The Stolper-Samuelson Theorem and the Problem of Aggregation

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1 Introduction

In its elegant and striking conclusion that a tariff on imports would raise real wages and lower the rental on capital if the import-competing industry is more labor-intensive than the export industry, the theorem established by Stolper and Samuelson (1941) demonstrated the power of the neoclassical general-equilibrium approach to international trade. It provided a reason why it was in labor's interest to press for tariff protection and thus the elements needed for an explanation of the existence of tariffs. While this had been basically (but vaguely) understood by the mercantilists and their successors, it was not until Stolper and Samuelson's contribution that the proposition had been put on a sound footing and was thus able to be incorporated into mainstream economic thought.

However, as it has come to be recognized[1] that the proposition does not generalize to models with many products and factors—i.e., that one cannot say that if the domestic price of commodity j rises, the rental of some factor i will necessarily rise more than proportionately—unless severe and rather unrealistic assumptions are made, the relevance of the proposition to the real world has come to be questioned. Are we then back where we started?

Some authors[2] have shown that it is possible to salvage at least one aspect of the Stolper-Samuelson theorem in a higher-dimensional setting: from elementary properties of matrix

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[1]Cf. Chipman (1969), Kemp & Wegge (1969), Wegge & Kemp (1969), Uekawa (1971), and Inada (1971).

[2]Ethier (1974a), Kemp & Wan (1976), and Jones & Scheinkman (1977).

multiplication it follows that, when the number of commodities is equal to the number of factors, for every factor one can specify (at least) one commodity such that if its price rises, the rental of the factor will fall. This is described by Jones & Scheinkman (1977, p. 919) in the words "every factor has at least one natural enemy." On the other hand, as they point out, it is not true that every factor has one natural friend; that is, one cannot say that for every factor there is a commodity a rise in whose price would cause a more than proportionate rise in the factor's rental. And in explaining why protectionist measures are introduced, it is clearly the latter kind of proposition that one would like to find.

In the present paper my approach is to look at this question as an aggregation problem. While it is true that different groups of workers push for tariffs and quotas on particular products, it is not clear that any one of these pressure groups would have enough political clout to influence the government if they acted separately rather than in combination. It makes sense, therefore, to ask whether there are conditions under which the separate labor factors might gain in the aggregate (or even separately) if uniform (or even non-uniform) tariffs are imposed simultaneously on all import goods.

In his seminal treatment of the theory of linear aggregation, Theil (1965) distinguished two types of conditions that would permit perfect aggregation of a model to a smaller number of dimensions. These may be illustrated by the simple Keynesian consumption function. Suppose that the ith household has a consumption function $c_i = a_i + b_i y_i$, where c_i is its consumption and y_i its income. Let aggregate consumption and income be denoted $C = \sum_{i=1}^{n} c_i$ and $Y = \sum_{i=1}^{n} y_i$ respectively. Then clearly there are two alternative conditions that will make it possible to express the aggregate consumption function as $C = a + bY$, where $a = \sum_{i=1}^{n} a_i$: (1) $b_i = b$ for $i = 1, 2, \ldots, n$; this may be called the case of *structural similarity*. (2) $y_i = \lambda_i Y$ where $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$ (so that $b = \sum_{i=1}^{n} \lambda_i b_i$); this is the case of *multicollinearity*. Either one of these assumptions (or a combination of the two) will lead to the desired result.

In an elegant article, Neary (1985) has shown how the multicollinearity approach can be used to aggregate a high-dimensional trade model to a 2×2 model, by assuming that import and export prices always move in proportion. In fact, such an approach had already been introduced by Kemp & Wan (1976). Neary obtained close analogues of the Stolper-Samuelson and Rybczynski theorems for the aggregative model.

In the present paper I will follow the structural approach, in which no constraints are placed on the variables under consideration (prices in the Stolper-Samuelson case, endowments in the Rybczynski case). It is evident that this is simply a generalization of the nonlinear aggregation problem as formulated by Solow (1956). Solow posed the question: when can a production function whose arguments include several types of capital be consolidated into a production function in which the capital inputs enter as an index of capital? In the present problem, the generalization is two-fold: first, all factors are simultaneously aggregated into groups; secondly, there are several outputs, and these are simultaneously aggregated into groups as well. The analysis is carried out in terms of the minimum-unit-cost functions dual to the production functions.

The main results are these: If commodity prices and factor rentals are aggregated by Laspeyres price indices, then the conditions for perfect aggregation of the Stolper-Samuelson mapping are that each aggregated industry (e.g., the export industry, or the import-competing industry), must absorb the endowments of each of the different types of labor (resp. capital) in the same proportions. The fact that these conditions are stated in terms of allocative shares of factor endowments, which are the elasticities of factor demand with respect to outputs, shows that they depend on properties of the dual Rybczynski mapping. Likewise, if commodity outputs and factor endowments are aggregated by Laspeyres quantity indices, the conditions for perfect aggregation of the Rybczynski mapping are that each aggregated factor (e.g., total labor, or total capital) should contribute the same fraction of unit costs in each of the component industries of the aggregated industry. The fact that these conditions are stated in terms of shares of factors in unit costs, which are the elasticities of unit costs with respect to factor rentals, shows that they depend on properties of the dual Stolper-Samuelson mapping. Since the Rybczynski mapping is linear (when the country diversifies and prices are given), the above conditions for its perfect aggregation are global; however, except for the case of fixed technical coefficients, the above conditions for perfect aggregation of the Stolper-Samuelson mapping are only local. As shown in Section 3, under a Cobb-Douglas technology global conditions may be obtained for the latter by using geometric rather than arithmetic means for the price and rental indices; but these would have to be combined with arithmetic quantity indices.

Before treating the aggregation problem as such, it is necessary to tackle the question of whether in the general model to be aggregated, one can allow for unequal numbers of commodities and factors, or whether these should be equal. I shall argue that if world prices are truly exogenous, a country will produce no more commodities than it has factors. It could of course produce fewer; but factor rentals would then depend on endowments as well as prices. I will go through this analysis in the next section; the final section will deal with the aggregation problem.

2 Equal or Unequal Numbers of Commodities and Factors

Defining our country's production-possibility set by

$$\mathcal{Y}(l) = \left\{ y = (y_1, y_2, \ldots, y_n) \left| \begin{array}{rcll} y_j & = & f_j(v_{1j}, v_{2j}, \ldots, v_{mj}) & (j = 1, 2, \ldots, n) \\ \sum_{j=1}^{n} v_{ij} & \leqq & l_i & (i = 1, 2, \ldots, m) \end{array} \right. \right\}$$

where y_j is the output of the jth commodity, l_i is the endowment of the ith factor, v_{ij} is the input of the ith factor into the production of the jth commodity, and f_j is the production function for the jth commodity, assumed concave and homogeneous of degree 1,[3] the domestic-product function is defined, for any price vector p, as

$$\Pi(p, l) = \max\{p \cdot y | y \in \mathcal{Y}(l)\}.$$

If Π is differentiable with respect to p, we know that (cf., e.g., Chipman 1987)

$$\frac{\partial \Pi(p, l)}{\partial p} = \hat{y}(p, l)$$

gives the single-valued Rybczynski function. A necessary condition for this differentiability is $m \geqq n$; if $m < n$ the country's production-possibility frontier is a ruled surface as illustrated in Figure 1 for the case $n = 3$ and $m = 2$. Any hyperplane tangential to this surface at an interior point necessarily touches it along a one-dimensional line segment (in the illustration), and in general, along a manifold of dimension $m - n$.

If Π is differentiable with respect to l, we know that

$$\frac{\partial \Pi(p, l)}{\partial l} = \hat{w}(p, l)$$

[3]This assumption is readily relaxed if economies of scale are external to individual firms, as shown by Inoue (1981).

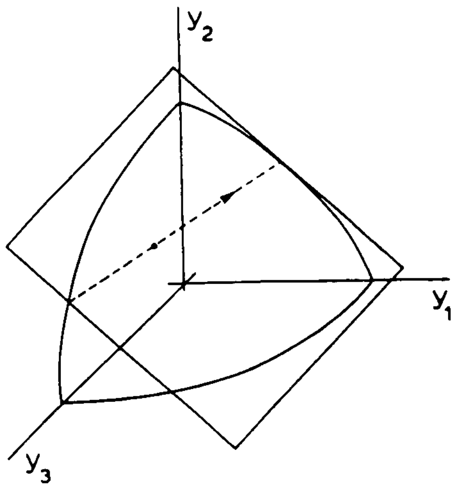

Figure 1: Ruled Production-Possibility Surface

gives the Stolper-Samuelson mapping. Unlike the case of the Rybczynski function, this is generally single-valued so long as the production functions are differentiable. However, Π, and thus $\hat{w}$, is in general not differentiable with respect to p if $n > m$ (this will be illustrated below). The Stolper-Samuelson function $\hat{w}$ has the important property (at the basis of Samuelson's (1953) factor-price equalization theorem) that, for $n \geqq m$, it is locally independent of l for certain values of p and l.

The standard 2×2 case is illustrated in Figure 2 showing cross-sections of the domestic-product function (the second panel also shows the Rybczynski lines). In the cones of diversification the contours of this function have flat segments; as endowments vary, the marginal value productivities of the factors remain constant.

If $m = 2$ and $n = 1$, $\hat{w}$ always depends on l so long as the single production function does not itself have any flat segments. If $m = 3$ and $n = 2$, the boundary (in the three-dimensional space of factor endowments) of the convex hull of the union of the sets $\{l | f_j(l_1, l_2, l_3) \geqq 1/p_j\}$ ($j = 1, 2$), will be a ruled surface. Hence, the vector of endowments may vary along a one-dimensional manifold on this surface without affecting factor rentals. However, such variation would constitute a "freak case" as one used to say, or would be "nongeneric" as one would say today. Essentially (generically), then, for the Stolper-Samuelson function to be locally independent of l we require $n \geqq m$.

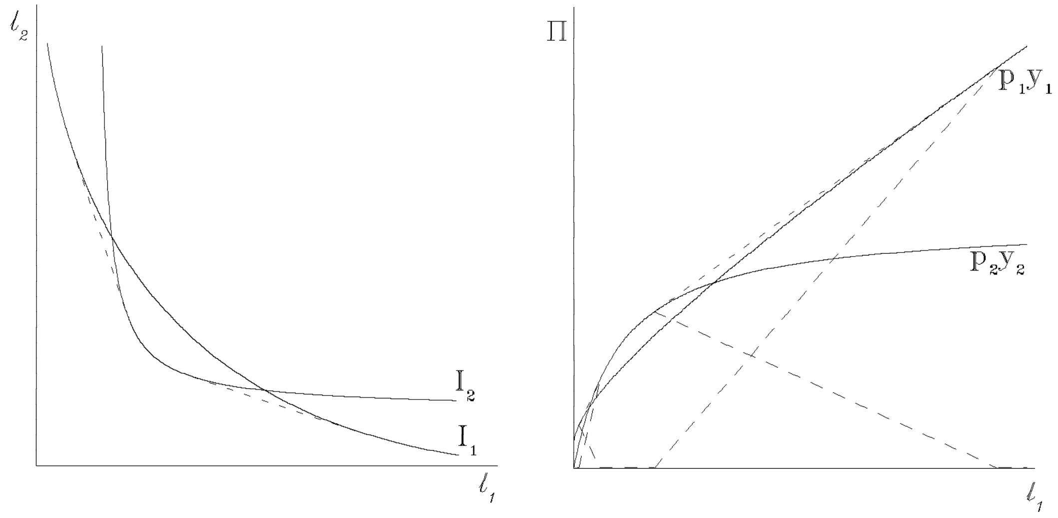

Figure 2: Cross-Sections of the Domestic-Product Function

Now let us consider the case $n > m$. Going back to Figure 1 we see that if there was an initial interior solution, world prices would have to change in a nongeneric way for a new equilibrium to remain an interior solution close to the initial one. An arbitrary small change in one price would drive equilibrium discontinuously to a corner. The situation can be seen more precisely in Figure 3 for $m = 2$ and $n = 3$. Isoquant I_j indicates the input combinations that will produce a dollar's worth of commodity j at the initial price p_j, i.e., the locus of points $\{(l_1, l_2) | f_j(l_1, l_2) = 1/p_j\}$. The arrow indicates the country's assumed endowment vector, enclosed in the large diversification cone shown by the solid rays from the origin. Suppose p_2 falls to p_2', so that the isoquant I_2 shifts upward to I_2'. Then the country will move discontinuously from producing all three goods to producing only commodities 1 and 3, yet factor rentals will remain unchanged. On the other hand if p_2 rises to p_2'', there will be two new cones of diversification in two commodities, one indicated by the dashed rays from the origin enclosing the country's endowment vector, in which commodities 1 and 2 are produced and the country ceases producing commodity 3, and w_2 rises relatively to w_1; if the endowment vector were in the other cone (not shown) w_2 would fall relatively to w_1. In either case, the left and right derivatives of each $\hat{w}_i$ with respect to p_2 are different.

The nondifferentiability of the Stolper-Samuelson function with respect to the other two prices could also easily be deduced from Figure 3. Thus, at the assumed position of the

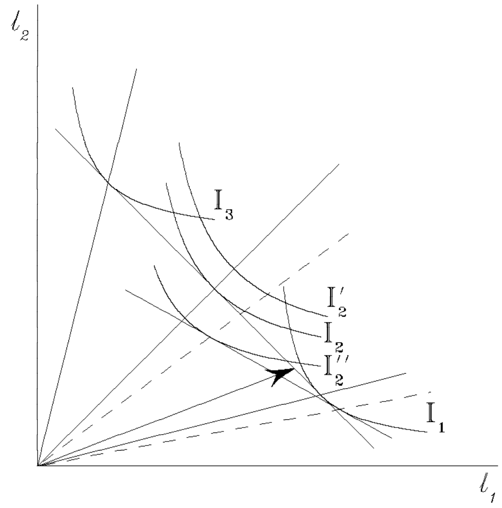

Figure 3: Effects of Price Changes with Three Commodities and Two Factors

endowment vector, if p_3 falls the country moves discontinuously from producing all three goods to producing only commodities 1 and 2; but if p_3 rises, the country will cease production of commodity 2 and produce only commodities 1 and 3. In the first case factor rentals remain unchanged, whereas in the second case w_2 rises relatively to w_1. Finally, if p_1 falls, production of commodity 3 will cease, whereas if p_1 rises production of commodity 2 will cease, and while in both cases w_1 moves in the same direction as p_1, the left and right derivatives will be different, since in the first case the model is equivalent to a two-commodity model in which only commodities 1 and 2 are produced, whereas in the second case it is equivalent to one in which only commodities 1 and 3 are produced.

The situation depicted in Figure 1 is clearly one in which our country strongly influences world prices. It is therefore illegitimate in this case to assume that world prices can be treated as exogenous. Denoting by $g_j(w) = g_j(w_1, w_2, \ldots, w_m)$ the minimum-unit-cost function dual to the production function f_j, the range of the mapping $g(w) = \big(g_1(w), g_2(w), \ldots, g_m(w)\big)$ has dimension at most m; if $n \geqq m$ it is an m-dimensional manifold (in fact, a cone) in n-dimensional space. If all n commodities are produced, the Stolper-Samuelson mapping $w = \hat{w}(p, l)$ must satisfy $p = g(w)$, i.e., p must be in the range of g. Any variation in an external price must be accompanied by suitable modifications in the remaining prices in order to maintain this restriction. It is not enough simply to "normalize" the prices to a unit simplex $\sum_{j=1}^{n} p_j = 1, p_j > 0$ (cf. Kemp & Wan 1976); in the case $n = 3$ and $m = 2$, for example, the price vector p would still have to be confined to a one-dimensional manifold in this simplex, namely the intersection of the simplex with the range of g.[4]

We may conclude that for external prices to be truly exogenous we must assume that $m \geqq n$, where n is the number of *produced* commodities in the country. Thus if our country is capable of producing three commodities, and started out specializing in commodities 1 and 2 and was not on the verge of producing commodity 3, we could use the standard two-commodity–two-factor apparatus.

How should we decide the question whether $m > n$ or $m = n$? At first glance it might seem absurd that if there are exactly 1,758,243 commodities, there must also be exactly 1,758,243 factors. However, I suggest that mere counting is not the right way to look at the problem. The fact is that our notions of "commodities" and "factors" are purely conventional. We consider tables and chairs to be two distinct commodities, because our language groups flat objects with four legs and calls them "tables," and similar objects (of appropriate dimensions) with backs and calls them "chairs." But we know that no two tables and no two chairs are exactly alike. Similarly with factors. The proper way to pose the question is: which model best represents reality? For example, if $m > n$ (where n is the number of produced tradable goods) we know that a unilateral transfer to a country with nontradable goods will affect the relative prices of tradables and nontradables, whereas this will not be the case if $m = n$. The null hypothesis $m = n$ can be tested by investigating whether capital inflows or outflows cause significant changes in relative prices.[5] In the simpler model with no nontradable goods, one can still—with any available grouped data on factor rentals, commodity prices, and factor endowments—test the null hypothesis that the Stolper-Samuelson function $\hat{w}(p, l)$ is independent of factor endowments, l.

[4]As a simple example, suppose the three minimum-unit-cost functions are given by

$$p_1 = g_1(w_1, w_2) = w_1^{.8} w_2^{.2}, \quad p_2 = g_2(w_1, w_2) = w_1^{.2} w_2^{.8}, \quad p_3 = g_3(w_1, w_2) = w_1^{.5} w_2^{.5}.$$

Solving the first two equations for the rentals and substituting them in the third we obtain $p_3 = p_1^{.5} p_2^{.5}$, which defines the range of g. Intersecting it with the unit simplex gives $p_1 + p_2 + p_1^{.5} p_2^{.5} = 1$.

[5]Such a test was carried out in Chipman (1985) and the null hypothesis was accepted.

3 The Aggregation Problem

We start with a minimum-unit-cost mapping $g : \mathcal{W} \to \mathcal{P}$ where $\mathcal{W}$ and $\mathcal{P}$ are n-dimensional spaces of vectors of factor rentals w and prices p; thus, $g(w) = p$. We postulate the existence of grouping mappings $\varphi : \mathcal{W} \to \overline{\mathcal{W}}$ and $\psi : \mathcal{P} \to \overline{\mathcal{P}}$, where $\overline{\mathcal{W}}$ and $\overline{\mathcal{P}}$ are $\bar{n}$-dimensional spaces of aggregate factor rentals and aggregate commodity prices, e.g., rental and price indices. These are defined as follows, where w and p are considered to be row vectors: the n factors are partitioned into $\bar{n}$ groups, and may be so numbered that w^{μ} is the row vector of rentals of the factors in the μth group; likewise for a partition of the n commodities into $\bar{n}$ groups. Thus $w = (w^1, w^2, \ldots, w^{\bar{n}})$ and $p = (p^1, p^2, \ldots, p^{\bar{n}})$. The mappings φ and ψ then have the form

$$\begin{aligned} \bar{w} &= \varphi(w) = \Big(\varphi_1(w^1), \varphi_2(w^2), \ldots, \varphi_{\bar{n}}(w^{\bar{n}})\Big) = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{\bar{n}}) \\ \bar{p} &= \psi(p) = \Big(\psi_1(p^1), \psi_2(p^2), \ldots, \psi_{\bar{n}}(p^{\bar{n}})\Big) = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_{\bar{n}}). \end{aligned}$$

The minimum-unit-cost mapping g will be said to satisfy the conditions for *perfect aggregation* (with unrestricted domain) if there exists a minimum-unit-cost mapping $\bar{g} : \overline{\mathcal{W}} \to \overline{\mathcal{P}}$ such that

$$\psi(g(w)) = \bar{g}(\varphi(w)) \quad \text{for all } w. \tag{3.1}$$

This is illustrated in Figure 4; perfect aggregation means that the diagram commutes, i.e., $\psi \circ g = \bar{g} \circ \varphi$. Under these conditions, the Stolper-Samuelson mapping may also be consolidated into a mapping between spaces of smaller dimension, i.e., $\varphi(g^{-1}(p)) = \bar{g}^{-1}(\psi(p))$ for all p.

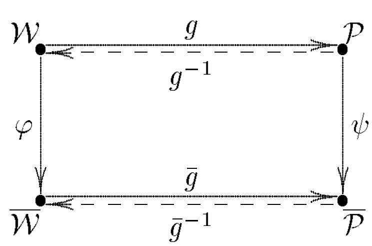

Figure 4: Commutative Diagram for the Stolper-Samuelson Mapping

In what follows, for simplicity of exposition and ease of notation I shall present the analysis in terms of an example of six commodities and factors which are to be aggregated to two of each; however, it will be evident that the same reasoning can handle the general case of aggregation from n to $\bar{n} < n$ groups. Let us assume in our illustration that there are three import-competing industries and three export industries (so that $n = 6$), and that commodities are so labelled that the first three are import-competing and the last three are exported; let them be aggregated into the corresponding export and import-competing groups (so that $\bar{n} = 2$). Likewise, let us assume that there are two labor factors (say, skilled and unskilled) and four capital factors, and that these are labelled so that the first two factors are kinds of labor and the last four are kinds of capital. The corresponding grouping mappings may then be written

$$\begin{aligned} \psi(p_1, p_2, p_3, p_4, p_5, p_6) &= \Big(\psi_1(p_1, p_2, p_3), \psi_2(p_4, p_5, p_6)\Big) = (\bar{p}_1, \bar{p}_2) \\ \varphi(w_1, w_2, w_3, w_4, w_5, w_6) &= \Big(\varphi_1(w_1, w_2), \varphi_2(w_3, w_4, w_5, w_6)\Big) = (\bar{w}_1, \bar{w}_2). \end{aligned} \tag{3.2}$$

From the above definitions,

$$\begin{aligned} \psi(g(w)) &= \Big(\psi_1\Big(g_1(w), g_2(w), g_3(w)\Big), \psi_2\Big(g_4(w), g_5(w), g_6(w)\Big)\Big) \\ \bar{g}(\varphi(w)) &= \Big(\bar{g}_1\Big(\varphi_1(w_1, w_2), \varphi_2(w_3, w_4, w_5, w_6)\Big), \bar{g}_2\Big(\varphi_1(w_1, w_2), \varphi_2(w_3, w_4, w_5, w_6)\Big)\Big). \end{aligned} \tag{3.3}$$

This implies that the composed function $\psi \circ g$ is separable in (w_1, w_2) and (w_3, w_4, w_5, w_6).[1]

Generalizing Solow's (1956) procedure, we may differentiate the two sets of two equations (3.3) with respect to the w_i and equate them, to obtain

(3.4)

$$\begin{bmatrix} \frac{\partial g_1}{\partial w_1} & \frac{\partial g_2}{\partial w_1} & \frac{\partial g_3}{\partial w_1} & \frac{\partial g_4}{\partial w_1} & \frac{\partial g_5}{\partial w_1} & \frac{\partial g_6}{\partial w_1} \\ \frac{\partial g_1}{\partial w_2} & \frac{\partial g_2}{\partial w_2} & \frac{\partial g_3}{\partial w_2} & \frac{\partial g_4}{\partial w_2} & \frac{\partial g_5}{\partial w_2} & \frac{\partial g_6}{\partial w_2} \\ \frac{\partial g_1}{\partial w_3} & \frac{\partial g_2}{\partial w_3} & \frac{\partial g_3}{\partial w_3} & \frac{\partial g_4}{\partial w_3} & \frac{\partial g_5}{\partial w_3} & \frac{\partial g_6}{\partial w_3} \\ \frac{\partial g_1}{\partial w_4} & \frac{\partial g_2}{\partial w_4} & \frac{\partial g_3}{\partial w_4} & \frac{\partial g_4}{\partial w_4} & \frac{\partial g_5}{\partial w_4} & \frac{\partial g_6}{\partial w_4} \\ \frac{\partial g_1}{\partial w_5} & \frac{\partial g_2}{\partial w_5} & \frac{\partial g_3}{\partial w_5} & \frac{\partial g_4}{\partial w_5} & \frac{\partial g_5}{\partial w_5} & \frac{\partial g_6}{\partial w_5} \\ \frac{\partial g_1}{\partial w_6} & \frac{\partial g_2}{\partial w_6} & \frac{\partial g_3}{\partial w_6} & \frac{\partial g_4}{\partial w_6} & \frac{\partial g_5}{\partial w_6} & \frac{\partial g_6}{\partial w_6} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi_1}{\partial p_1} & 0 \\ \frac{\partial \psi_1}{\partial p_2} & 0 \\ \frac{\partial \psi_1}{\partial p_3} & 0 \\ 0 & \frac{\partial \psi_2}{\partial p_4} \\ 0 & \frac{\partial \psi_2}{\partial p_5} \\ 0 & \frac{\partial \psi_2}{\partial p_6} \end{bmatrix} = \begin{bmatrix} \frac{\partial \varphi_1}{\partial w_1} & 0 \\ \frac{\partial \varphi_1}{\partial w_2} & 0 \\ 0 & \frac{\partial \varphi_2}{\partial w_3} \\ 0 & \frac{\partial \varphi_2}{\partial w_4} \\ 0 & \frac{\partial \varphi_2}{\partial w_5} \\ 0 & \frac{\partial \varphi_2}{\partial w_6} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{g}_1}{\partial \bar{w}_1} & \frac{\partial \bar{g}_2}{\partial \bar{w}_1} \\ \frac{\partial \bar{g}_1}{\partial \bar{w}_2} & \frac{\partial \bar{g}_2}{\partial \bar{w}_2} \end{bmatrix}.$$

[1] When the aggregator functions φ and ψ are linear as in (3.6) below, a sufficient condition for this is that the individual cost functions be themselves separable, i.e.,

$$g_j(w) = c_j(\chi_{j1}(w_1, w_2), \chi_{j2}(w_3, w_4, w_5, w_6)).$$

From the formula

$$\begin{bmatrix} 0 & \partial g_j / \partial w \\ \frac{\partial g_j}{\partial w'} & \frac{\partial^2 g_j}{\partial w' \partial w} \end{bmatrix} = \begin{bmatrix} 0 & \partial f_j / \partial v_j \\ \frac{\partial f_j}{\partial v_j'} & \frac{\partial^2 f_j}{\partial v_j' \partial v_j} \end{bmatrix}^{-1}$$

for the bordered Hessians of the cost function $g_j(w)$ and its dual production function $f_j(v_j) = f_j(v_{1j}, \ldots, v_{nj})$, and by application of Jacobi's theorem and use of Leontief's (1947) conditions, we see that the corresponding separability properties hold for the production functions.

Since by Shephard's duality theorem, $\partial g_j/\partial w_i = b_{ij}$ where b_{ij} is the amount of factor i needed per unit of output of commodity j, and similarly $\partial \bar{g}_\nu/\partial \bar{w}_\mu = \bar{b}_{\mu\nu}$, this equation may be written compactly as

$$B\Psi = \Phi\bar{B}, \tag{3.5}$$

where Ψ and Φ are grouping matrices, i.e., matrices with exactly one nonzero (in fact positive) element in each row. It is clear that condition (3.5) holds quite generally for the case of aggregating from n to $\bar{n}$ commodities and factors. From the assumption of constant returns to scale, the cost mapping $g(w)$ and its inverse Stolper-Samuelson mapping $g^{-1}(p)$ may be written as the matrix transformations between the row vectors w and p:

$$wB(w) = p \quad \text{and} \quad pB(g^{-1}(p))^{-1} = w.$$

The commutativity condition (3.5) is illustrated in Figure 5.

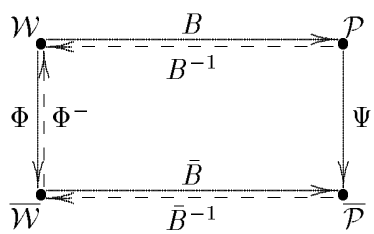

Figure 5: Commutative Diagram for the Stolper-Samuelson Transformation

In order to interpret conditions (3.4) let us consider the usual case in which the aggregator functions are linear-homogeneous, i.e., numerators of Laspeyres price indices:

$$\begin{aligned} \varphi(w) &= (l_1w_1 + l_2w_2,\ l_3w_3 + l_4w_4 + l_5w_5 + l_6w_6) \\ \psi(p) &= (y_1p_1 + y_2p_2 + y_3p_3,\ y_4p_4 + y_5p_5 + y_6p_6) \end{aligned} \tag{3.6}$$

where the l_i and y_j are respectively factor endowments and commodity outputs in some base period, which will be identified with the initial period. Then the above system of equations (3.4) may be written as

(3.7)

$$\begin{bmatrix} \frac{b_{11}y_1}{l_1} & \frac{b_{12}y_2}{l_1} & \frac{b_{13}y_3}{l_1} & \vdots & \frac{b_{14}y_1}{l_1} & \frac{b_{15}y_2}{l_1} & \frac{b_{16}y_3}{l_1} \\ \frac{b_{21}y_1}{l_2} & \frac{b_{22}y_2}{l_2} & \frac{b_{23}y_3}{l_2} & \vdots & \frac{b_{24}y_1}{l_2} & \frac{b_{25}y_2}{l_2} & \frac{b_{26}y_3}{l_2} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ \frac{b_{31}y_1}{l_3} & \frac{b_{32}y_2}{l_3} & \frac{b_{33}y_3}{l_3} & \vdots & \frac{b_{34}y_1}{l_3} & \frac{b_{35}y_2}{l_3} & \frac{b_{36}y_3}{l_3} \\ \frac{b_{41}y_1}{l_4} & \frac{b_{42}y_2}{l_4} & \frac{b_{43}y_3}{l_4} & \vdots & \frac{b_{44}y_1}{l_4} & \frac{b_{45}y_2}{l_4} & \frac{b_{46}y_3}{l_4} \\ \frac{b_{51}y_1}{l_5} & \frac{b_{52}y_2}{l_5} & \frac{b_{53}y_3}{l_5} & \vdots & \frac{b_{54}y_1}{l_5} & \frac{b_{55}y_2}{l_5} & \frac{b_{56}y_3}{l_5} \\ \frac{b_{61}y_1}{l_6} & \frac{b_{62}y_2}{l_6} & \frac{b_{63}y_3}{l_6} & \vdots & \frac{b_{64}y_1}{l_6} & \frac{b_{65}y_2}{l_6} & \frac{b_{66}y_3}{l_6} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ \cdots & \cdots \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \cdots & \cdots \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{bmatrix},$$

or, defining the diagonal matrices $Y = \text{diag}\,\{y_j\}$ and $L = \text{diag}\,\{l_i\}$,

$$RH = G\bar{B} \quad \text{where } R = L^{-1}BY \text{ and } G = L^{-1}\Phi,\ H = Y^{-1}\Phi. \tag{3.8}$$

The elements of the matrix R are simply the proportions of the factors allocated to the various industries, i.e., the elasticities of factor demands with respect to commodity outputs; the inverse matrix R^{-1} is the matrix of elasticities of the inverse (Rybczynski) transformation from endowments to outputs. The condition (3.8) is known in the literature on aggregation in input-output models as the "Hatanaka condition" (cf. Hatanaka 1952). What it states is that in each block of the matrix R, the row sums are equal to one another:[2] That is, the proportion of the total endowment of skilled labor allocated among the three import-competing industries must be the same as the proportion of the total endowment of unskilled labor allocated among these same industries (northwest block of R); and the same for the export industries (northeast block of R), which—in this case of only two aggregated industries—follows from the fact that R has unit row sums. Similarly (southwest block of R), the three import-competing industries must together employ the country's endowments in each of the four types of capital in the same proportion, and similarly for the export industries (southeast block of R).

Another way to interpret these conditions is as follows. Let $\mathcal{I}_\mu$ denote the set of integers i such that factor i is aggregated into the μth group of factors, and let $\mathcal{J}_\nu$ denote the set of integers j such that commodity j is aggregated into the νth group of commodities. In our

[2]This characterization was noted by Ara (1959); for a detailed exposition see Charnes & Cooper (1961, I, Appendix E). See also Chipman (1976, pp. 651–3, 745–8).

illustration, $\mathcal{I}_1 = \{1,2\}$, $\mathcal{I}_2 = \{3,4,5,6\}$, and $\mathcal{J}_1 = \{1,2,3\}$, $\mathcal{J}_2 = \{4,5,6\}$. Then for any aggregated industry $\mathcal{J}_\nu$ and any aggregated combination of factors $\mathcal{I}_\mu$,

$$\frac{\sum_{j\in\mathcal{J}_\nu} b_{ij}y_j}{\sum_{j\in\mathcal{J}_\nu} b_{i'j}y_j} = \frac{l_i}{l_{i'}} \quad \text{for} \quad i, i' \in \mathcal{I}_\mu,\ i \neq i'.$$

That is to say, the ratio of skilled to unskilled labor employed in each aggregated industry (export and import-competing) is the same as the endowment ratio, and similarly for the ratio of any two types of capital. Thus we see that the conditions for perfect aggregation involve a very strong form of degeneracy in the technology, which should not surprise us.

It should be noted that since the units of measurement in the aggregate rental and price indices are a dollar's worth, the matrix $\bar{B}$ is also the matrix of elasticities of factor demands with respect to outputs, hence $\bar{B}$ has unit row sums. This may be seen from the explicit solution of (3.8),

$$\bar{B} = G^{-}RH$$

where G^{-} is a left inverse of G (see Figure 5). In our illustration this may be written out as

$$(3.9)\quad \begin{bmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \vdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \vdots & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{b_{11}y_1}{l_1} & \frac{b_{12}y_2}{l_1} & \frac{b_{13}y_3}{l_1} & \vdots & \frac{b_{14}y_1}{l_1} & \frac{b_{15}y_2}{l_1} & \frac{b_{16}y_3}{l_1} \\ \frac{b_{21}y_1}{l_2} & \frac{b_{22}y_2}{l_2} & \frac{b_{23}y_3}{l_2} & \vdots & \frac{b_{24}y_1}{l_2} & \frac{b_{25}y_2}{l_2} & \frac{b_{26}y_3}{l_2} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ \frac{b_{31}y_1}{l_3} & \frac{b_{32}y_2}{l_3} & \frac{b_{33}y_3}{l_3} & \vdots & \frac{b_{34}y_1}{l_3} & \frac{b_{35}y_2}{l_3} & \frac{b_{36}y_3}{l_3} \\ \frac{b_{41}y_1}{l_4} & \frac{b_{42}y_2}{l_4} & \frac{b_{43}y_3}{l_4} & \vdots & \frac{b_{44}y_1}{l_4} & \frac{b_{45}y_2}{l_4} & \frac{b_{46}y_3}{l_4} \\ \frac{b_{51}y_1}{l_5} & \frac{b_{52}y_2}{l_5} & \frac{b_{53}y_3}{l_5} & \vdots & \frac{b_{54}y_1}{l_5} & \frac{b_{55}y_2}{l_5} & \frac{b_{56}y_3}{l_5} \\ \frac{b_{61}y_1}{l_6} & \frac{b_{62}y_2}{l_6} & \frac{b_{63}y_3}{l_6} & \vdots & \frac{b_{64}y_1}{l_6} & \frac{b_{65}y_2}{l_6} & \frac{b_{66}y_3}{l_6} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ \cdots & \cdots \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{3}\frac{b_{ij}y_j}{l_i} & \frac{1}{2}\sum_{i=1}^{2}\sum_{j=4}^{6}\frac{b_{ij}y_j}{l_i} \\ \frac{1}{4}\sum_{i=3}^{6}\sum_{j=1}^{3}\frac{b_{ij}y_j}{l_i} & \frac{1}{4}\sum_{i=3}^{6}\sum_{j=4}^{6}\frac{b_{ij}y_j}{l_i} \end{bmatrix}.$$

This states that each $\bar{b}_{\mu\nu}$ is equal to the average of the (equal) row sums (and therefore the common value of these row sums) of the elements in block $\mu\nu$ of R. More formally, denoting by ι_n the column vector of n ones, since R, H, and G have unit row sums, and any left inverse

G^- of G has unit row sums[3] (since $G\iota_{\bar{n}} = \iota_n$ implies $G^-\iota_n = G^-G\iota_{\bar{n}} = \iota_{\bar{n}}$), it follows from (3.8) that

$$\bar{B}\iota_{\bar{n}} = G^-G\bar{B}\iota_{\bar{n}} = G^-RH\iota_{\bar{n}} = G^-R\iota_n = G^-\iota_n = \iota_{\bar{n}}.$$

From these developments we see that the conditions for perfect aggregation of the Stolper-Samuelson mapping depend on properties of the Rybczynski mapping. It is evident that a complete analysis requires us to investigate the conditions for perfect aggregation of the dual Rybczynski mapping.

We have denoted by $B(w)$ the transpose of the Jacobian $\partial g(w)/\partial w$ of the system $g(w)$ of minimum-unit cost functions. $B(w)$ itself is the Jacobian of the system of resource-allocation equations

$$B(w)y' = l', \quad \text{or} \quad l = r(y) \equiv yB(w)' \tag{3.10}$$

where y' and l' are the column vectors of outputs and factor endowments respectively, and $r(y)$ denotes the resource requirements for outputs y at any fixed w (the argument w being suppressed). For each fixed w this defines a mapping from the n-dimensional space $\mathcal{Y}$ of output vectors y to the n-dimensional space $\mathcal{L}$ of endowment vectors l (y and l being considered as row vectors) whose inverse for each fixed p is the Rybczynski mapping $y' = B(g^{-1}(p))^{-1}l'$, where $g^{-1}(p) = w$. We may investigate the conditions under which this system may be aggregated to an $\bar{n} \times \bar{n}$ system

$$\bar{B}(\bar{w})\bar{y}' = \bar{l}', \quad \text{or} \quad \bar{l} = \bar{r}(y) = \bar{y}\bar{B}(\bar{w})' \tag{3.11}$$

defining for each $\bar{w}$ (already determined from the previous aggregation) a mapping from an $\bar{n}$-dimensional space $\overline{\mathcal{Y}}$ to an $\bar{n}$-dimensional space $\overline{\mathcal{L}}$, where $\bar{y} = \psi^*(y)$ and $\bar{l} = \varphi^*(l)$ are grouping mappings from $\mathcal{Y}$ to $\overline{\mathcal{Y}}$ and $\mathcal{L}$ to $\overline{\mathcal{Y}}$ conformable to the mappings $\bar{p} = \psi(p)$ and $\bar{w} = \varphi(w)$, i.e.,

$$\begin{aligned} \psi^*(y_1,y_2,y_3,y_4,y_5,y_6) &= \Big(\psi^*(y_1,y_2,y_3),\ \psi_2^*(y_4,y_5,y_6)\Big) = (\bar{y}_1,\bar{y}_2) \\ \varphi^*(l_1,l_2,l_3,l_4,l_5,l_6) &= \Big((\varphi_1^*(l_1,l_2),\ \varphi_2^*(l_3,l_4,l_5,l_6)\Big) = (\bar{l}_1,\bar{l}_2). \end{aligned} \tag{3.12}$$

Note that for consistency, the aggregate structural matrix $\bar{B}(\bar{w})$ should correspond to the one obtained by aggregating the transformation from factor rentals to commodity prices.

[3] Note that this is simply a generalization of the result that the inverse of a matrix with unit row (resp. column) sums itself has unit row (resp. column) sums; cf. Chipman (1969), p. 402, formula (1.10).

Proceeding as before we arrive at the condition

$$\begin{bmatrix} \frac{\partial\varphi_1^*}{\partial l_1} & \frac{\partial\varphi_1^*}{\partial l_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial\varphi_2^*}{\partial l_3} & \frac{\partial\varphi_2^*}{\partial l_4} & \frac{\partial\varphi_2^*}{\partial l_5} & \frac{\partial\varphi_2^*}{\partial l_6} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial\psi_1^*}{\partial y_1} & \frac{\partial\psi_1^*}{\partial y_2} & \frac{\partial\psi_1^*}{\partial y_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial\psi_2^*}{\partial y_4} & \frac{\partial\psi_2^*}{\partial y_5} & \frac{\partial\psi_2^*}{\partial y_6} \end{bmatrix},$$

or, in compact notation,

(3.13) $$\Phi^* B = \bar{B}\Psi^*.$$

This commutativity condition is illustrated in Figure 6, where, interpreting l and p as row vectors, the Jacobian of the transformation is the transpose matrix $B(w)'$.

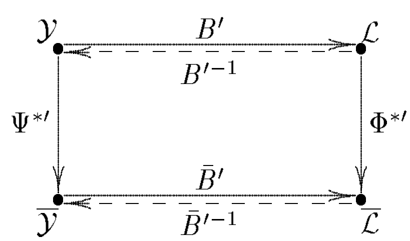

Figure 6: Commutative Diagram for the Rybczynski Transformation

As before, we may take the case in which the functions φ^* and ψ^* are linear-homogeneous, e.g., numerators of Laspeyres quantity indices:

(3.14) $$\begin{aligned} \varphi^*(l) &= (w_1 l_1 + w_2 l_2,\ w_3 l_3 + w_4 l_4 + w_5 l_5 + w_6 l_6) \\ \psi^*(y) &= (p_1 y_1 + p_2 y_2 + p_3 y_3,\ p_4 y_4 + p_5 y_5 + p_6 y_6) \end{aligned}$$

where the l_i and y_j are now variable and the w_i and p_j are fixed weights, say corresponding to the rentals and prices in the initial base period. Then we may write the above system in the form

$$
(3.15) \qquad \begin{bmatrix} 1 & 1 & \vdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \vdots & 1 & 1 & 1 & 1 \end{bmatrix}
\begin{bmatrix}
\frac{w_1 b_{11}}{p_1} & \frac{w_1 b_{12}}{p_2} & \frac{w_1 b_{13}}{p_3} & \vdots & \frac{w_1 b_{14}}{p_4} & \frac{w_1 b_{15}}{p_5} & \frac{w_1 b_{16}}{p_6} \\
\frac{w_2 b_{21}}{p_1} & \frac{w_2 b_{22}}{p_2} & \frac{w_2 b_{23}}{p_3} & \vdots & \frac{w_2 b_{24}}{p_4} & \frac{w_2 b_{25}}{p_5} & \frac{w_2 b_{26}}{p_6} \\
\cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
\frac{w_3 b_{31}}{p_1} & \frac{w_3 b_{32}}{p_2} & \frac{w_3 b_{33}}{p_3} & \vdots & \frac{w_3 b_{34}}{p_4} & \frac{w_3 b_{35}}{p_5} & \frac{w_3 b_{36}}{p_6} \\
\frac{w_4 b_{41}}{p_1} & \frac{w_4 b_{42}}{p_2} & \frac{w_4 b_{43}}{p_3} & \vdots & \frac{w_4 b_{44}}{p_4} & \frac{w_4 b_{45}}{p_5} & \frac{w_4 b_{46}}{p_6} \\
\frac{w_5 b_{51}}{p_1} & \frac{w_5 b_{52}}{p_2} & \frac{w_5 b_{53}}{p_3} & \vdots & \frac{w_5 b_{54}}{p_4} & \frac{w_5 b_{55}}{p_5} & \frac{w_5 b_{56}}{p_6} \\
\frac{w_6 b_{61}}{p_1} & \frac{w_6 b_{62}}{p_2} & \frac{w_6 b_{63}}{p_3} & \vdots & \frac{w_6 b_{64}}{p_4} & \frac{w_6 b_{65}}{p_5} & \frac{w_6 b_{66}}{p_6}
\end{bmatrix}
$$

$$
= \begin{bmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 1 & 1 & 1 \end{bmatrix}
$$

or, defining the diagonal matrices $P = \text{diag}\ \{p_j\}$ and $W = \text{diag}\ \{w_i\}$,

$$
(3.16) \qquad G^* S = \bar{B} H^* \quad \text{where} \quad S = WBP^{-1} \text{ and } G^* = \Phi^* W^{-1},\ H^* = \Psi^* P^{-1}.
$$

(Note that G^* and H^* are the transposes of the respective matrices G and H of (3.8).) The elements of the matrix S are simply the shares (in the initial period) of the various factors in the costs of production in the several industries, i.e., the elasticities of minimum-unit costs with respect to factor rentals; thus the columns of S sum to unity. The inverse matrix S^{-1} is the matrix of elasticities of the inverse (Stolper-Samuelson) transformation from commodity prices to factor rentals. What the dual Hatanaka condition (3.16) states is that is that in each block of the matrix S, the column sums are equal to one another: that is, the share of labor (skilled and unskilled) in unit costs (northwest block of S), and thus the share in unit costs of the aggregate of the four kinds of capital (southwest block of S), must be the same in each of the three import-competing industries; likewise in the three export industries (northeast and southeast blocks of S). An argument similar to the preceding shows that $\bar{B}$ has unit column sums.

Now let us consider the Stolper-Samuelson transformation, given by the inverse of the system of minimum-unit-cost functions. From (3.5) we have (see also Figure fig:SST)

$$
(3.17) \qquad \Phi = B \Psi \bar{B}^{-1} \quad \text{hence} \quad B^{-1} \Phi = \Psi \bar{B}^{-1}.
$$

Written out in the case of our example, with the linear-homogeneous aggregator functions, this is, denoting $B^{-1} = [b^{ij}]$ and $\bar{B}^{-1} = [\bar{b}^{\mu\nu}]$,

$$\begin{bmatrix} b^{11} & b^{12} & b^{13} & b^{14} & b^{15} & b^{16} \\ b^{21} & b^{22} & b^{23} & b^{24} & b^{25} & b^{26} \\ b^{31} & b^{32} & b^{33} & b^{34} & b^{35} & b^{36} \\ b^{41} & b^{42} & b^{43} & b^{44} & b^{45} & b^{46} \\ b^{51} & b^{52} & b^{53} & b^{54} & b^{55} & b^{56} \\ b^{61} & b^{62} & b^{63} & b^{64} & b^{65} & b^{66} \end{bmatrix} \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \\ 0 & l_3 \\ 0 & l_4 \\ 0 & l_5 \\ 0 & l_6 \end{bmatrix} = \begin{bmatrix} y_1 & 0 \\ y_2 & 0 \\ y_3 & 0 \\ 0 & y_3 \\ 0 & y_4 \\ 0 & y_5 \end{bmatrix} \begin{bmatrix} \bar{b}^{11} & \bar{b}^{12} \\ \bar{b}^{21} & \bar{b}^{22} \end{bmatrix}.$$

Since the b^{ji} are the Stolper-Samuelson derivatives $\partial\hat{w}_i/\partial p_j$, the above may be summarized by the equations

$$\begin{aligned} \frac{\partial}{\partial p_j}\sum_{i=1}^{2} l_i\hat{w}_i &= y_j\bar{b}^{11} \quad \text{and} \quad & \frac{\partial}{\partial p_j}\sum_{i=3}^{6} l_i\hat{w}_i &= y_j\bar{b}^{12} \qquad \text{for } j = 1,2,3 \\ \frac{\partial}{\partial p_j}\sum_{i=1}^{2} l_i\hat{w}_i &= y_j\bar{b}^{21} \quad \text{and} \quad & \frac{\partial}{\partial p_j}\sum_{i=3}^{6} l_i\hat{w}_i &= y_j\bar{b}^{22} \qquad \text{for } j = 4,5,6. \end{aligned}$$

If we now assume (referring to (3.9)) that in each block of B, each row has at least one positive b_{ij}, and that all outputs and factor endowments are positive, then the elements of $\bar{B}$ are all positive. It follows that the matrix $\bar{B}^{-1}$ has either positive diagonal and negative off-diagonal elements or negative diagonal and positive off-diagonal elements. Supposing the former, so that the import-competing industries are labor intensive in the aggregate and the export industries capital intensive in the aggregate, the above equations state that a rise in any one of the import prices will lead to a rise in a Laspeyres index of wages (with the labor endowments as weights) and a fall in a Laspeyres index of rentals of capital (with the capital endowments as weights); likewise, a rise in any one of the export prices will have the opposite effect. Moreover, these changes will be proportionate to the outputs of the respective import-competing or export goods. It follows that a rise in a price index of importables (with any positive weights—e.g., amounts of imports rather than outputs in the base period) will lead to a rise in the wage index and a fall in the rental index, and similarly for a price index of exportables. If the weights in the price indices are outputs, however, and if the conditions (3.5) for perfect aggregation hold then—since $\bar{B}^{-1}$ defines precisely the matrix of elasticities of rental indices with respect to price indices—these elasticities are respectively greater than unity and less than zero.

Development of the Rybczynski transformation is entirely similar. From (3.13) we have

$$\Psi^* = \bar{B}^{-1}\Phi^* B \quad \text{hence} \quad \Psi^* B^{-1} = \bar{B}^{-1}\Phi^*. \tag{3.18}$$

In the case of our example with aggregator functions (3.14), this gives

$$\begin{bmatrix} p_1 & p_2 & p_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_4 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} b^{11} & b^{12} & b^{13} & b^{14} & b^{15} & b^{16} \\ b^{21} & b^{22} & b^{23} & b^{24} & b^{25} & b^{26} \\ b^{31} & b^{32} & b^{33} & b^{34} & b^{35} & b^{36} \\ b^{41} & b^{42} & b^{43} & b^{44} & b^{45} & b^{46} \\ b^{51} & b^{52} & b^{53} & b^{54} & b^{55} & b^{56} \\ b^{61} & b^{62} & b^{63} & b^{64} & b^{65} & b^{66} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{b}^{11} & \bar{b}^{12} \\ \bar{b}^{21} & \bar{b}^{22} \end{bmatrix} \begin{bmatrix} w_1 & w_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & w_3 & w_4 & w_5 & w_6 \end{bmatrix}.$$

Since the b^{ji} are the Rybczynski derivatives $\partial\hat{y}_j/\partial l_i$, this reduces to

$$\begin{aligned} \frac{\partial}{\partial l_i}\sum_{j=1}^{3} p_j\hat{y}_j &= w_i\bar{b}^{11} \quad \text{and} \quad & \frac{\partial}{\partial l_i}\sum_{j=4}^{6} p_j\hat{y}_j &= w_i\bar{b}^{21} \qquad \text{for } i = 1,2 \\ \frac{\partial}{\partial l_i}\sum_{j=1}^{3} p_j\hat{y}_j &= w_i\bar{b}^{12} \quad \text{and} \quad & \frac{\partial}{\partial l_i}\sum_{j=4}^{6} p_j\hat{y}_j &= w_i\bar{b}^{22} \qquad \text{for } i = 3,4,5,6. \end{aligned}$$

From the solution $\bar{B} = G^*SH^{*-}$ (where H^{*-} is a right inverse of H^*) it is clear that as long as in each block of B, each column has at least one positive element, and all prices and rentals are positive, the elements of $\bar{B}$ are all positive. Supposing as before that $\bar{B}^{-1}$ has negative off-diagonal elements, the above equations imply that a rise in the endowment of either skilled or unskilled labor will lead to a rise in a Laspeyres index of output of importables (with prices of importables as weights) and a fall in a Laspeyres index of output of exportables (with prices of exportables as weights), and a rise in any of the capital endowments will have the opposite effect. The remaining conditions analogous to the Stolper-Samuelson case also hold.

Putting all this together we see that in order for both the Stolper-Samuelson and Rybczynski mappings to be perfectly aggregable, the two sets of conditions (3.5) and (3.13) must both hold, hence the matrix B of factor-output coefficients must be subject to the double bilinear restriction

$$\Phi^- B\Psi = \bar{B} = \Phi^* B\Psi^{*-} \tag{3.19}$$

where Φ^- and Ψ^{*-} are left and right inverses of Φ and Ψ^* respectively. This is quite a stringent requirement. However, it is not an unreasonable one provided the matrices are empirically stable. Recall that in (3.7) the l_i and y_j are fixed weights corresponding to the initial endowments and outputs, and the $b_{ij}(w)$ are evaluated at the factor rentals prevailing in the initial equilibrium. (3.7) then expresses a local condition for perfect aggregation of the Stolper-Samuelson mapping. Of course, if the technical coefficients b_{ij} are fixed as in Leontief's (1951) model, then it is a global condition. Turning to (3.15), we recall that the w_i and p_j appearing in the formula are now fixed weights corresponding to the initial rentals and prices, and the $b_{ij}(w)$ are evaluated at these fixed rentals; the share matrix S is therefore fixed, and (3.15) expresses a global condition for perfect aggregation of the Rybczynski mapping.

It would of course be preferable if we could find some global conditions, expressed in terms of the underlying parameters of the model, for simultaneous perfect aggregation of both the Stolper-Samuelson and Rybczynski mappings. This would require us of course to make some parametric specifications concerning the production functions and thus their dual minimum-unit-cost functions.

If production functions are of the Cobb-Douglas type then we know that the dual minimum-unit-cost functions are also of the Cobb-Douglas type:

$$g_j(w_1, w_2, \ldots, w_6) = \nu_j w_1^{\beta_{1j}} w_2^{\beta_{2j}} \ldots w_6^{\beta_{6j}} \quad (j = 1, 2, \ldots, 6).$$

In this case it is clearly appropriate to replace the arithmetic means specified by the forms (3.6) and (3.14) chosen for the price and quantity indices (3.2) and (3.12) by geometric means:

$$\begin{aligned} \varphi(w) &= (w_1^{\theta_1} w_2^{\theta_2},\, w_3^{\theta_3} w_4^{\theta_4} w_5^{\theta_5} w_6^{\theta_6}) = (\bar{w}_1, \bar{w}_2) \qquad (\theta_i > 0) \\ \psi(p) &= (p_1^{\upsilon_1} p_2^{\upsilon_2} p_3^{\upsilon_3},\, p_4^{\upsilon_4} p_5^{\upsilon_5} p_6^{\upsilon_6}) = (\bar{p}_1, \bar{p}_2) \qquad (\upsilon_j > 0). \end{aligned} \tag{3.20}$$

We then have

$$\begin{aligned} \psi(g(w)) &= \left(\prod_{j=1}^{3} \nu_j^{\upsilon_j} \prod_{i=1}^{6} w_i^{\sum_{j=1}^{3} \beta_{ij}\upsilon_j},\, \prod_{j=4}^{6} \nu_j^{\upsilon_j} \prod_{i=1}^{6} w_i^{\sum_{j=4}^{6} \beta_{ij}\upsilon_j} \right) \\ \bar{g}(\varphi(w)) &= \left(\bar{\nu}_1 \prod_{i=1}^{2} w_i^{\theta_i \bar{\beta}_{11}} \prod_{i=3}^{6} w_i^{\theta_i \bar{\beta}_{21}},\, \bar{\nu}_2 \prod_{i=1}^{2} w_i^{\theta_i \bar{\beta}_{12}} \prod_{i=3}^{6} w_i^{\theta_i \bar{\beta}_{22}} \right). \end{aligned}$$

For these to be equal for all w, the following equation must be satisfied:

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & \beta_{14} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} & \beta_{24} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} & \beta_{34} & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} & \beta_{44} & \beta_{44} \\ \beta_{51} & \beta_{52} & \beta_{53} & \beta_{54} & \beta_{54} & \beta_{54} \\ \beta_{61} & \beta_{62} & \beta_{63} & \beta_{64} & \beta_{64} & \beta_{64} \end{bmatrix} \begin{bmatrix} \upsilon_1 & 0 \\ \upsilon_2 & 0 \\ \upsilon_3 & 0 \\ 0 & \upsilon_4 \\ 0 & \upsilon_5 \\ 0 & \upsilon_5 \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ \theta_2 & 0 \\ 0 & \theta_3 \\ 0 & \theta_4 \\ 0 & \theta_5 \\ 0 & \theta_6 \end{bmatrix} \begin{bmatrix} \bar{\beta}_{11} & \bar{\beta}_{12} \\ \bar{\beta}_{21} & \bar{\beta}_{22} \end{bmatrix},$$

or, since the share matrix S is now the fixed matrix $[\beta_{ij}]$,

$$S\Upsilon = \Theta \bar{B}. \tag{3.21}$$

This expresses the conditions for perfect aggregation of the Stolper-Samuelson mapping in terms of weighted averages of the exponents of the Cobb-Douglas production functions.

We could try replacing the arithmetic endowment and output indices (3.14) by the geometric ones

$$\begin{aligned} \varphi^*(l) &= (l_1^{\theta_1^*} l_2^{\theta_2^*},\, l_3^{\theta_3^*} l_4^{\theta_4^*} l_5^{\theta_5^*} l_6^{\theta_6^*}) = (\bar{l}_1, \bar{l}_2) \qquad (\theta_i^* > 0) \\ \psi^*(y) &= (y_1^{\upsilon_1^*} y_2^{\upsilon_2^*} y_3^{\upsilon_3^*},\, y_4^{\upsilon_4^*} y_5^{\upsilon_5^*} y_6^{\upsilon_6^*}) = (\bar{y}_1, \bar{y}_2) \quad (\upsilon_j^* > 0) \end{aligned} \tag{3.22}$$

in order to aggregate the factor-demand mapping (3.10) to (3.11), to obtain the composed functions

$$\begin{aligned} \varphi^*(r(y)) &= \left(\prod_{i=1}^{2} \Big(\sum_{j=1}^{6} b_{ij} y_j \Big)^{\theta_i^*},\, \prod_{i=3}^{6} \Big(\sum_{j=1}^{6} b_{ij} y_j \Big)^{\theta_i^*} \right) \\ \bar{r}(\psi^*(y)) &= \left(\bar{b}_{11} \prod_{j=1}^{3} y_j^{\upsilon_j^*} + \bar{b}_{12} \prod_{j=4}^{6} y_j^{\upsilon_j^*},\, \bar{b}_{21} \prod_{j=1}^{3} y_j^{\upsilon_j^*} + \bar{b}_{22} \prod_{j=4}^{6} y_j^{\upsilon_j^*} \right). \end{aligned}$$

However, upon differentiating these two composite functions with respect to the y_j and equating, one finds that the restrictions are in the form of extremely complicated nonlinear equations; hence these conditions could not be expected to be empirically stable. We are therefore compelled to fall back on the linear quantity indices (3.14) for the aggregation of the Rybczynski relations.

With Cobb-Douglas technology, geometric price and rental indices, and arithmetic output and endowment indices, we now have the following double bilinear restriction on the constant share matrix S,

$$G^* S H^{*-} = \bar{B} = \Theta^- S \Upsilon, \tag{3.23}$$

where all the matrices entering the condition are constant. While this restriction has the advantage of being a global one (provided the technology is Cobb-Douglas), it has the distinct disadvantage of asymmetry in the choice of geometric means for price indices and arithmetic means for quantity indices, and it may involve considerable specification error if the technology departs at all significantly from the Cobb-Douglas type.

The condition (3.19) combined with the assumption of Laspeyres price and quantity indices thus seems to be a sounder criterion in general for simultaneous perfect aggregation of the Stolper-Samuelson and Rybczynski mappings. Of course, nobody believes such a condition to be literally true; its virtue is that it is capable of exact interpretation, and forms a benchmark by which one may assess the goodness of approximation of estimated production coefficients to the conditions for simultaneous perfect aggregation of the Stolper-Samuelson and Rybczynski mappings. Aggregation may be justified provided the distance between the matrices $\Phi^{-}B\Psi$ and $\Phi^{*}B\Psi^{*-}$ (a concept which can be made perfectly precise—see Chipman 1976) is sufficiently small. And recent advances in the development of heuristic algorithms for integer programming make it possible to find close-to-optimal modes of aggregation of commodities and factors into groups so as to achieve a suitably low level of aggregation error (cf. Chipman and Winker 1992). In the last analysis, the robustness of the Stolper-Samuelson theorem must be tested by empirical application, and the framework developed here is offered as a means to that end.

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