}\left(\frac{l_2^1}{l_1^1}\right) = \pi_1^1.$$

Analogously to (25), let us define world output of commodity j by

(27) $$y_j = y_j^1 + y_j^2,$$

and let the world Rybczynski function for commodity j be defined by

(28) $$\hat{y}_j(p_1, p_2, l_1^1, l_2^1, l_1^2, l_2^2) = \sum_{k=1}^{2} \hat{y}_j^k(p_1, p_2, l_1^k, l_2^k),$$

where $\hat{y}_j^k$ is the Rybczynski function for commodity j in country k. Then, if and only if (26) holds, since when (and only when) country k produces both commodities the Rybczynski function can be written in the form (10), for p_1/p_2 in the interval (26) we have

$$
\begin{aligned}
\hat{y}_1(p_1,p_2,l_1^1,l_2^1,l_1^2,l_2^2) &= b^{11}(p_1,p_2)(l_1^1+l_1^2)+b^{12}(p_1,p_2)(l_2^1+l_2^2);\\
\hat{y}_2(p_1,p_2,l_1^1,l_2^1,l_1^2,l_2^2) &= b^{21}(p_1,p_2)(l_1^1+l_1^2)+b^{22}(p_1,p_2)(l_2^1+l_2^2).
\end{aligned} \tag{29}
$$

Notice that it crucial here in being able to aggregate the countries' factor endowments that both countries be assumed to have identical technologies and to have their factor endowments in the same diversification cone.

Since the functions (29) are homogeneous of degree 0 in the prices, the ratio y_1/y_2 may be expressed as a function of the price ratio p_1/p_2 and the world endowments $l_1 = l_2^1 + l_1^2$ and $l_2 = l_2^1 + l_2^2$. If (as is usually necessary for aggregability) preferences are assumed to be homothetic, the ratio x_1/x_2 will also be a function of the price ratio p_1/p_2. Equating these ratios at the critical price ratios π_2^2 and π_1^1 gives us the conditions on preferences that are required for factor-rental equalization. To illustrate the required procedure, a precise calculation will be carried out in the next section.

4 An Illustration for the Case of Cobb-Douglas Technologies and Preferences

Let world preferences be represented by the utility function

$$U(x_1,x_2) = x_1^{\theta}x_2^{1-\theta} \quad (\theta > 0). \tag{30}$$

Setting marginal utilities proportional to prices we have

$$\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{\theta}{1-\theta}\frac{x_2}{x_1} = \frac{p_1}{p_2}.$$

Setting world supply y_j equal to world demand x_j we have

$$\frac{\hat{y}_2(p_1,p_2,l_1^1,l_2^1,l_1^2,l_2^2)}{\hat{y}_1(p_1,p_2,l_1^1,l_2^1,l_1^2,l_2^2)} = \frac{1-\theta}{\theta}\frac{p_1}{p_2}.$$

For price ratios $\pi = p_1/p_2$ satisfying (26) we have, from (29) and the homogeneity of the functions b^{ij},

$$R(\pi;l_1,l_2) \equiv \frac{b^{21}(\pi,1)l_1 + b^{22}(\pi,1)l_2}{b^{11}(\pi,1)l_1 + b^{12}(\pi,1)l_2} = \frac{1-\theta}{\theta}\pi, \tag{31}$$

or

$$\theta = [R(\pi;l_1,l_2)/\pi + 1]^{-1} \equiv \bar{\theta}(\pi;l_1,l_2), \tag{32}$$

where the function R is defined by (31).

Now, letting the production functions (1) have the Cobb-Douglas form

$$f_j(v_{1j},v_{2j}) = \mu_j v_{1j}^{\beta_{1j}} v_{2j}^{\beta_{2j}} \quad (\beta_{ij} > 0, \beta_{1j}+\beta_{2j} = 1), \tag{33}$$

the minimum-unit-cost functions are readily seen to have the form

$$g_j(w_1, w_2) = \nu_j w_1^{\beta_{1j}} w_2^{\beta_{2j}} \quad \text{where} \quad \nu_j = 1/f_j(\beta_{1j}, \beta_{2j}). \tag{34}$$

Corresponding to (5) we have the log-linear system

$$\begin{aligned} \beta_{11} \log w_1 + \beta_{21} \log w_2 &= \log p_1 - \log \nu_1 \\ \beta_{12} \log w_1 + \beta_{22} \log w_2 &= \log p_2 - \log \nu_2 \end{aligned} \tag{35}$$

which, when solved, yields the Stolper-Samuelson functions

$$\begin{aligned} w_1 = g_1^{-1}(p_1, p_2) = \nu_1^{-\beta_{22}/\delta} \nu_2^{\beta_{21}/\delta} p_1^{\beta_{22}/\delta} p_2^{-\beta_{21}/\delta} \\ w_2 = g_2^{-1}(p_1, p_2) = \nu_1^{\beta_{12}/\delta} \nu_2^{-\beta_{11}/\delta} p_1^{-\beta_{12}/\delta} p_2^{\beta_{11}/\delta} \end{aligned} \tag{36}$$

where $\delta = \det[\beta_{ij}] = \beta_{11} - \beta_{12}$. From (4), $\delta > 0$. Differentiating these functions we obtain the desired elements of the Jacobian matrix (7), as follows:

$$\begin{aligned} b^{11}(p_1, p_2) &= \frac{\beta_{22}}{\delta} \nu_1^{-\beta_{22}/\delta} \nu_2^{\beta_{21}/\delta} \left(\frac{p_1}{p_2}\right)^{\beta_{21}/\delta}; \\ b^{21}(p_1, p_2) &= \frac{-\beta_{21}}{\delta} \nu_1^{-\beta_{22}/\delta} \nu_2^{\beta_{21}/\delta} \left(\frac{p_1}{p_2}\right)^{\beta_{22}/\delta}; \\ b^{12}(p_1, p_2) &= \frac{-\beta_{12}}{\delta} \nu_1^{\beta_{12}/\delta} \nu_2^{-\beta_{11}/\delta} \left(\frac{p_1}{p_2}\right)^{-\beta_{11}/\delta}; \\ b^{22}(p_1, p_2) &= \frac{\beta_{11}}{\delta} \nu_1^{\beta_{12}/\delta} \nu_2^{-\beta_{11}/\delta} \left(\frac{p_1}{p_2}\right)^{-\beta_{12}/\delta}. \end{aligned} \tag{37}$$

Substituting these expressions in (16a) and (16b) we obtain

$$L\left(\frac{p_1}{p_2}\right) = \frac{\beta_{21}}{\beta_{11}} \left(\frac{\nu_1}{\nu_2}\right)^{-1/\delta} \left(\frac{p_1}{p_2}\right)^{1/\delta}; \quad U\left(\frac{p_1}{p_2}\right) = \frac{\beta_{22}}{\beta_{12}} \left(\frac{\nu_1}{\nu_2}\right)^{-1/\delta} \left(\frac{p_1}{p_2}\right)^{1/\delta}, \tag{38}$$

and thus

$$L^{-1}\left(\frac{l_2}{l_1}\right) = \frac{\nu_1}{\nu_2} \left(\frac{\beta_{11}}{\beta_{21}}\right)^{\delta} \left(\frac{l_2}{l_1}\right)^{\delta}; \quad U^{-1}\left(\frac{l_2}{l_1}\right) = \frac{\nu_1}{\nu_2} \left(\frac{\beta_{12}}{\beta_{22}}\right)^{\delta} \left(\frac{l_2}{l_1}\right)^{\delta}. \tag{39}$$

The condition

$$\pi_2^2 = U^{-1}\left(\frac{l_2^2}{l_1^2}\right) < L^{-1}\left(\frac{l_2^1}{l_1^1}\right) = \pi_1^1 \tag{40}$$

given by configuration (24c), namely that there exist a nondegenerate segment of the world production-possibility frontier such that both countries produce both commodities, reduces to

$$\frac{l_2^1}{l_1^1} > \frac{\beta_{21}}{\beta_{11}} \frac{\beta_{12}}{\beta_{22}} \frac{l_2^2}{l_1^2}. \tag{41}$$

Consider now a numerical example with

$$\beta_{11} = .8, \quad \beta_{21} = .2, \quad \beta_{12} = .3, \quad \beta_{22} = .7, \quad \mu_1 = \mu_2 = 1 \tag{42}$$

and

$$l_1^1 = 9, \quad l_2^1 = 3, \quad l_1^2 = 5, \quad l_2^2 = 2. \tag{43}$$

Since $\beta_{ij} = \partial \log g_j / \partial \log w_i = (w_i/g_j)\partial g_j/\partial w_i$, (4) is satisfied. Moreover, (18) and (40) are both satisfied, so we are dealing with configuration (24*c*). From (34) we compute $\nu_1 = 1.64938$ and $\nu_2 = 1.84202$. Recalling that $\delta = \beta_{11} - \beta_{12} = .5$, the critical price ratios (40) are then computed from (39) to be

$$\pi_2^2 = .37074, \quad \pi_1^1 = 1.03394.$$

Finally, for the expressions (37) we compute

$$\begin{aligned} b^{11}(\pi,1) &= .88715\pi^{.4}; & b^{21}(\pi,1) &= -.25347\pi^{1.4}; \\ b^{12}(\pi,1) &= -.30484\pi^{-1.6}; & b^{22}(\pi,1) &= .81291\pi^{-.6}. \end{aligned}$$

From (31) we then compute

$$R(\pi; 14, 5) = \frac{-3.54858\pi^{1.4} + 4.06454\pi^{-.6}}{12.42005\pi^{.4} - 1.52420\pi^{-1.6}}$$

so that

$$R(\pi_2^2; 14, 5) = \frac{6.48713}{.89478} = 7.25001 \quad \text{and} \quad R(\pi_1^1; 14, 5) = \frac{.26560}{11.14205} = .02384.$$

Formula (32) then yields

$$\bar{\theta}(\pi_2^2; 14, 5) = .04865 \quad \text{and} \quad \bar{\theta}(\pi_1^1; 14, 5) = .97746.$$

Since $\hat{y}_2/\hat{y}_1$ is a monotone decreasing function of p_1, we conclude that for all θ in the interval

$$.04865 \leqq \theta \leqq .97746,$$

factor rentals will be equalized as between the two countries.

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