

A Three-Commodity Model of Tariffs, and The Dynamics of Protectionism*

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Introduction

There has been much interest in recent years in the attempt to endogenize the protectionist process. The standard two-commodity models of international trade are, however, rather ill-suited to the task, since they necessarily overlook the struggle between competing interest groups vying for protection.

The best dynamic treatment of the protectionist process is still that of Johnson (1954), who formulated the tariff game as one in which each country imposed an optimal tariff on the assumption that the foreign country's tariff rate was given, and found that the process converged to an equilibrium in which either both countries were worse off than under free trade, or one of them was actually better off than under free trade.

It has often been remarked, however, that it is difficult to document a case in which a government deliberately used tariff policy to improve, let alone optimize, the country's terms of trade. Rather, tariffs and other trade barriers are the outcome of lobbying efforts of special interest groups. Nevertheless, the results of their exertions might be fleeting and unviable unless they happened to give rise to improved terms of trade which alone could provide the real benefits they seek.

In this paper, I develop a three-commodity model of trade in which the country of interest exports one of the commodities and imports the remaining two. The question of interest is whether a tariff imposed on one of the commodities harms or benefits the other import-competing industry, and whether it alone can improve, or on the contrary may worsen, the country's potential welfare. Not surprisingly it turns out that either case can occur. It is assumed that what triggers the protectionist process is an exogenous decline in one of the import prices; political forces in the industry will seek measures which will restore the initial internal price. However, this may have the effect of reducing the internal price faced by the other import-competing industry; in this case, an internal tariff war takes place in which both tariffs rise and the country's terms of trade improve (it is assumed as a first approximation that foreign countries act passively and do not retaliate, except possibly at the end of the process).

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If protection of one industry actually helps the other industry, no internal tariff war need ensue; however, since in this case the single tariff has raised the internal prices of both import goods, it may be conjectured (but has not been formally proved in this paper) that it improves the country's terms of trade.

1 The effect of tariffs on external prices

Let us assume that country 1 exports one good—commodity 1—to country 2 and imports two goods—commodities 2 and 3—from country 2. (We may think of country 2 as “the rest of the world.”) Country 1 imposes tariffs on these two import goods, but no restrictions on its export good. It may or may not produce a nontradable good.

Letting $T_j = 1 + \tau_j$ denote the tariff factor, and τ_j the tariff rate, imposed by country 1 on the import of commodity j ($j = 2, 3$), and let p_j^k denote the price of commodity j on country k 's markets. Denoting country k 's trade-demand function for commodity j by

$$z_j^k = \hat{h}_j^k(p_1^k, p_2^k, p_3^k, D^k; l^k),$$

where D^k is the deficit in country k 's balance of trade and l^k is the vector of country k 's factor endowments, we may define country 1's tariff-inclusive demand-for-import functions

$$(1.1) \quad \hat{z}_j^1(p_1^2, p_2^2, p_3^2, T_2, T_3, l^1) \quad (j = 2, 3)$$

implicitly by the two equations

$$(1.2) \quad \begin{aligned} \hat{z}_2^1(\cdot) &= \hat{h}_2^1(p_1^2, T_2 p_2^2, T_3 p_3^2, (T_2 - 1)p_2^2 \hat{z}_2^1(\cdot) + (T_3 - 1)p_3^2 \hat{z}_3^1(\cdot); l^1) \\ \hat{z}_3^1(\cdot) &= \hat{h}_3^1(p_1^2, T_2 p_2^2, T_3 p_3^2, (T_2 - 1)p_2^2 \hat{z}_2^1(\cdot) + (T_3 - 1)p_3^2 \hat{z}_3^1(\cdot); l^1). \end{aligned}$$

Assuming country 2 not to retaliate, its excess-demand functions for commodities 2 and 3 (its two export goods) are defined by

$$(1.3) \quad \hat{z}_j^2(p_1^2, p_2^2, p_3^2, l^2) = \hat{h}_j^2(p_1^2, p_2^2, p_3^2, 0; l^2) \quad (j = 2, 3).$$

The functions $p_i^2 = \bar{p}_i^2(T_2, T_3)$ ($i = 2, 3$) are defined implicitly by the following equations of world equilibrium, where the price $p_1^1 = p_1^2 = \bar{p}_1^2$ of commodity 1 is taken as numéraire and held fixed:

$$(1.4) \quad \begin{aligned} \hat{z}_2^1(\bar{p}_1^2, \bar{p}_2^2(\cdot), \bar{p}_3^2(\cdot), T_2, T_3, l^1) + \hat{z}_2^2(\bar{p}_1^2, \bar{p}_2^2(\cdot), \bar{p}_3^2(\cdot), l^2) &= 0 \\ \hat{z}_3^1(\bar{p}_1^2, \bar{p}_2^2(\cdot), \bar{p}_3^2(\cdot), T_2, T_3, l^1) + \hat{z}_3^2(\bar{p}_1^2, \bar{p}_2^2(\cdot), \bar{p}_3^2(\cdot), l^2) &= 0. \end{aligned}$$

From these equations we obtain

$$(1.5) \quad \begin{bmatrix} \frac{\partial \bar{p}_2^2}{\partial T_2} & \frac{\partial \bar{p}_2^2}{\partial T_3} \\ \frac{\partial \bar{p}_3^2}{\partial T_2} & \frac{\partial \bar{p}_3^2}{\partial T_3} \end{bmatrix} = - \begin{bmatrix} \frac{\partial \hat{z}_2^1}{\partial p_2^2} + \frac{\partial \hat{z}_2^2}{\partial p_2^2} & \frac{\partial \hat{z}_2^1}{\partial p_3^2} + \frac{\partial \hat{z}_2^2}{\partial p_3^2} \\ \frac{\partial \hat{z}_3^1}{\partial p_2^2} + \frac{\partial \hat{z}_3^2}{\partial p_2^2} & \frac{\partial \hat{z}_3^1}{\partial p_3^2} + \frac{\partial \hat{z}_3^2}{\partial p_3^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \hat{z}_2^1}{\partial T_2} & \frac{\partial \hat{z}_2^1}{\partial T_3} \\ \frac{\partial \hat{z}_3^1}{\partial T_2} & \frac{\partial \hat{z}_3^1}{\partial T_3} \end{bmatrix}$$

provided the inverse matrix exists.

From (1.5) we may conclude that the negative inverse matrix, which is precisely $-A^{-1}$ (see (A.4)), has positive principal minors and has all its elements nonnegative. Each $\partial \bar{p}_i^2 / \partial T_j$ ($i = 2, 3$) is thus a positive weighted average of the $\partial \hat{z}_i^1 / \partial T_j$ ($i = 2, 3$).

Let us now obtain expressions for the $\partial \hat{z}_i^1 / \partial T_j$ for $i, j = 2, 3$. Differentiating equations (1.2) with respect to T_2 and T_3 we obtain

$$(1.6) \quad \begin{bmatrix} \frac{\partial \hat{z}_2^1}{\partial T_2} & \frac{\partial \hat{z}_2^1}{\partial T_3} \\ \frac{\partial \hat{z}_3^1}{\partial T_2} & \frac{\partial \hat{z}_3^1}{\partial T_3} \end{bmatrix} = \begin{bmatrix} 1 - \frac{T_2 - 1}{T_2} p_2^1 \frac{\partial \hat{h}_2^1}{\partial D^1} & -\frac{T_3 - 1}{T_3} p_3^1 \frac{\partial \hat{h}_2^1}{\partial D^1} \\ -\frac{T_2 - 1}{T_2} p_2^1 \frac{\partial \hat{h}_3^1}{\partial D^1} & 1 - \frac{T_3 - 1}{T_3} p_3^1 \frac{\partial \hat{h}_3^1}{\partial D^1} \end{bmatrix}^{-1} \begin{bmatrix} \hat{s}_{22}^1 & \hat{s}_{23}^1 \\ \hat{s}_{32}^1 & \hat{s}_{33}^1 \end{bmatrix} \begin{bmatrix} \frac{p_2^1}{T_2} & 0 \\ 0 & \frac{p_3^1}{T_3} \end{bmatrix}$$

(where the

$$\hat{s}_{ij}^1 = \frac{\partial \hat{h}_i^1}{\partial p_j^1} + \frac{\partial \hat{h}_i^1}{\partial D^1} \hat{h}_j^1$$

are country 1's trade-Slutsky terms), provided the inverse matrix exists. That it does exist follows by computation of the determinant, which is

$$(1.7) \quad \Delta = \begin{vmatrix} 1 - \frac{T_2 - 1}{T_2} p_2^1 \frac{\partial \hat{h}_2^1}{\partial D^1} & -\frac{T_3 - 1}{T_3} p_3^1 \frac{\partial \hat{h}_2^1}{\partial D^1} \\ -\frac{T_2 - 1}{T_2} p_2^1 \frac{\partial \hat{h}_3^1}{\partial D^1} & 1 - \frac{T_3 - 1}{T_3} p_3^1 \frac{\partial \hat{h}_3^1}{\partial D^1} \end{vmatrix} = 1 - \frac{T_2 - 1}{T_2} p_2^1 \frac{\partial \hat{h}_2^1}{\partial D^1} - \frac{T_3 - 1}{T_3} p_3^1 \frac{\partial \hat{h}_3^1}{\partial D^1}.$$

Since $(1 - 1/T_i) \in [0, 1)$ for $T_i \in [1, \infty)$, and since if we assume that

$$p_1^1 \frac{\partial \hat{h}_1^1}{\partial D^1} > 0 \quad \text{and} \quad p_i^1 \frac{\partial \hat{h}_i^1}{\partial D^1} \geq 0 \quad \text{for} \quad i = 2, 3,$$

it follows that

$$p_i^1 \frac{\partial \hat{h}_i^1}{\partial D^1} < 1 \quad \text{for} \quad i = 2, 3,$$

therefore the determinant (1.7) is necessarily positive. The inverse matrix in (1.6) is then

$$\frac{1}{\Delta} \begin{bmatrix} 1 - \frac{T_3 - 1}{T_3} p_3^1 \frac{\partial \hat{h}_3^1}{\partial D^1} & \frac{T_3 - 1}{T_3} p_3^1 \frac{\partial \hat{h}_2^1}{\partial D^1} \\ \frac{T_2 - 1}{T_2} p_2^1 \frac{\partial \hat{h}_3^1}{\partial D^1} & 1 - \frac{T_2 - 1}{T_2} p_2^1 \frac{\partial \hat{h}_2^1}{\partial D^1} \end{bmatrix},$$

which has all its elements nonnegative.

Putting together (1.5) and (1.6) we then have

$$\begin{bmatrix} \frac{\partial \bar{p}_2^2}{\partial T_2} & \frac{\partial \bar{p}_2^2}{\partial T_3} \\ \frac{\partial \bar{p}_3^2}{\partial T_2} & \frac{\partial \bar{p}_3^2}{\partial T_3} \end{bmatrix} =$$

$$(1.8) \quad \begin{bmatrix} \frac{\partial \hat{z}_2^1}{\partial p_2^2} + \frac{\partial \hat{z}_2^2}{\partial p_2^2} & \frac{\partial \hat{z}_2^1}{\partial p_3^2} + \frac{\partial \hat{z}_2^2}{\partial p_3^2} \\ \frac{\partial \hat{z}_3^1}{\partial p_2^2} + \frac{\partial \hat{z}_3^2}{\partial p_2^2} & \frac{\partial \hat{z}_3^1}{\partial p_3^2} + \frac{\partial \hat{z}_3^2}{\partial p_3^2} \end{bmatrix}^{-1} \begin{bmatrix} 1 - \frac{T_2 - 1}{T_2} p_2^1 \frac{\partial \hat{h}_2^1}{\partial D^1} & -\frac{T_3 - 1}{T_3} p_3^1 \frac{\partial \hat{h}_2^1}{\partial D^1} \\ -\frac{T_2 - 1}{T_2} p_2^1 \frac{\partial \hat{h}_3^1}{\partial D^1} & 1 - \frac{T_3 - 1}{T_3} p_3^1 \frac{\partial \hat{h}_3^1}{\partial D^1} \end{bmatrix}^{-1} \times$$

$$\begin{bmatrix} \hat{s}_{22}^1 & \hat{s}_{23}^1 \\ \hat{s}_{32}^1 & \hat{s}_{33}^1 \end{bmatrix} \begin{bmatrix} \frac{p_2^1}{T_2} & 0 \\ 0 & \frac{p_3^1}{T_3} \end{bmatrix},$$

where the first inverse matrix has all its elements nonpositive and the second has all its elements nonnegative. Thus, the $\partial \bar{p}_i^2 / \partial T_j$ are proportional to nonnegative weighted averages of the trade-Slutsky terms $\hat{s}_{2j}, \hat{s}_{3j}$ for $j = 2, 3$. It follows immediately that if for each j both trade-Slutsky terms are negative, then a tariff on either commodity 2 or commodity 3 will lower the world prices of both these commodities. However, owing to the relation $p_1^1 \hat{s}_{1j}^1 + p_2^1 \hat{s}_{2j}^1 + p_3^1 \hat{s}_{3j}^1 = 0$, this would imply that $\hat{s}_{1j}^1 > 0$ for $j = 2, 3$, i.e., that country 1's export good is a trade-Hicksian substitute of both import goods, while the import goods are trade-Hicksian complements of each other. It would seem more natural to assume that $\hat{s}_{ij}^1 > 0$ for all $i \neq j$, i.e., that all three goods are trade-Hicksian substitutes of each other; in that case the \hat{s}_{ij}^1 have opposite signs for $i = 2, 3$ and any $j = 2, 3, j \neq i$, and the sign of $\partial \bar{p}_i^2 / \partial T_j$ is indeterminate in the absence of further assumptions (but see below).

2 Welfare analysis

What we are really interested in, rather than the effect on the two import prices of a tariff imposed on either import good, is whether country 1 gains from a tariff on either of its import goods, in the sense that the gainers could compensate the losers. Since we have assumed aggregable preferences, this amounts to determining whether a tariff will increase the country's utility.

The indirect trade-utility function of country 1 is defined by

$$\hat{V}^1(p_1^1, p_2^1, p_3^1, D^1; l^1) = \hat{U}^1(\hat{h}^1(p_1^1, p_2^1, p_3^1, D^1; l^1))$$

where $\hat{h}^1(\cdot) = (\hat{h}_1^1(\cdot), \hat{h}_2^1(\cdot), \hat{h}_3^1(\cdot))$. Accordingly, we may define country 1's potential welfare as a function of the two tariff factors by

$$(2.1) \quad W^1(T_2, T_3) = \hat{V}^1(\bar{p}_1^2, T_2 \bar{p}_2^2(\cdot), T_3 \bar{p}_3^2(\cdot), (T_2 - 1) \bar{p}_2^2(\cdot) \bar{z}_2^1(\cdot) + (T_3 - 1) \bar{p}_3^2(\cdot) \bar{z}_3^1(\cdot); l^1),$$

where $\bar{p}_i^2(\cdot)$ denotes $\bar{p}_i^2(T_2, T_3)$ and likewise $\bar{z}_i^1(\cdot)$ denotes

$$(2.2) \quad \bar{z}_i^1(T_2, T_3) = \hat{z}_i^1(\bar{p}_1^2, \bar{p}_2^2(T_2, T_3), \bar{p}_3^2(T_2, T_3), T_2, T_3, l^1).$$

Differentiating (2.1) with respect to T_j while making use of Antonelli's partial differential equation

$$(2.3) \quad \frac{\partial \hat{V}^1}{\partial p_j^1} = -\frac{\partial \hat{V}^1}{\partial D^1} \hat{h}_j^1,$$

we obtain after cancelling like terms

$$(2.4) \quad \frac{\partial W^1}{\partial T_j} = \frac{\partial \hat{V}^1}{\partial D^1} \left(-\bar{z}_2^1 \frac{\partial \bar{p}_2^2}{\partial T_j} - \bar{z}_3^1 \frac{\partial \bar{p}_3^2}{\partial T_j} + \tau_2 \bar{p}_2^2 \frac{\partial \bar{z}_2^1}{\partial T_j} + \tau_3 \bar{p}_3^2 \frac{\partial \bar{z}_3^1}{\partial T_j} \right).$$

Starting from free trade, the last two terms of (2.4) vanish. Since $\partial \hat{V}^1 / \partial D^1 > 0$, an increase in T_j raises potential welfare if and only if it lowers a Laspeyres world price index of imports, $\bar{z}_2^1 \bar{p}_2^2(T_2, T_3) + \bar{z}_3^1 \bar{p}_3^2(T_2, T_3)$, where the quantity weights are the volumes of country 1's imports in the initial free-trade situation. One of the questions we must investigate is the conditions under which this holds, making possible a generalization of Bickerdike's first theorem, that a sufficiently small tariff will raise a country's potential welfare (cf. Bickerdike 1906, 1907).

The second question to investigate is conditions under which the expressions (2.4) vanish for $j = 2, 3$, providing the first-order conditions for a set of "optimal tariffs," generalizing Bickerdike's second theorem. Let us now tackle these two questions.

2.1 Extension of Bickerdike's first theorem

From (2.4) it is clear as we have just seen that a necessary and sufficient condition for a small tariff on either import good to improve country 1's potential welfare, starting from free trade, is that

$$(\bar{z}_2^1, \bar{z}_3^1) \begin{bmatrix} \frac{\partial \bar{p}_2^2}{\partial T_2} & \frac{\partial \bar{p}_2^2}{\partial T_3} \\ \frac{\partial \bar{p}_3^2}{\partial T_2} & \frac{\partial \bar{p}_3^2}{\partial T_3} \end{bmatrix} < (0, 0).$$

From (1.8) this is equivalent to

$$(2.5) \quad -(\bar{z}_2^1, \bar{z}_3^1) \begin{bmatrix} \frac{\partial \hat{z}_2^1}{\partial p_2^2} + \frac{\partial \hat{z}_2^2}{\partial p_2^2} & \frac{\partial \hat{z}_2^1}{\partial p_3^2} + \frac{\partial \hat{z}_2^2}{\partial p_3^2} \\ \frac{\partial \hat{z}_3^1}{\partial p_2^2} + \frac{\partial \hat{z}_3^2}{\partial p_2^2} & \frac{\partial \hat{z}_3^1}{\partial p_3^2} + \frac{\partial \hat{z}_3^2}{\partial p_3^2} \end{bmatrix}^{-1} \begin{bmatrix} \hat{s}_{22}^1 & \hat{s}_{23}^1 \\ \hat{s}_{32}^1 & \hat{s}_{33}^1 \end{bmatrix} \begin{bmatrix} p_2^1 & 0 \\ 0 & p_3^1 \end{bmatrix} < (0, 0).$$

Since it is natural to require that this hold independently of the initial levels of imports, and since the diagonal matrix of prices may obviously be cancelled from (2.5), our required condition is

$$(2.6) \quad - \begin{bmatrix} \frac{\partial \hat{z}_2^1}{\partial p_2^2} + \frac{\partial \hat{z}_2^2}{\partial p_2^2} & \frac{\partial \hat{z}_2^1}{\partial p_3^2} + \frac{\partial \hat{z}_2^2}{\partial p_3^2} \\ \frac{\partial \hat{z}_3^1}{\partial p_2^2} + \frac{\partial \hat{z}_3^2}{\partial p_2^2} & \frac{\partial \hat{z}_3^1}{\partial p_3^2} + \frac{\partial \hat{z}_3^2}{\partial p_3^2} \end{bmatrix}^{-1} \begin{bmatrix} \hat{s}_{22}^1 & \hat{s}_{23}^1 \\ \hat{s}_{32}^1 & \hat{s}_{33}^1 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

or, in matrix notation,

$$-A^{-1}S \equiv B^{-1}S < 0,$$

where $B = -A$ (A being defined by (A.1)) and S is the trade-Slutsky submatrix).

If both import goods are trade-Hicksian complements, then since $B^{-1} > 0$ and $S < 0$, it is immediate that the product is negative. It remains only to consider the case in which the import goods are trade-Hicksian substitutes, i.e., $\hat{s}_{ij} > 0$ for $i \neq j, j = 2, 3$.

Denoting $B^{-1} = [b^{ij}]$, for the top left term of (2.6) to be negative we require (recalling that the b^{ij} are positive for $i, j = 2, 3$)

$$b^{22}\hat{s}_{22} + b^{23}\hat{s}_{32} < 0 \iff \frac{|\hat{s}_{22}|}{\hat{s}_{32}} > \frac{b^{23}}{b^{22}} = \frac{-b_{23}}{b_{33}} = \frac{a_{23}}{|a_{33}|}.$$

As long as we assume that both A and S have dominant diagonals (by rows), so that $|a_{33}| > a_{23}$ and $|\hat{s}_{22}| > \hat{s}_{32}$, the above inequality follows automatically. The same reasoning applies to the bottom right term of (2.6). Proceeding to the off-diagonal terms, we see that

$$b^{32}\hat{s}_{22} + b^{33}\hat{s}_{32} < 0 \iff \frac{|\hat{s}_{22}|}{\hat{s}_{32}} > \frac{b^{33}}{b^{32}} = \frac{b_{22}}{-b_{32}} = \frac{|a_{22}|}{a_{32}}.$$

Similarly we see that

$$b^{23}\hat{s}_{23} + b^{33}\hat{s}_{33} < 0 \iff \frac{|\hat{s}_{33}|}{\hat{s}_{23}} > \frac{b^{22}}{b^{23}} = \frac{b_{33}}{-b_{23}} = \frac{|a_{33}|}{a_{23}}.$$

Thus, in order for the off-diagonal elements of $B^{-1}S$ to be negative we require that the ratios of the absolute values of the diagonal elements of S to the other elements in the same column be greater than the corresponding ratios of the absolute values of the diagonal elements of A to the corresponding elements in the same column, or in other words, that S have a stronger relative column-diagonal-dominance property than A . This is certainly a strong and rather arbitrary assumption; thus we cannot expect Bickerdike's first theorem to generalize without fairly strong assumptions.

On the other hand, it is a straightforward matter to show that a small *uniform* tariff $T = T_2 = T_3$ will necessarily lower a Laspeyres price index of imports and thus improve country 1's potential welfare. The proof of this is left to the reader.

2.2 Extension of Bickerdike's second theorem

Country 1's potential-welfare function can also be expressed in terms of its *direct* trade-utility function, composed with the negatives of the foreign excess-demand functions:

$$W^1(T_2, T_3) = \hat{U}^1\left(-\hat{z}_1^2(\bar{p}_1^2, \bar{p}_2^2(\cdot), \bar{p}_3^2(\cdot), l^2), -\hat{z}_2^2(\bar{p}_1^2, \bar{p}_2^2(\cdot), \bar{p}_3^2(\cdot), l^2), -\hat{z}_3^2(\bar{p}_1^2, \bar{p}_2^2(\cdot), \bar{p}_3^2(\cdot), l^2)\right). \quad (2.7)$$

Differentiating with respect to T_j and using the first-order conditions for maximum trade-utility subject to the balance-of-trade constraint,

$$\frac{\partial \hat{U}^1}{\partial z_i^1} = \lambda p_i^1 \quad \left(\text{where } \lambda = \frac{\partial \hat{V}^1}{\partial D^1}\right),$$

we have

$$\frac{\partial W^1}{\partial T_j} = -\sum_{i=1}^3 \frac{\partial \hat{U}^1}{\partial z_i^1} \left[\sum_{k=2}^3 \frac{\partial \hat{z}_i^2}{\partial p_k^2} \frac{\partial \bar{p}_k^2}{\partial T_j} \right]$$

$$\begin{aligned}
&= -\lambda \sum_{k=2}^3 \left[\sum_{i=1}^3 p_i \frac{\partial \hat{z}_i^2}{\partial p_k^2} \right] \frac{\partial \bar{p}_k^2}{\partial T_j} \\
(2.8) \quad &= -\lambda \sum_{k=2}^3 \left[\sum_{i=1}^3 T_i p_i^2 \frac{\partial \hat{z}_i^2}{\partial p_k^2} \right] \frac{\partial \bar{p}_k^2}{\partial T_j}.
\end{aligned}$$

Now,

$$\begin{aligned}
(2.9) \quad \sum_{i=1}^3 T_i p_i^2 \frac{\partial \hat{z}_i^2}{\partial p_k^2} &= p_1^2 \frac{\partial \hat{z}_1^2}{\partial p_k^2} + (1 + \tau_2) p_2^2 \frac{\partial \hat{z}_2^2}{\partial p_k^2} + (1 + \tau_3) p_3^2 \frac{\partial \hat{z}_3^2}{\partial p_k^2} \\
&= p_1^2 \frac{\partial \hat{z}_1^2}{\partial p_k^2} + p_2^2 \frac{\partial \hat{z}_2^2}{\partial p_k^2} + p_3^2 \frac{\partial \hat{z}_3^2}{\partial p_k^2} + \tau_2 p_2^2 \frac{\partial \hat{z}_2^2}{\partial p_k^2} + \tau_3 p_3^2 \frac{\partial \hat{z}_3^2}{\partial p_k^2}.
\end{aligned}$$

From the budget (balance-of-trade) identity

$$p_1^2 \hat{z}_1^2 + p_2^2 \hat{z}_2^2 + p_3^2 \hat{z}_3^2 = 0$$

we have

$$\frac{\partial}{\partial p_k^2} (p_1^2 \hat{z}_1^2 + p_2^2 \hat{z}_2^2 + p_3^2 \hat{z}_3^2) = \hat{z}_k^2 + \sum_{i=1}^3 p_i^2 \frac{\partial \hat{z}_i^2}{\partial p_k^2} = 0.$$

Therefore from (2.9),

$$\sum_{i=1}^3 T_i p_i^2 \frac{\partial \hat{z}_i^2}{\partial p_k^2} = -\hat{z}_k^2 + \tau_2 p_2^2 \frac{\partial \hat{z}_2^2}{\partial p_k^2} + \tau_3 p_3^2 \frac{\partial \hat{z}_3^2}{\partial p_k^2} \quad (k = 2, 3).$$

Setting (2.8) equal to zero for $j = 2, 3$ we then have

$$\begin{bmatrix} -\hat{z}_2^2 + \tau_2 p_2^2 \frac{\partial \hat{z}_2^2}{\partial p_2^2} + \tau_3 p_3^2 \frac{\partial \hat{z}_3^2}{\partial p_2^2}, & -\hat{z}_3^2 + \tau_2 p_2^2 \frac{\partial \hat{z}_2^2}{\partial p_3^2} + \tau_3 p_3^2 \frac{\partial \hat{z}_3^2}{\partial p_3^2} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{p}_2^2}{\partial T_2} & \frac{\partial \bar{p}_2^2}{\partial T_3} \\ \frac{\partial \bar{p}_3^2}{\partial T_2} & \frac{\partial \bar{p}_3^2}{\partial T_3} \end{bmatrix} = 0.$$

But the matrix $[\partial \bar{p}_i^2 / \partial T_j]$ is nonsingular provided the 2×2 submatrix $S = [\hat{s}_{ij}^1]$ ($i, j = 2, 3$) of the trade-Slutsky matrix is nonsingular,¹ as is seen from (1.8), hence the row vector on the left must vanish, i.e.,

$$(2.10) \quad \begin{bmatrix} p_2^2 \frac{\partial \hat{z}_2^2}{\partial p_2^2} & p_3^2 \frac{\partial \hat{z}_3^2}{\partial p_2^2} \\ p_2^2 \frac{\partial \hat{z}_2^2}{\partial p_3^2} & p_3^2 \frac{\partial \hat{z}_3^2}{\partial p_3^2} \end{bmatrix} \begin{bmatrix} \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} \hat{z}_2^2 \\ \hat{z}_3^2 \end{bmatrix}.$$

These are the equations of the optimal tariffs.

¹This requires only that the trade-indifference surfaces in 3-dimensional space be smooth, i.e., that the direct trade-utility function be twice differentiable.

3 The effect of tariffs on internal prices

In order to trace the welfare effects on individual factors of production it is desirable to focus on country 1's internal prices. Accordingly, with country-1 prices as arguments, we may denote country 1's tariff-inclusive demand-for-import functions by

$$(3.1) \quad \tilde{z}_j^1(p_1^1, p_2^1, p_3^1, T_2, T_3, l^1) \quad (j = 2, 3).$$

These are defined implicitly by the two equations

$$(3.2) \quad \begin{aligned} \tilde{z}_2^1(\cdot) &= \hat{h}_2^1(p_1^1, p_2^1, p_3^1, [(T_2 - 1)/T_2]p_2^1\tilde{z}_2^1(\cdot) + [(T_3 - 1)/T_3]p_3^2\tilde{z}_3^1(\cdot), l^1) \\ \tilde{z}_3^1(\cdot) &= \hat{h}_3^1(p_1^1, p_2^1, p_3^1, [(T_2 - 1)/T_2]p_2^2\tilde{z}_2^1(\cdot) + [(T_3 - 1)/T_3]p_3^2\tilde{z}_3^1(\cdot), l^1). \end{aligned}$$

Assuming country 2 not to retaliate, its excess-demand functions for commodities 2 and 3 (its two export goods), as functions of country 1's prices, are defined by

$$(3.3) \quad \tilde{z}_i^2(p_1^1, p_2^1, p_3^1, T_2, T_3, l^2) = \hat{h}_i^2(p_1^1, p_2^1/T_2, p_3^1/T_3, 0; l^2) \quad (i = 2, 3).$$

The functions $p_i^2 = \bar{p}_i^2(T_2, T_3)$ ($i = 2, 3$) are defined implicitly by the following equations of world equilibrium, where as before the price $p_1^1 = p_1^2 = \bar{p}_1^2$ of commodity 1 is taken as numéraire and held fixed:

$$(3.4) \quad \begin{aligned} \tilde{z}_2^1(\bar{p}_1^1, \bar{p}_2^1(\cdot), \bar{p}_3^1(\cdot), T_2, T_3, l^1) + \tilde{z}_2^2(\bar{p}_1^1, \bar{p}_2^1(\cdot), \bar{p}_3^1(\cdot), T_2, T_3, l^2) &= 0 \\ \tilde{z}_3^1(\bar{p}_1^1, \bar{p}_2^1(\cdot), \bar{p}_3^1(\cdot), T_2, T_3, l^1) + \tilde{z}_3^2(\bar{p}_1^1, \bar{p}_2^1(\cdot), \bar{p}_3^1(\cdot), T_2, T_3, l^2) &= 0. \end{aligned}$$

From these equations we obtain

$$(3.5) \quad \begin{bmatrix} \frac{\partial \bar{p}_2^1}{\partial T_2} & \frac{\partial \bar{p}_2^1}{\partial T_3} \\ \frac{\partial \bar{p}_3^1}{\partial T_2} & \frac{\partial \bar{p}_3^1}{\partial T_3} \end{bmatrix} = - \begin{bmatrix} \frac{\partial \tilde{z}_2^1}{\partial p_2^1} + \frac{\partial \tilde{z}_2^2}{\partial p_2^1} & \frac{\partial \tilde{z}_2^1}{\partial p_3^1} + \frac{\partial \tilde{z}_2^2}{\partial p_3^1} \\ \frac{\partial \tilde{z}_3^1}{\partial p_2^1} + \frac{\partial \tilde{z}_3^2}{\partial p_2^1} & \frac{\partial \tilde{z}_3^1}{\partial p_3^1} + \frac{\partial \tilde{z}_3^2}{\partial p_3^1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \tilde{z}_2^1}{\partial T_2} + \frac{\partial \tilde{z}_2^2}{\partial T_2} & \frac{\partial \tilde{z}_2^1}{\partial T_3} + \frac{\partial \tilde{z}_2^2}{\partial T_3} \\ \frac{\partial \tilde{z}_3^1}{\partial T_2} + \frac{\partial \tilde{z}_3^2}{\partial T_2} & \frac{\partial \tilde{z}_3^1}{\partial T_3} + \frac{\partial \tilde{z}_3^2}{\partial T_3} \end{bmatrix}$$

provided (as before) the inverse matrix exists.

Let us now obtain expressions for the $\partial \tilde{z}_i^k / \partial T_j$ for $i, j = 2, 3$ and $k = 1, 2$. Differentiating equations (3.2) with respect to T_2 and T_3 we obtain

$$(3.6) \quad \begin{bmatrix} \frac{\partial \tilde{z}_2^1}{\partial T_2} & \frac{\partial \tilde{z}_2^1}{\partial T_3} \\ \frac{\partial \tilde{z}_3^1}{\partial T_2} & \frac{\partial \tilde{z}_3^1}{\partial T_3} \end{bmatrix} = \begin{bmatrix} 1 - \frac{T_2 - 1}{T_2} p_2^1 \hat{c}_2^1 & - \frac{T_3 - 1}{T_3} p_3^1 \hat{c}_2^1 \\ - \frac{T_2 - 1}{T_2} p_2^1 \hat{c}_3^1 & 1 - \frac{T_3 - 1}{T_3} p_3^1 \hat{c}_3^1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{c}_2^1 & \hat{c}_2^1 \\ \hat{c}_3^1 & \hat{c}_3^1 \end{bmatrix} \begin{bmatrix} \frac{p_2^1 \tilde{z}_2^1}{(T_2)^2} & 0 \\ 0 & \frac{p_3^1 \tilde{z}_3^1}{(T_3)^2} \end{bmatrix},$$

where the

$$\hat{c}_i^1 = \frac{\partial \hat{h}_i^1}{\partial D^1}$$

are the “deficit effects” on country 1’s excess demands. Likewise, differentiating equations (3.3) with respect to T_2 and T_3 we obtain

$$(3.7) \quad \begin{bmatrix} \frac{\partial z_2^2}{\partial T_2} & \frac{\partial z_2^2}{\partial T_3} \\ \frac{\partial z_3^2}{\partial T_2} & \frac{\partial z_3^2}{\partial T_3} \end{bmatrix} = - \begin{bmatrix} \frac{\partial \hat{h}_2^2}{\partial p_2^2} & \frac{\partial \hat{h}_2^2}{\partial p_3^2} \\ \frac{\partial \hat{h}_3^2}{\partial p_2^2} & \frac{\partial \hat{h}_3^2}{\partial p_3^2} \end{bmatrix} \begin{bmatrix} \frac{p_2^1}{(T_2)^2} & 0 \\ 0 & \frac{p_3^1}{(T_3)^2} \end{bmatrix}.$$

Putting together (3.5), (3.6), and (3.7) we then have

$$(3.8) \quad \begin{bmatrix} \frac{\partial \bar{p}_2^1}{\partial T_2} & \frac{\partial \bar{p}_2^1}{\partial T_3} \\ \frac{\partial \bar{p}_3^1}{\partial T_2} & \frac{\partial \bar{p}_3^1}{\partial T_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial z_2^1}{\partial p_2^2} + \frac{\partial z_2^2}{\partial p_2^2} & \frac{\partial z_2^1}{\partial p_3^2} + \frac{\partial z_2^2}{\partial p_3^2} \\ \frac{\partial z_3^1}{\partial p_2^2} + \frac{\partial z_3^2}{\partial p_2^2} & \frac{\partial z_3^1}{\partial p_3^2} + \frac{\partial z_3^2}{\partial p_3^2} \end{bmatrix}^{-1} \begin{bmatrix} 1 - \frac{T_2 - 1}{T_2} p_2^1 \hat{c}_2^1 & -\frac{T_3 - 1}{T_3} p_3^1 \hat{c}_2^1 \\ -\frac{T_2 - 1}{T_2} p_2^1 \hat{c}_3^1 & 1 - \frac{T_3 - 1}{T_3} p_3^1 \hat{c}_3^1 \end{bmatrix}^{-1} \times \left(\begin{bmatrix} \hat{c}_2^1 z_2^1 & \hat{c}_2^1 z_3^1 \\ \hat{c}_3^1 z_2^1 & \hat{c}_3^1 z_3^1 \end{bmatrix} - \begin{bmatrix} 1 - \frac{T_2 - 1}{T_2} p_2^1 \hat{c}_2^1 & -\frac{T_3 - 1}{T_3} p_3^1 \hat{c}_2^1 \\ -\frac{T_2 - 1}{T_2} p_2^1 \hat{c}_3^1 & 1 - \frac{T_3 - 1}{T_3} p_3^1 \hat{c}_3^1 \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{h}_2^2}{\partial p_2^2} & \frac{\partial \hat{h}_2^2}{\partial p_3^2} \\ \frac{\partial \hat{h}_3^2}{\partial p_2^2} & \frac{\partial \hat{h}_3^2}{\partial p_3^2} \end{bmatrix} \right) \times \begin{bmatrix} \frac{p_2^1}{(T_2)^2} & 0 \\ 0 & \frac{p_3^1}{(T_3)^2} \end{bmatrix},$$

where by our assumptions the first inverse matrix has all its elements nonpositive and the second has all its elements nonnegative.

This formula simplifies greatly in the case $T_2 = T_3 = 1$ of initial free trade, in which case (3.8) reduces to

$$(3.9) \quad \begin{bmatrix} \frac{\partial \bar{p}_2^1}{\partial T_2} & \frac{\partial \bar{p}_2^1}{\partial T_3} \\ \frac{\partial \bar{p}_3^1}{\partial T_2} & \frac{\partial \bar{p}_3^1}{\partial T_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial z_2^1}{\partial p_2^1} + \frac{\partial z_2^2}{\partial p_2^1} & \frac{\partial z_2^1}{\partial p_3^1} + \frac{\partial z_2^2}{\partial p_3^1} \\ \frac{\partial z_3^1}{\partial p_2^1} + \frac{\partial z_3^2}{\partial p_2^1} & \frac{\partial z_3^1}{\partial p_3^1} + \frac{\partial z_3^2}{\partial p_3^1} \end{bmatrix}^{-1} \begin{bmatrix} \hat{c}_2^1 z_2^1 - \frac{\partial \hat{h}_2^2}{\partial p_2^2} & \hat{c}_2^1 z_3^1 - \frac{\partial \hat{h}_2^2}{\partial p_3^2} \\ \hat{c}_3^1 z_2^1 - \frac{\partial \hat{h}_3^2}{\partial p_2^2} & \hat{c}_3^1 z_3^1 - \frac{\partial \hat{h}_3^2}{\partial p_3^2} \end{bmatrix} \begin{bmatrix} p_2^1 & 0 \\ 0 & p_3^1 \end{bmatrix}.$$

This generalizes Metzler's (1949) famous formula.² The problem that now concerns us is: what are the signs of these partial derivatives?

Denoting

$$\tilde{A} = \begin{bmatrix} \frac{\partial \tilde{z}_2^1}{\partial p_2^1} + \frac{\partial \tilde{z}_2^2}{\partial p_2^1} & \frac{\partial \tilde{z}_2^1}{\partial p_3^1} + \frac{\partial \tilde{z}_2^2}{\partial p_3^1} \\ \frac{\partial \tilde{z}_3^1}{\partial p_2^1} + \frac{\partial \tilde{z}_3^2}{\partial p_2^1} & \frac{\partial \tilde{z}_3^1}{\partial p_3^1} + \frac{\partial \tilde{z}_3^2}{\partial p_3^1} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \hat{c}_2^1 z_2^1 - \frac{\partial \hat{h}_2^2}{\partial p_2^2} & \hat{c}_2^1 z_3^1 - \frac{\partial \hat{h}_2^2}{\partial p_3^2} \\ \hat{c}_3^1 z_2^1 - \frac{\partial \hat{h}_3^2}{\partial p_2^2} & \hat{c}_3^1 z_3^1 - \frac{\partial \hat{h}_3^2}{\partial p_3^2} \end{bmatrix},$$

and $\tilde{B} = -\tilde{A}$, we wish to assess the probable signs of the elements of the product $\tilde{B}^{-1}R$. The diagonal elements of R correspond to Metzler's expressions as shown in the preceding footnote. The "Metzler paradox" occurs when $r_{jj} < 0$, which can happen only if $\partial \hat{h}_j^j / \partial p_j^2 > 0$, i.e., if a rise in the price of commodity j causes country 2 to *reduce* its supply of exports of commodity j (recall that $\hat{h}_j^j < 0$). I shall assume that this does not occur. On the other hand, if country 2's two export goods (2 and 3) are gross substitutes, we shall have $\partial \hat{h}_3^2 / \partial p_2^2 > 0$ and $\partial \hat{h}_2^2 / \partial p_3^2 > 0$, i.e., a rise in the price of commodity 2 will induce a *fall* in country 2's net exports of commodity 3, and likewise a rise in the price of commodity 3 will induce a *fall* in its net exports of commodity of commodity 2. Thus, we can expect R to have positive diagonal and negative off-diagonal elements. Since

$$r_{22} - r_{23} = \hat{c}_2^1(z_2^1 - z_3^1) - \frac{\partial \hat{h}_2^2}{\partial p_2^2} - \frac{\partial \hat{h}_2^2}{\partial p_3^2},$$

and we may expect the last two terms to be negative and positive respectively whereas the first term on the right is likely to be quite small in absolute value, and the same applies to the difference $r_{33} - r_{32}$, we may expect R to have a strong row-diagonal-dominance property. We may then pursue the same reasoning as was applied in trying to extend Bickerdike's first theorem. For the top left element of $\tilde{B}^{-1}R$ to be positive we require

$$\tilde{b}^{22}r_{22} + \tilde{b}^{23}r_{32} > 0 \iff \frac{r_{22}}{-r_{32}} > \frac{\tilde{b}^{23}}{\tilde{b}^{22}} = \frac{-\tilde{b}_{23}}{\tilde{b}_{33}} = \frac{\tilde{a}_{23}}{|\tilde{a}_{33}|},$$

²If commodity 2 is the only import good, (3.9) reduces to

$$\frac{1}{p_2^1} \frac{d\bar{p}_2^1}{dT_2} = -\frac{\hat{c}_2^1 z_2^1 - \frac{\partial \hat{h}_2^2}{\partial p_2^2}}{\frac{\partial z_2^1}{\partial p_2^1} + \frac{\partial z_2^2}{\partial p_2^1}} = -\frac{p_2^1 \hat{c}_2^1 + \frac{p_2^2}{z_2^2} \frac{\partial \hat{h}_2^2}{\partial p_2^2}}{\frac{p_2^1}{z_2^1} \frac{\partial z_2^1}{\partial p_2^1} - \frac{p_2^1}{z_2^2} \frac{\partial z_2^2}{\partial p_2^1}} = \frac{\hat{\eta}_2^1 + \eta^2 - 1}{\eta^1 + \eta^2 - 1},$$

since, by the homogeneity of degree zero of the excess-demand functions and by using and differentiating the balance-of-trade constraints,

$$\frac{p_2^2}{\hat{h}_2^2} \frac{\partial \hat{h}_2^2}{\partial p_2^2} = -\frac{p_1^2}{\hat{h}_1^2} \frac{\partial \hat{h}_1^2}{\partial p_1^2} - 1,$$

where

$$\eta^2 = -\frac{p_1^2}{\hat{h}_1^2} \frac{\partial \hat{h}_1^2}{\partial p_1^2}$$

is country 2's elasticity of demand for imports (and likewise for country 1's).

but since both \tilde{A} and R are assumed to have row-dominant diagonals this follows automatically from

$$\frac{r_{22}}{-r_{23}} > 1 > \frac{\tilde{a}_{23}}{|\tilde{a}_{33}|}.$$

Similarly for the bottom right element of $\tilde{B}^{-1}R$,

$$\tilde{b}^{32}r_{23} + \tilde{b}^{33}r_{33} > 0 \iff \frac{r_{33}}{-r_{23}} > \frac{\tilde{b}^{32}}{\tilde{b}^{33}} = \frac{-\tilde{b}_{32}}{\tilde{b}_{22}} = \frac{\tilde{a}_{32}}{|\tilde{a}_{22}|},$$

which follows from

$$\frac{r_{33}}{-r_{23}} > 1 > \frac{\tilde{a}_{32}}{|\tilde{a}_{22}|}.$$

For the off-diagonal elements we have

$$\tilde{b}^{32}r_{22} + \tilde{b}^{33}r_{32} < 0 \iff \frac{r_{22}}{-r_{32}} < \frac{\tilde{b}^{33}}{\tilde{b}^{32}} = \frac{\tilde{b}_{22}}{-\tilde{b}_{23}} = \frac{|\tilde{a}_{22}|}{\tilde{a}_{32}}$$

and

$$\tilde{b}^{22}r_{23} + \tilde{b}^{23}r_{33} < 0 \iff \frac{r_{33}}{-r_{23}} < \frac{\tilde{b}^{22}}{\tilde{b}^{23}} = \frac{\tilde{b}_{33}}{-\tilde{b}_{23}} = \frac{|\tilde{a}_{33}|}{\tilde{a}_{23}}.$$

These conditions state that the ratios of the absolute values of the diagonal elements of \tilde{A} to the off-diagonal elements in the same *column* exceed the corresponding ratios of the diagonal elements of R to the absolute values of the off-diagonal elements in the same column. Since a rise in the price of commodity 3 may be expected to have a stronger effect on the world excess demand for commodity 3 than for either commodity 1 or 2, and similarly for a rise in the price of commodity 2, the matrix \tilde{A} may be expected to have a dominant column diagonal. Since the diagonal elements of R consist in the sum of two positive terms, whereas the off-diagonal elements consist in the difference of two positive terms, we may also expect R to have a dominant column diagonal. Thus,

$$\frac{r_{22}}{-r_{32}} > 1 < \frac{|\tilde{a}_{22}|}{\tilde{a}_{32}} \quad \text{and} \quad \frac{r_{33}}{-r_{23}} > 1 < \frac{|\tilde{a}_{33}|}{\tilde{a}_{23}}.$$

Which of the two ratios is greater cannot therefore be determined on *a priori* grounds. If the column-diagonal dominance of \tilde{A} exceeds that of R , a tariff on one of the import goods will hurt the other import-competing industry, i.e., lower its domestic price; this will be called the *competitive case*. If the column-diagonal dominance of R exceeds that of \tilde{A} , a tariff on one of the import goods will benefit the other import-competing industry, i.e., raise its domestic price; this will be called the *complementary case*. Which of these conditions holds will greatly affect the nature of the protectionist dynamics.

4 The dynamics of protectionism

I shall merely sketch some possible dynamic processes corresponding to the two possible cases.

4.1 The competitive case

This case is the most interesting one. One may postulate that the main motivating source for protection is that of Simon's "satisficing"; that is, protection is considered as a defense mechanism against random shocks. Such a shock may be a fall in an import price, hurting the corresponding import-competing industry and lowering the welfare of the factor used most intensively in its production. The industry responds by lobbying for tariff protection so as to bring the domestic price back to its former level. But this results in a fall in the domestic price of the other import-competing industry, which then responds by lobbying for tariff protection of its own, and so on. Eventually, the effect of the inverse matrix of (3.6) will be felt and the situation will be transformed into a complementary one and converge to some equilibrium. In this equilibrium, tariff levels will have been raised in both industries, and as a result the country's terms of trade will have improved; the new equilibrium is thus sustainable, with each desired domestic price being possible. Thus, the country will have improved its terms of trade without consciously doing so, but the improvement in the terms of trade was necessary for the new equilibrium to be viable.

4.2 The complementary case

In this case there is no reason why an initial fall in an import price should trigger a tariff war. If the tariff succeeds in raising the internal prices of both import goods, there is a presumption that it has improved the country's terms of trade, though this result has not been proved formally in the paper.

Appendix: Conditions for dynamic stability

To obtain needed properties of the matrix

$$(A.1) \quad A = \begin{bmatrix} \frac{\partial \hat{z}_2^1}{\partial p_2^2} + \frac{\partial \hat{z}_2^2}{\partial p_2^2} & \frac{\partial \hat{z}_2^1}{\partial p_3^2} + \frac{\partial \hat{z}_2^2}{\partial p_3^2} \\ \frac{\partial \hat{z}_3^1}{\partial p_2^2} + \frac{\partial \hat{z}_3^2}{\partial p_2^2} & \frac{\partial \hat{z}_3^1}{\partial p_3^2} + \frac{\partial \hat{z}_3^2}{\partial p_3^2} \end{bmatrix},$$

we appeal to the property of dynamic stability. Given any fixed tariff factors T_2, T_3 , and any fixed values of l^1, l^2 , and \bar{p}_1^2 , let us denote the world excess demand for commodity i by

$$Z_i(p_2^2, p_3^2) = \hat{z}_i^1(\bar{p}_1^2, p_2^2, p_3^2, T_2, T_3, l^1) + \hat{z}_i^2(\bar{p}_1^2, p_2^2, p_3^2, l^2) \quad (i = 2, 3).$$

Then we may postulate the dynamic-adjustment system

$$(A.2) \quad \begin{aligned} \dot{p}_2^2 &= \kappa_2 Z_2(p_2^2, p_3^2) \\ \dot{p}_3^2 &= \kappa_3 Z_3(p_2^2, p_3^2) \end{aligned}$$

where $\dot{p}_i^2 = dp_i^2/dt$ ($t = \text{time}$) and the κ_i denote positive speeds of adjustment. Denoting for brevity $\bar{p}_i^2 = \bar{p}_i^2(T_2, T_3)$ for the given fixed T_2, T_3 , let us take a first-order Taylor approximation of the $Z_i(\cdot)$ around the equilibrium values $(\bar{p}_2^2, \bar{p}_3^2)$, where $Z_{ij}(\bar{p}_2^2, \bar{p}_3^2)$ denotes $\partial Z_i/\partial p_j^2$ evaluated at $(p_2^2, p_3^2) = (\bar{p}_2^2, \bar{p}_3^2)$ and $R_i(\cdot)$ denotes the remainder term:

$$(A.3) \quad \begin{aligned} Z_2(p_2^2, p_3^2) &= Z_2(\bar{p}_2^2, \bar{p}_3^2) + Z_{22}(\bar{p}_2^2, \bar{p}_3^2)(p_2^2 - \bar{p}_2^2) + Z_{23}(\bar{p}_2^2, \bar{p}_3^2)(p_3^2 - \bar{p}_3^2) + R_2(\bar{p}_2^2, \bar{p}_3^2) \\ Z_3(p_2^2, p_3^2) &= Z_3(\bar{p}_2^2, \bar{p}_3^2) + Z_{32}(\bar{p}_2^2, \bar{p}_3^2)(p_2^2 - \bar{p}_2^2) + Z_{33}(\bar{p}_2^2, \bar{p}_3^2)(p_3^2 - \bar{p}_3^2) + R_3(\bar{p}_2^2, \bar{p}_3^2). \end{aligned}$$

The first terms on the right vanish by definition of equilibrium. Ignoring the remainder terms, defining the deviation of the j th price from its equilibrium value by

$$u_j = p_j^2 - \bar{p}_j^2,$$

and denoting

$$(A.4) \quad a_{ij} = Z_{ij}(\bar{p}_2^2, \bar{p}_3^2) = \frac{\partial \hat{z}_i^1}{\partial p_j^2} + \frac{\partial \hat{z}_i^2}{\partial p_j^2},$$

where the derivatives in the third expression are evaluated at the given T_2, T_3, l^1, l^2 , and p_1^2 , we have from (A.2) and (A.3):

$$\begin{bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} \kappa_2 & 0 \\ 0 & \kappa_3 \end{bmatrix} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}, \quad \text{or} \quad \dot{u} = KA u.$$

Assume that the eigenvalues λ_2, λ_3 of KA are distinct, and let v^2, v^3 denote the corresponding eigenvectors, which are then linearly independent. Then, denoting $V = [v^2, v^3]$ and $\Lambda = \text{diag}\{\lambda_2, \lambda_3\}$, we have

$$KAV = V\Lambda, \quad \text{hence} \quad V^{-1}KAV = \Lambda.$$

Defining $u^* = V^{-1}u$, it follows that

$$(A.5) \quad \dot{u}^* = V^{-1}\dot{u} = V^{-1}KAu = V^{-1}KAVu^* = \Lambda u^*.$$

Dynamic stability requires that $u \rightarrow 0$, hence $u^* \rightarrow 0$, as $t \rightarrow \infty$. From (A.5) we have

$$\frac{du_i^*}{dt} = \lambda_i u_i^*, \quad \text{or} \quad d \log u_i^* = \frac{du_i^*}{u_i^*} = \lambda_i dt \quad (i = 2, 3),$$

which integrates to

$$u_i^* = \nu_i \exp\{\lambda_i t\} \quad (i = 2, 3).$$

This converges to zero as $t \rightarrow \infty$ if and only if the real part of λ_i is negative.

The eigenvalues of KA are the solutions of the characteristic equation

$$f(\lambda) = |I\lambda - KA| = \begin{vmatrix} \lambda - \kappa_2 a_{22} & -\kappa_2 a_{23} \\ -\kappa_3 a_{32} & \lambda - \kappa_3 a_{33} \end{vmatrix} = \lambda^2 - \text{tr}(KA)\lambda + \det(KA) = 0,$$

namely

$$\lambda = \frac{\text{tr}(KA) \pm \sqrt{[\text{tr}(KA)]^2 - 4\det(KA)}}{2}.$$

If the discriminant is negative (which can happen only if $\det(KA) > 0$) then $\Re(\lambda) = \text{tr}(KA)/2$, hence we must have $\text{tr}(KA) < 0$. If the discriminant is positive then we must have

$$\text{tr}(KA) + \sqrt{[\text{tr}(KA)]^2 - 4\det(KA)} < 0, \quad \text{or} \quad \text{tr}(KA) < -\sqrt{[\text{tr}(KA)]^2 - 4\det(KA)},$$

so that again we have $\text{tr}(KA) < 0$. Squaring both sides of the latter inequality we see that we again have $\det(KA) > 0$. But $\det(KA) = \det(K)\det(A)$ and $\det(K) > 0$ hence $\det(A) > 0$. Now if $\text{tr}(KA) < 0$ for *all* (positive) speeds of adjustment κ_2, κ_3 , then we must have $a_{22} \leq 0$ and $a_{33} \leq 0$. These conditions are essentially the ‘‘Hicks conditions’’ for dynamic stability, i.e., that the principal minors of A should be alternately nonpositive and nonnegative.³

Another sharper stability condition was also obtained by Metzler (1945), namely that if the off-diagonal elements of A are nonnegative, then the system is stable if and only if its principal minors are alternately negative and positive. Such matrices are known as Metzler matrices; stability implies that the diagonal elements of A are nonpositive. Finally, McKenzie (1960, p. 50) showed that if a matrix B has positive diagonal elements and nonpositive off-diagonal elements, then B^{-1} has all its elements nonnegative if and only if, for some diagonal matrix K with positive diagonal elements, the matrix KB has a dominant diagonal, i.e., the absolute value of each diagonal element is greater than the sum of the absolute values of the remaining elements in the same column.

Putting together these results, we may say that if A is a Metzler matrix, and the system is stable, and if KA has a dominant diagonal for some K , then $-A^{-1}$ has positive principal minors and has all its elements nonnegative. We shall assume these conditions to hold.

³As stated by Hicks (1939, pp. 315–16, 325), the principal minors should be alternately negative and positive. See also Metzler (1945) and Arrow (1974); Arrow pointed out that the weaker inequalities were all that one could prove for the the minors of the matrix A .

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