Notes on the Duality between Production and Cost Functions: A Differentiable Approach

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PRIMAL PROBLEM

Minimize $\sum_{j=1}^{m} w_j v_j$ subject to

$$f(v) \geq y$$

where $y, w = (w_1, w_2, \ldots, w_m)$ are parameters, $y > 0, w > 0$, to obtain the functions

$$v_j = \tilde{v}_j(w, y) \quad (j = 1, 2, \ldots, m)$$

for which

$$\begin{aligned} & \sum_{j=1}^{m} w_j \tilde{v}_j(w, y) \\ = \ & \min_v \{w \cdot v : f(v) \geq y\} \\ \equiv \ & G(w, y). \end{aligned}$$

Form the Lagrangean function

$$L(v, p^*; w, y) = \sum_{j=1}^{m} w_j v_j - p^*[f(v) - y]$$

where p^* is a Lagrange multiplier. Differentiate L partially with respect to the v_i and p^* and set equal to zero:

$$\text{(I)} \qquad w_i = p^* \frac{\partial f}{\partial v_i}; \quad f(v) = y.$$

The $m + 1$ equations (I) define a mapping

$$\mathcal{F}(v, p^*) = (w, y)$$

from the positive orthant of $(m+1)$-dimensional Euclidean space into itself. The differentiability and strict quasi-concavity properties of f imply that $\mathcal{F}$ has an inverse

DUAL PROBLEM

Minimize $\sum_{j=1}^{m} v_j w_j$ subject to

$$g(w) \geq p$$

where $p, v = (v_1, v_2, \ldots, v_m)$ are parameters, $p > 0, v > 0$, to obtain the functions

$$w_j = \tilde{w}_j(v, p) \quad (j = 1, 2, \ldots, m)$$

for which

$$\begin{aligned} & \sum_{j=1}^{m} v_j \tilde{w}_j(v, p) \\ = \ & \min_w \{v \cdot w : g(w) \geq p\} \\ \equiv \ & F(v, p). \end{aligned}$$

Form the Lagrangean function

$$L^*(w, y^*; v, p) = \sum_{j=1}^{m} v_j w_j - y^*[g(w) - p]$$

where y^* is a Lagrange multiplier. Differentiate L^* partially with respect to the w_i and y^* and set equal to zero:

$$\text{(I}^*\text{)} \qquad v_i = y^* \frac{\partial g}{\partial w_i}; \quad g(w) = p.$$

The $m + 1$ equations (I*) define a mapping

$$\mathcal{G}(w, y^*) = (v, p)$$

from the positive orthant of $(m+1)$-dimensional Euclidean space into itself. The differentiability and strict quasi-concavity properties of g imply that $\mathcal{G}$ has an inverse

$$\mathcal{F}^{-1}(w,y) = (v,p^*)$$

which is defined by the functions

$$v_j = \tilde{v}_j(w,y) \quad (j = 1,2,\ldots,m)$$

obtained by solving the m equations

(I$'$)
$$\frac{w_j}{w_k} = \frac{\partial f/\partial v_j}{\partial f/\partial v_k}\ (j \neq k); \quad f(v) = y,$$

and by the function

$$p^* = \frac{w_k}{f_k(\tilde{v}_1(w,y),\ldots,\tilde{v}_m(w,y))} \equiv \tilde{p}^*(w,y)$$

where $f_k = \partial f/\partial v_k$.

Define the indirect production function by

$$\tilde{f}(w,y) = f(\tilde{v}_1(w,y),\ldots,\tilde{v}_m(w,y)).$$

This function satisfies the identity

$$\text{(I}''\text{)} \quad \tilde{f}(w,y) = y \quad \text{for all } w,y.$$

This will be used to establish the following basic relations:

$$\text{(II)} \qquad \frac{\partial G(w,y)}{\partial w_j} = \tilde{v}_j(w,y),$$

$$\text{(III)} \qquad \frac{\partial G(w,y)}{\partial y} = \tilde{p}^*(w,y).$$

To obtain (II), differentiate G with respect to w_j:

$$\mathcal{G}^{-1}(v,p) = (w,y^*)$$

which is defined by the functions

$$w_j = \tilde{w}_j(v,p) \quad (j = 1,2,\ldots,m)$$

obtained by solving the m equations

(I$^{*\prime}$)
$$\frac{v_j}{v_k} = \frac{\partial g/\partial w_j}{\partial g/\partial w_k}\ (j \neq k); \quad g(w) = p,$$

and by the function

$$y^* = \frac{v_k}{g_k(\tilde{w}_1(v,p),\ldots,\tilde{w}_m(v,p))} \equiv \tilde{y}^*(v,p)$$

where $g_k = \partial g/\partial w_k$.

Define the indirect minimum-unit-cost function by

$$\tilde{g}(v,p) = g(\tilde{w}_1(v,p),\ldots,\tilde{w}_m(v,p)).$$

This function satisfies the identity

$$\text{(I}^{*\prime\prime}\text{)} \quad \tilde{g}(v,p) = p \quad \text{for all } v,p.$$

This will be used to establish the following basic relations:

$$\text{(II}^*\text{)} \qquad \frac{\partial F(v,p)}{\partial v_j} = \tilde{w}_j(v,p),$$

$$\text{(III}^*\text{)} \qquad \frac{\partial F(v,p)}{\partial p} = \tilde{y}^*(v,p).$$

To obtain (II*), differentiate F with respect to v_j:

$$\frac{\partial G(w,y)}{\partial w_j} = \frac{\partial}{\partial w_j}\sum_{k=1}^{m} w_k \tilde{v}_k(w,y) = \tilde{v}_j(w,y) + \sum_{k=1}^{m} w_k \frac{\partial v_k(w,y)}{\partial w_j}.$$

This is equal to (II) if and only if the second term on the right vanishes, and this follows from (I″):

$$\begin{aligned} 0 &= \frac{\partial \tilde{f}(w,y)}{\partial w_j} \\ &= \sum_{k=1}^{m} \frac{\partial f}{\partial v_k}\bigg|_{v=\tilde{v}(w,y)} \cdot \frac{\partial \tilde{v}_k(w,y)}{\partial w_j} \\ &= \frac{1}{\tilde{p}^*(w,y)} \sum_{k=1}^{m} w_k \frac{\partial \tilde{v}_k(w,y)}{\partial w_j}, \end{aligned}$$

where use has been made of (I) and the definition of $\tilde{p}^*$.

To obtain (III), differentiate G with respect to y:

$$\frac{\partial G(w,y)}{\partial y} = \sum_{k=1}^{m} w_k \frac{\partial \tilde{v}_k(w,y)}{\partial y}.$$

Now we have from (I″)

$$\begin{aligned} 1 &= \frac{\partial \tilde{f}(w,y)}{\partial y} \\ &= \sum_{k=1}^{m} \frac{\partial f}{\partial v_k}\bigg|_{v=\tilde{v}(w,y)} \cdot \frac{\partial \tilde{v}_k(w,y)}{\partial y} \\ &= \frac{1}{\tilde{p}^*(w,y)} \sum_{k=1}^{m} w_k \frac{\partial \tilde{v}_k(w,y)}{\partial y}, \end{aligned}$$

so (III) follows.

Let f be homogeneous of degree 1; then by Euler's theorem,

$$\begin{aligned} &\frac{\partial F(v,p)}{\partial v_j} \\ &= \frac{\partial}{\partial v_j} \sum_{k=1}^{m} v_k \tilde{w}_k(v,p) \\ &= \tilde{w}_j(v,p) + \sum_{k=1}^{m} v_k \frac{\partial w_k(v,p)}{\partial v_j}. \end{aligned}$$

This is equal to (II*) if and only if the second term on the right vanishes, and this follows from (I$^{*\prime\prime}$):

$$\begin{aligned} 0 &= \frac{\partial \tilde{g}(v,p)}{\partial v_j} \\ &= \sum_{k=1}^{m} \frac{\partial g}{\partial w_k}\bigg|_{w=\tilde{w}(v,p)} \cdot \frac{\partial \tilde{w}_k(v,p)}{\partial v_j} \\ &= \frac{1}{\tilde{y}^*(v,p)} \sum_{k=1}^{m} v_k \frac{\partial \tilde{w}_k(v,p)}{\partial v_j}, \end{aligned}$$

where use has been made of (I*) and the definition of $\tilde{y}^*$.

To obtain (III*), differentiate F with respect to p:

$$\frac{\partial F(v,p)}{\partial p} = \sum_{k=1}^{m} v_k \frac{\partial \tilde{w}_k(v,p)}{\partial p}.$$

Now we have from (I$^{*\prime\prime}$)

$$\begin{aligned} 1 &= \frac{\partial \tilde{g}(v,p)}{\partial p} \\ &= \sum_{k=1}^{m} \frac{\partial g}{\partial w_k}\bigg|_{w=\tilde{w}(v,p)} \cdot \frac{\partial \tilde{w}_k(v,p)}{\partial p} \\ &= \frac{1}{\tilde{y}^*(v,p)} \sum_{k=1}^{m} v_k \frac{\partial \tilde{w}_k(v,p)}{\partial p}, \end{aligned}$$

so (III*) follows.

Let g be homogeneous of degree 1; then by Euler's theorem,

$$
\begin{aligned}
&G(w, y) \\
&= \sum_{k=1}^{m} w_k \tilde{v}_k(w, y) \\
&= p^* \sum_{k=1}^{m} \frac{\partial f}{\partial v_k}\bigg|_{v=\tilde{v}_k(w,y)} \cdot \tilde{v}_k(w, y) \\
&= \tilde{p}^*(w, y) y
\end{aligned}
$$

whence

$$\tilde{p}^*(w, y) = \frac{G(w, y)}{y}.$$

We verify that $\partial \tilde{p}^* / \partial y = 0$ from (III), hence we may write

$$\text{(IV)} \qquad p^* = \tilde{p}^*(w, y) = g^*(w)$$

for some function g^*. Denote

$$b_j(w) = \frac{\partial g^*(w)}{\partial w_j}.$$

Then from (II) we have

$$
\begin{aligned}
\tilde{v}_j(w, y) &= \frac{\partial G(w, y)}{\partial w_j} \\
&= y \frac{\partial g^*(w)}{\partial w_j} \\
&= y b_j(w)
\end{aligned}
$$

so that the factor-product ratios are determined by

$$\text{(V)} \qquad \frac{v_j}{y} = b_j(w) = \frac{\partial g^*(w)}{\partial w_j}.$$

$$
\begin{aligned}
&F(w, y) \\
&= \sum_{k=1}^{m} v_k \tilde{w}_k(v, p) \\
&= y^* \sum_{k=1}^{m} \frac{\partial g}{\partial w_k}\bigg|_{w=\tilde{w}_k(v,p)} \cdot \tilde{w}_k(v, p) \\
&= \tilde{y}^*(v, p) p
\end{aligned}
$$

whence

$$\tilde{y}^*(v, p) = \frac{F(v, p)}{p}.$$

We verify that $\partial \tilde{y}^* / \partial p = 0$ from (III*), hence we may write

$$\text{(IV}^*\text{)} \qquad y^* = \tilde{y}^*(v, p) = f^*(v)$$

for some function f^*. Denote

$$r_j(v) = \frac{\partial f^*(v)}{\partial v_j}.$$

Then from (II*) we have

$$
\begin{aligned}
\tilde{w}_j(v, p) &= \frac{\partial F(v, p)}{\partial v_j} \\
&= p \frac{\partial f^*(v)}{\partial v_j} \\
&= p r_j(v)
\end{aligned}
$$

so that the real factor rentals are determined by

$$\text{(V}^*\text{)} \qquad \frac{w_j}{p} = r_j(v) = \frac{\partial f^*(v)}{\partial v_j}.$$

From the above development we may conclude that

$$\mathcal{F}^{-1} = \mathcal{G}, \quad \mathcal{G}^{-1} = \mathcal{F}, \quad f^* = f, \quad \text{and } g^* = g.$$

For, from (IV*) and (V*) we have

$$\text{(VI)} \qquad w_i = p\frac{\partial f^*(v)}{\partial v_i}, \quad F^*(v) = y^*,$$

which is precisely the same as (I). It therefore determines a mapping

$$\mathcal{F}^*(v, p) = (w, y^*);$$

but (v, p) was obtained by (I) from the mapping

$$\mathcal{G}(w, y^*) = (p, v).$$

Therefore,

$$(v, p) = \mathcal{G}(w, y^*) = \mathcal{G}(\mathcal{F}^*(v, p)$$

so $\mathcal{G} \circ \mathcal{F}^*$ is the identity mapping, i.e., $\mathcal{F}^* = \mathcal{G}^{-1}$. But since (VI) and (I) are defined by the same conditions, the mappings $\mathcal{F}$ and $\mathcal{F}^*$ coincide.

Similarly, (IV) and (V) coincide with (I*), hence define the same mapping. Thus the mappings defined by (I) and (I*) are inverse to one another.

Starting from a concave homogeneous-of-degree-one production function f, one obtains a unique concave homogeneous-of-degree-one minimum-unit-cost function g; and starting from the function g one recovers the unique production function f which gave rise to it.