International Trade*

John S. Chipman
University of Minnesota

Edgeworth (1894) opened his survey of the theory of international values with the provocative statement: "International trade meaning in plain English trade between nations, it is not surprising that the term should mean something else in Political Economy." This could equally well be said today. What distinguishes international from domestic trade is the greater prevalence of barriers (both natural and artificial) to trade and factor movements in the former; different currencies; and (perhaps most important) autonomous governments, leading to a pattern of shocks which impact different countries in different ways. Because of these differences, a different type of theoretical model is called for. For example, international immobility of factors results in greater disparity in relative factor endowments among countries than among regions of the same country; these disparities may make it reasonable, as a first approximation, to ignore variations in supplies of factor services that come about in response to changes in factor rentals and commodity prices, if these variations are small in comparison to the differences in endowments. Likewise, great differences among resource endowments and productive techniques may make it reasonable to disregard differences in consumers' tastes within and across countries, even though this might be a very inappropriate type of simplification for purposes of analyzing domestic trade.

The fact that national governments act independently leads to the need to analyse the effects of country-specific shocks, which take the form of intensification or liberalization of restrictions on trade or capital movements, unilateral transfers such as reparation payments, gifts, or loans, and disparities in monetary and fiscal policies. For this reason the emphasis in international-trade theory has from the

*Work supported by National Science Foundation grant SES-8607652.

beginning (Mill, 1848; Marshall, 1879) been on comparative statics: one wants to ascertain the qualitative, if not the quantitative, effect of a tariff or quota or transfer on the various quantities involved. To obtain unambiguous qualitative results one needs fairly drastic simplifications and strong assumptions. On the other hand, the emphasis in general-equilibrium theory (Walras, 1874; Pareto, 1896-97; Debreu, 1959) has been on proving the existence, stability, and Pareto-optimality of competitive equilibrium, for which much milder assumptions are required. A good definition of international-trade theory as it has evolved would therefore be: "general-equilibrium theory with structure."

The requirements of "simplicity" in a theory are not absolute, but vary with the goals of the theory and the technical resources available to researchers at the time. There is not much virtue in simplicity if a result that holds in a model of two countries, two commodities, and two factors does not generalize in any meaningful way to higher dimensions. With the increasing possibilities of handling large-scale models and data sets and estimating their parameters numerically, it is natural to expect a movement of both general-equilibrium traditions towards each other.

Attention will be focussed here on the neoclassical model developed by Haberler (1930, 1933), Lerner (1932, 1933, 1934), Ohlin (1928, 1933), Stolper and Samuelson (1941), Samuelson (1953), and Rybczynski (1955), which Baldwin (1982) has described as the "Haberler-Lerner-Samuelson model"—an appellation which is more accurate than the usual "Heckscher-Ohlin theory," since the model commonly employed makes the simplifying assumption—rejected by Ohlin (1933, Ch. VII) except in his illustrative Appendix I—that factors of production are inelastic in supply and indifferent among alternative occupations, allowing one to define unambiguously a country's production-possibility frontier. This model has in recent years come to lose some of its hold on the profession—just as the Ricardian theory had in the 1930s—in favor of models that stress imperfect competition (see, e.g., Helpman and Krugman, 1985). However, these latter models have so far not been successfully formulated as general-equilibrium models, and are thus still in a formative stage. It goes without saying that, in the nature of the case, a partial-equilibrium model is incapable of explaining or predicting trade patterns or analyzing the effect on prices and resource allocation of trade restrictions and transfers.

The material that follows is divided into two parts. Part 1 covers the mathematical foundations of the received theory, and deals with the duality between production functions and cost functions, the concept of a national-product function, the Stolper-Samuelson and Rybczynski relations between factor rentals and commodity prices and between commodity outputs and factor endowments, the concepts of trade-demand functions and trade-utility functions, world equilibrium and its dynamic stability. Part 2 covers the applications of these basic concepts to the most noteworthy problems that have been the object of attention in the theory of international trade since its beginnings: the explanation of trade flows, the effect of unilateral transfers on sectoral prices and resource allocation, and the effect of trade restrictions such as tariffs and quotas. The reader who is interested in substantive questions is advised to proceed directly to Part 2.

Part 1. The Mathematical Foundations

1 Duality of cost functions and production functions

Let an industry produce a positive amount of y of output of a particular product, with the aid of non-negative amounts v_j of m primary factors of production, determining the vector $v = (v_1, v_2, \ldots, v_m)$. A *production function* f is defined over the non-negative orthant E_m^+ of $m-$ dimensional Euclidean space, with values $y = f(v)$ on the non-negative real line E_1^+. We assume that f has the following properties:

(a) *upper semi-continuity*: for each y the set

$$A(y) = \{v : f(v) \geq y\} \tag{1.1}$$

is closed;

(b) *quasi-concavity*: for each y, the set $A(y)$ defined by (1.1) is convex;

(c) *monotonicity*: if $v, v' \in E_m^+$ are such that $v' \geq v$, then $f(v') \geq f(v)$.

Further properties of f will be specified later on.

We shall denote by $w = (w_1, w_2, \ldots, w_m)$ a vector of *factor rentals*, i.e., prices of the services of the m factors of production. The following conventional notation will be adhered to:

$$\begin{array}{lll} w \geq 0 & \text{means} & w_i \geq 0 \quad \text{for all } i = 1, 2, \ldots, m; \\ w \geq 0 & \text{means} & \begin{cases} w_i \geq 0 & \text{for all } i = 1, 2, \ldots m, \quad \text{and} \\ w_i > 0 & \text{for some } i; \end{cases} \\ w > 0 & \text{means} & w_i > 0 \quad \text{for all } i = 1, 2, \ldots, m. \end{array}$$

For each $y > 0$ and all $w \geq 0$ we define the *minimum-total-cost function* G by

$$G(w, y) = \min_v \{w \cdot v : f(v) \geq y\}, \tag{1.2}$$

where $w \cdot v$ denotes the inner product $\sum_{j=1}^m w_j v_j$. Mathematically, for each fixed y the function $G(\cdot, y)$ is the *support function* of the convex set $A(y)$ (cf. Fenchel, 1953). It has the following properties:

(a*) *continuity* in w: for each $y, G(w, y)$ is continuous;

(b*) *concavity* in w: if $0 < \theta < 1$ then

$$(1 - \theta)G(w^0, y) + \theta G(w^1, y) \leq G[(1 - \theta)w^0 + \theta w^1, y];$$

(c*) *monotonicity*: $y' \geq y$ implies $G(w, y') \geq G(w, y)$ and $w' \geq w \geq 0$ implies $G(w', y) \geq G(w, y)$;

(d*) *positive homogeneity* in w: $G(\lambda w, y) = \lambda G(w, y)$ for all $\lambda > 0$.

Property (a*) follows from (a) and the definition of G; property (c*) follows the definition of G and the fact that $y' \geq y$ implies $A(y') \subseteq A(y)$; property (d*) follows immediately from the definition of G. To prove (b*), let $w^0, w^1 \geq 0$ and denote $w^\theta = (1-\theta)w^0 + \theta w^1$; from the definitions of G and $A(y)$ in (1.2) and (1.1), we have

$$\begin{array}{ll} G(w^0, y) \leq w^0 \cdot v & \text{for all } v \in A(y); \\ G(w^1, y) \leq w^1 \cdot v & \text{for all } v \in A(y); \end{array}$$

consequently,

$$(1 - \theta)G(w^0, y) + \theta G(w^1, y) \leq w^0 \cdot v \quad \text{for all } v \in A(y).$$

Hence, in particular,

$$\begin{aligned}&(1-\theta)G(w^0,y)+\theta G(w^1,y)\\&\qquad\le \min_v\{w^\theta\cdot v : v\in A(y)\}\equiv G(w^\theta,y),\end{aligned}$$

which is the result sought (cf. Uzawa, 1964b).

Of fundamental importance in international trade theory is the following *duality theorem* first proved by Shephard (1953). The formulation and proof contained in Theorem 1 to follow are due to Uzawa (1964b).

Theorem 1 (Duality Theorem). Define the set

$$B(y)=\{v:(\forall w\ge 0)w\cdot v\ge G(w,y)\}, \tag{1.3}$$

where G is defined by (1.2) and f satisfies properties (a), (b), (c). Then $B(y)=A(y)$, where $A(y)$ is defined by (1.1).

Proof. Let $v^0\in A(y)$; then $f(v^0)\ge y$, hence from (1.2),

$$G(w,y)\equiv\min_v\{w\cdot v: f(v)\ge y\}\le w\cdot v^0$$

for all $w\ge 0$; that is, $v^0\in B(y)$.

Conversely, suppose $v^0\notin A(y)$. Since $A(y)$ is closed and convex by properties (a) and (b) of f, it follows from the separating-hyperplane theorem of closed convex sets (cf. Fenchel 1953, p. 48) that there exists a vector $w^0\ne 0$ such that

$$w^0\cdot v^0<\min_v\{w^0\cdot v: v\in A(y)\} \tag{1.4}$$

(see Figure 1). Now if w^0 had a negative component, it follows from property (c) that the corresponding component of $v\in A(y)$ may be chosen to be artitrarily large, hence no minimum of $w^0\cdot v$ over $A(y)$ exists; consequently, $w\ge 0$. But then the expression on the right of the inequality sign in (1.4) is just $G(w^0,y)$. From the definition of $B(y)$ in (1.3), it follows that $v^0\notin B(y)$. $\square$

The duality theorem may be stated in words as follows: given the function G, the set $A(y)$ may be identified with the set of all factor combinations v which, at each constellation $w\ge 0$ of factor rentals, are at least as expensive as the minimal total cost of producing output y at factor rentals w.

1.2. Let us now explore the consequences of imposing a further condition on the production function f:

(d) *positive homogeneity*: for all $\lambda > 0, f(\lambda v) = \lambda f(v)$.

From the definition of G in (1.2), we now have

$$\begin{aligned} G(w,y) &= \min_v \left\{ w \cdot v : f\left(\frac{v}{y}\right) \geq 1 \right\} \\ &= \min_b \{ w \cdot by : f(b) \geq 1 \} \quad \left(b = \frac{v}{y} \right) \\ &= y \cdot \min_b \{ w \cdot b : f(b) \geq 1 \}. \end{aligned}$$

Thus, $G(w,y)$ factors into two terms, of which the second depends only on $w \geq 0$ and may be denoted

$$g(w) = \min_v \{ w \cdot v : f(v) \geq 1 \}. \tag{1.5}$$

We therefore have

Theorem 2. If f satisfies properties (a), (b), (c), (d), then the function of G of (1.3) factors into

$$G(w,y) = yg(w) \tag{1.6}$$

where g is defined by (1.5) and is continuous, concave, monotone, and positively homogeneous of first degree.

The properties of g specified in Theorem 2 follow directly from those of the function G.

We may now state a special form of the duality theorem for the case of homogeneous production functions.

Theorem 3. Let g be defined by (1.5) where f satisfies properties (a), (b), (c), (d), and let the function f^* be defined by

$$f^*(v) = \min_w \{ w \cdot v : g(w) \geq 1 \}. \tag{1.7}$$

Then $f^* = f$.

Proof. Define the set

$$C(y) = \{ v : (\forall w \in A^*(1)) w \cdot v \geq y \} \tag{1.8}$$

where for convenience we define

$$A^*(p) = \{ w : g(w) \geq p \}. \tag{1.9}$$

(Since g is defined only for $w \geq 0, w \in A^*(p)$ implies $w \geq 0$.) First we shall show that $C(y) = B(y)$, where $B(y)$ is defined by (1.3).

From (1.3) and (1.6), if $v^0 \in B(y)$ then for all $w \in A^*(1)$, $w \cdot v^0 \geq G(w, y) = yg(w) \geq y$, so $B(y) \subseteq C(y)$. Conversely suppose $v^0 \in C(y)$ and take any $w^0 \geq 0$. Then from the homogeneity of g we have $g(w^0/g(w^0)) = 1$, hence from the definition (1.8) of $C(y)$ if follows that

$$\frac{w^0}{g(w^0)} \cdot v^0 \geq y,$$

i.e., $w^0 \cdot v^0 \geq yg(w^0)$; thus $v^0 \in B(y)$. Therefore $B(y) = C(y)$, and by Theorem 1, $C(y) = A(y)$.

Now denote $r = w/g(w)$ and consider the set

$$C^*(y) = \{v : \min_r \{r \cdot v : r \in A^*(1)\} \geq y\}. \tag{1.8$'$}$$

If $r \cdot v \geq y$ for all $r \in A^*(1)$, then a fortiori $r \cdot v \geq y$ for the $r \in A^*(1)$ which minimizes $r \cdot v$; hence $C(y) \subseteq C^*(y)$. Conversely, for all $r \in A^*(1)$ we have $r \cdot v \geq \min_r \{r \cdot v : r \in A^*(1)\}$, so $C^*(y) \subseteq C(y)$. Thus $C^*(y) = C(y) = A(y)$. But from (1.7), (1.9) and (l.8′) we have

$$C^*(y) = \{v : f^*(v) \geq y\}. \tag{1.8$''$}$$

Since $A(y) = C^*(y)$ for all y, therefore f and f^* coincide. □

1.3. Let us consider the consequences of adding to the properties (a), (b), (c) of f given in §1.1 the following further properties: (b_1) strict quasi-concavity: for each y, the set $A(y)$ defined by (1.1) is strictly convex: (e) differentiability: f has continuous first-order partial derivatives.

For the time being, property (d) of §1.2 will not be used, but will be introduced again later on.

The problem of deriving the minimum total cost function $G(w, y)$ may be posed in terms of the following non-linear programming problem:

$$\text{minimize } \sum_{j=1}^{m} w_j v_j \quad \text{subject to} \quad f(v) \geq y, v \geq 0. \tag{1.10}$$

Form the Lagrangean function

$$L(p^*, v; y, w) = \sum_{j=1}^{m} w_j v_j - p^*[f(v) - y] \tag{1.11}$$

where y, w are parameters and $p*$ is a Lagrangean multiplier. In accordance with the Kuhn-Tucker theorem (cf. Kuhn and Tucker

1951, p. 486) in order for $v^0 = (v_1^0, v_2^0, \dots, v_m^0)$ to be a solution of the minimum problem (1.10), it is necessary and sufficient that v^0 and some $p^* \geq 0$ satisfy

$$(1.12a) \qquad \left.\frac{\partial L}{\partial v_j}\right|_{v=v^0} = w_j - p^* \left.\frac{\partial f}{\partial v_j}\right|_{v=v^0} \geq 0; \quad v_j \frac{\partial L}{\partial v_j} = 0$$

and

$$(1.12b) \qquad \sum_{j=1}^{m} v_j^0 \left.\frac{\partial L}{\partial v_j}\right|_{v_j=v_j^0} = \sum_{j=1}^{m} v_j^0 \left(w_j - p^* \left.\frac{\partial f}{\partial v_j}\right|_{v=v^0} \right) = 0$$

as well as

$$(1.12c) \qquad \frac{\partial L}{\partial p^*} = -[f(v^0) - y] \leq 0; p^* \frac{\partial L}{\partial p^*} = 0.$$

In the above we have used (e), but so far property (b_1) has not yet been used: Let us introduce the further properties:

(f) *indispensability*: $f(0) = 0$.

(f_1) *strict indispensability*: if v has a component $v_j = 0$, then $f(v) = 0$.

Now suppose the solution v^0 to (1.10) is such that $f(v^0) > y$ (see Figure 2). This violates (b_1), since strict quasi-concavity requires that if $v^0, v^1 \in A(y)$ and $0 < \theta < 1$, the point $v^\theta = (1-\theta)v^0 + \theta v^1$ should be in the interior of $A(y)$. Suppose, however, that property (b_1) is not assumed, and that $f(v^0) > y > 0$; then $p^* = 0$ from (1.12c) hence $w \cdot v^0 = 0$ from (1.12b), and since $w \leq 0$ this implies that v^0 has a zero component. Thus, if (f_1) is assumed, we have $0 = f(v^0) > y > 0$—a contradiction. Thus either (b_1) or (f_1) is sufficient—in conjunction with (a), (c), (e), to guarantee $f(v^0) = y$. If $w > 0$, a similar argument shows that (f) implies $f(v^0) = y$.

Now suppose that v^0 is such that strict inequality holds in (1.12a) for some j. Then $v_j^0 = 0$ from (1.12b). If (f_1) holds this would lead to a contradiction, since then $0 = f(v^0) \geq y > 0$. If (f_1) is not assumed, but if (b_1) holds, then strict inequality in (1.12a) implies that v^0 has a zero component, so v^0 is on the boundary of $A(y)$; but $2v^0$ is also on the boundary of $A(y)$, by property (c), and consequently the mid-point $1\frac{1}{2}v^0$ is as well, contradicting (b_1). Thus, if (a), (b), (c),

(e) hold, then either (b_1) or (f_1) implies that equality holds in (1.12a) for all $j = 1, 2, \ldots, m$.

Consider a solution v^0 to (1.10) corresponding to a w^0 which has some zero components. Let $J = \{j : w^0_j = 0\}$. Then if $w^0 \cdot v^0 = C^0$, certainly $A(y) \subseteq \{v : w^0 \cdot v^0 \geq C^0\}$. Let v^1 be such that $v^1_j > v^0_j$ for $j \in J$ and $v^1_j = v^0_j$ for $j \notin J$. Then $w^0 \cdot v^1 = w^0 \cdot v^0$, hence $v^1 \in \{v : w^0 \cdot v = C^0\}$. But by condition (c), $v^1 \in A(y)$; thus v^1 and v^0 are both on the boundary of $A(y)$, as is $(1-\theta)v^0 + \theta v^1$ for $0 < \theta < 1$ (see Figure 3). This contradicts (b_1). Therefore under (b_1), a solution to (1.10) exists only if $w > 0$.

It should be noted that even if the function $G(w, y)$ of (1.2) is well-defined in the sense that

$$G(w, y) = \inf_v \{w \cdot v : f(v) \geq y\}, \tag{1.2$'$}$$

a solution of (1.10) need not exist. For example, if

$$f(v_1, v_2) = \frac{1}{1/v_1 + 1/v_2}$$

then

$$G(0, w_2; y) = y w_2$$

but the *infimum* is achieved as $(v_1, v_2) \to (\infty, y)$. On the other hand a solution to (1.10) always exists if $w > 0$; for, choosing any $v^0 \in \mathrm{int}A(y)$ and $w^0 > 0$, the set

$$A(y) \cap \{v : w^0 \cdot v \leq w^0 v^0, v \geq 0\}$$

is compact by virtue of condition (a), and from (b) and (c) the minimum of $w^0 \cdot v$ over this set is the minimum over $A(y)$.

An immediate consequence of (b_1) is that if (1.10) has a solution, it is *unique*. Since (1.10) need not have a solution unless $w > 0$, it is of some advantage to replace (b_1) by a weaker condition which still ensures uniqueness provided $w > 0$. Such a condition is:

(b_2) if $v^0 \neq v^1$ and neither $v^0 \geq v^1$ nor $v^1 \geq v^0$, and if $0 < \theta < 1$, then $f((1-\theta)v^0 + \theta v^1) > \min[f(v^0), f(v^1)]$.

The above discussion may now be summarized in the following theorem.

Theorem 4. Let conditions (a), (b), (c), (e), (f) hold. Then if either (b_1) or (f_1) holds, any solution v^0 to (1.10) has the property

$$w_j = p^* \left.\frac{\partial f}{\partial v_j}\right|_{v=v^0} \quad (j = 1, 2, \ldots, m); \quad f(v^0) = y. \tag{1.12d}$$

If (b_1) holds, this solution is unique. If (b_2) holds and if $w > 0$, then a unique solution to (1.10) exists, and it satisfies (1.12d).

1.4. We now proceed with an analysis of the solution v of the programming problem (1.10) regarded as a function of the parameters $y > 0, w > 0$, when conditions (a), (b_2), (c), (e), (f) are assumed to hold.

In accordance with Theorem 4, the solution satisfies (1.12d) and is unique, given y and w. Thus we have the functions

$$v_j = \tilde{v}_j(w, y) \quad (j = 1, 2, \ldots, m). \tag{1.13a}$$

It is shown in Fenchel (1953, pp. 102-4) that these functions are differentiable. Substituting (1.13a) into (1.12d) we obtain

$$p^* = w_j \Big/ \frac{\partial}{\partial v_j} f\Big(\tilde{v}_1(w, y), \tilde{v}_2(w, y), \ldots, \tilde{v}_m(w, y)\Big) \equiv \tilde{p}^*(w, y). \tag{1.13b}$$

The system of equations (1.12d) defines a mapping F from the non-negative orthant of $(m + 1)$-dimensional space into itself:

$$F(v, p^*) = (w, y). \tag{1.14a}$$

Equations (1.13a) and (1.13b) define the inverse mapping:

$$F^{-1}(w, y) = (v, p^*). \tag{1.14b}$$

In accordance with (1.2) we define

$$G(w, y) = \sum_{k=1}^{m} w_k \tilde{v}_k(w, y). \tag{1.15}$$

We shall also define the *indirect production function* $\tilde{f}$ by

$$\tilde{f}(w, y) = f(\tilde{v}_1(w, y), \tilde{v}_2(w, y), \ldots, \tilde{v}_m(w, y)) \tag{1.16a}$$

which satisfies the identity

$$\tilde{f}(w, y) = y \quad \text{for all } w, y. \tag{1.16b}$$

Theorem 5. (Fundamental Envelope Theorem of Production Theory). The functions $G, \tilde{v}_j, \tilde{p}^*$ of (1.15), (1.13a), (1.13b) are related by

$$(1.17a) \qquad \frac{\partial G(w,y)}{\partial w_j} = \tilde{v}_j(w,y) \quad (j = 1, 2, \ldots, m)$$

and

$$(1.17b) \qquad \frac{\partial G(w,y)}{\partial y} = \tilde{p}^*(w,y).$$

Proof. Differentiating (15) with respect to w_j, we obtain

$$(1.12) \qquad \frac{\partial G(w,y)}{\partial w_j} = \tilde{v}_j(w,y) + \sum_{k=1}^{m} w_k \frac{\partial \tilde{v}_k(w,y)}{\partial w_j}.$$

To prove (1.17a) we must show that the second term on the right of (1.18) vanishes. Differentiating (1.16a) with respect to w_j and making use of the identity (1.16b) and the chain rule, we obtain upon substitution of (1.13b),

$$\begin{aligned} 0 = \frac{\partial \tilde{f}(w,y)}{\partial w_j} &= \sum_{k=1}^{m} \frac{\partial f}{\partial v_k}\bigg|_{v_k = \tilde{v}_k(w,y)} \cdot \frac{\partial \tilde{v}_k(w,y)}{\partial w_j} \\ &= \frac{1}{\tilde{p}^*(w,y)} \sum_{k=1}^{m} w_k \frac{\partial \tilde{v}_k(w,y)}{\partial w_j} \end{aligned}$$

and (1.17a) follows. Likewise, differentiating (1.16a) with respect to y and using the identity (1.16b) and the chain rule, we have, upon making use once again of (1.13b),

$$\begin{aligned} 1 = \frac{\partial \tilde{f}(w,y)}{\partial y} &= \sum_{k=1}^{m} \frac{\partial f}{\partial v_k}\bigg|_{v_k = \tilde{v}_k(w,y)} \cdot \frac{\partial \tilde{v}_k(w,y)}{\partial y} \\ &= \frac{1}{\tilde{p}^*(w,y)} \sum_{k=1}^{m} w_k \frac{\partial \tilde{v}_k(w,y)}{\partial y}. \end{aligned}$$

Thus, from this result and (1.15),

$$\frac{\partial G(w,y)}{\partial y} = \sum_{k=1}^{m} w_k \frac{\partial \tilde{v}_k(w,y)}{\partial y} = \tilde{p}^*(w,y),$$

establishing (1.17b). □

An alternative proof of (1) proceeds as follows: Given the output level y and a vector of factor rentals w^0, let the vector v^0 of factor inputs minimize $w^0 \cdot v$ subject to $v \in A(y)$. Define the function

$$\Psi(w, y) = G(w, y) - w \cdot v^0.$$

Then $\Psi(w^0, y) = 0$ and $\Psi(w, y) \leq 0$ for $w \neq w^0$ (by definition of G), hence $\Psi(w, y)$ is a maximizer with respect to w at $w = w^0$. Therefore, assuming $G(w, y)$ to be differentiable,

$$\frac{\partial \Psi(w^0, y)}{\partial w_i} = \frac{\partial G(w^0, y)}{\partial w_i} - v^0 = 0. \quad \square$$

It may be noted immediately from (1.15) and (1.17a) that

$$G(w, y) = \sum_{k=1}^{m} w_k \frac{\partial G(w, y)}{\partial w_k},$$

providing the necessary and sufficient condition, by Euler's theorem, that G be homogeneous of degree 1 in w—a result already obtained in §1.1. Using (1.17a) again it follows that $\tilde{v}_j$ is homogeneous of degree zero in w.

Now let us introduce condition (d): the positive homogeneity (of degree 1) of the production function f. Using (1.15) and (1.13b) we have, by Euler's theorem,

$$\begin{aligned} G(w, y) &= \sum_{k=1}^{m} w_k \tilde{v}_k(w, y) \\ &= \tilde{p}^*(w, y) \sum_{k=1}^{m} \left.\frac{\partial f}{\partial v_k}\right|_{v_k = \tilde{v}_k(w,y)} \cdot \tilde{v}_k(w, y) \\ &= y \tilde{p}^*(w, y) \end{aligned}$$

whence from (1.6)

$$\tilde{p}^*(w, y) = \frac{G(w, y)}{y} = g(w). \tag{1.19}$$

Defining

$$b_j(w) = \frac{\partial g(w)}{\partial w_j} \quad (j = 1, 2, \ldots, m) \tag{1.20}$$

we have from (1.17a), (1.19), and (1.20),

$$\tilde{v}_j(w, y) = \frac{\partial G(w, y)}{\partial w_j} = y \frac{\partial g(w)}{\partial w_j} = y b_j(w) \tag{1.21}$$

hence the optimal factor-product ratios are given by

$$\frac{v_j}{y} = b_j(w). \tag{1.22}$$

From the differentiability assumption (e) imposed on the function f we can derive a strict quasi-concavity property of the function g. For suppose $w^0 > 0, w^1 > 0$, and $w^0 \neq \lambda w^1$; then from (b_2) and (e), we have $b(w^0) \neq b(w^1)$, where

$$b(w) = [b_1(w), b_2(w), \ldots, b_m(w)]. \tag{1.23}$$

Now by definition of g [see (1.5)]

$$\begin{array}{ll} g(w^0) \leq w^0 \cdot v & \text{for all } v \in A(1) \\ g(w^1) \leq w^1 \cdot v & \text{for all } v \in A(1) \end{array} \tag{1.24a}$$

and moreover

$$\begin{array}{ll} g(w^0) = w^0 \cdot v & \text{if and only if } v = b(w^0) \\ g(w^1) = w^1 \cdot v & \text{if and only if } v = b(w^1). \end{array} \tag{1.24b}$$

Furthermore, $b(w^0) \neq b(w^1)$, so strict inequality must hold in one of the inequalities (1.24a); thus if $0 < \theta < 1$,

$$(1-\theta)g(w^0) + \theta g(w^1) < [(1-\theta)w^0 + \theta w^1] \cdot v \quad \text{for all } v \in A(1)$$

and therefore in particular

$$\begin{aligned} (1-\theta)g(w^0) + \theta g(w^1) &< \min_{v \in A(1)} \{[(1-\theta)w^0 + \theta w^1] \cdot v\} \\ &= g((1-\theta)w^0 + \theta w^1). \end{aligned}$$

So we have

(b_3^*) if $w^0 > 0, w^1 > 0$, and $w^0 \neq \lambda w^1$, and if $0 < \theta < 1$, then $g((1-\theta)w^0 + \theta w^1) > (1-\theta)g(w^0) + \theta g(w^1)$.

It is not hard to see that a corresponding property (b_3) holds for f as well. Failure of (b_3^*) when f is not differentiable, allowing $b(w^0) = b(w^1)$ for $w^0 \neq \lambda w^1$, is illustrated in Figure 4.

In general, a flat segment on a production isoquant goes over into a kink on the dual cost isoquant, and vice versa. There is another still more subtle relationship, illustrated by the following function found in Katzner (1970, p. 54):

$$f(v_1, v_2) = (v_1^3 v_2 + v_1 v_2^3)^{1/4}.$$

Its dual minimum-unit-cost function is found to be

$$g(w_1, w_2) = 2^{-1/4}[(w_1 + w_2)^{4/3} - (w_1 - w_2)^{4/3}]^{3/4}.$$

The isoquants of f are extremely flat at $v_1 = v_2$, and as a result g is once but not twice differentiable at $w_1 = w_2$. A graph of

$$g(w_1, w_2) = w_1 b_1(w_1, w_2) + w_2 b_2(w_1, w_2)$$

for $w_2 = \bar{w}_2$ is shown in Figure 5. At $w_1 = \bar{w}_2, \bar{w}_2 b_2(w_1, \bar{w}_2)$ has a slope of $+\infty$ and $w_1 b_1(w_1, \bar{w}_2)$ has a slope of $-\infty$, yet their sum is differentiable. When the bordered Hessian of the production f function is invertible, its inverse is the bordered Hessian of the cost function g; in the above example, it is not invertible at $v_1 = v_2$.

A useful illustration of the duality of cost and production functions is given by the case of C.E.S. (constant-elasticity-of-substitution) production functions (cf. Arrow, Chenery, Minhas and Solow, 1961; Uzawa, 1962):

$$f(v) = \left[\sum_{i=1}^{m} \alpha_i v_i^{1-1/\sigma}\right]^{\sigma/(\sigma-1)}$$

The corresponding cost functions have the form

$$g(w) = \left[\sum_{i=1}^{m} \alpha_i^{\sigma} w_i^{1-\sigma}\right]^{1/(1-\sigma)}$$

whose elasticity of substitution is $\sigma^* = 1/\sigma$.

2 The production-possibility set

Suppose a country to be capable of producing n commodities with the aid of m primary factors of production. Denoting the output of commodity j by y_j , and the input of factor i into the production of commodity j by v_{ij} , the production function may be written

(2.1) $\qquad y_j = f_j(v_{1j}, v_{2j}, \ldots, v_{mj}) = f_j(v_{\cdot j}) \quad (j = 1, 2, \ldots, n),$

where

(2.2) $$v_{\cdot j} = (v_{1j}, v_{2j}, \ldots, v_{mj}).$$

It will be assumed that f_j is:

(a) *continuous*; i.e., $\lim_{v_{\cdot j} \to v^0_{\cdot j}} f_j(v_{\cdot j}) = f_j(v^0_{\cdot j})$;

(b) *weakly monotone*; i.e., if $v^1_{\cdot j} \geq v^2_{\cdot j}$ (meaning that $v^1_{ij} \geq v^2_{ij}$ for $i = 1, 2, \ldots, m$) then $f_j(v^1_{\cdot j}) \geq f_j(v^2_{\cdot j})$, and if $v^1_{\cdot j} > v^2_{\cdot j}$ (i.e., $v^1_{ij} > v^2_{ij}$ for $i = 1, 2, \ldots, m$) then $f_j(v^1_{\cdot j}) > f_j(v^2_{\cdot j})$;

(c) *concave*; i.e., if $v^0_{\cdot j}$ and $v^1_{\cdot j}$ are any two vectors of primary inputs into the production of commodity j, then for any t in the interval $0 < t < 1$,

(2.3) $$f_j((1-t)v^0_{\cdot j} + tv^1_{\cdot j}) \geq (1-t)f_j(v^0_{\cdot j}) + tf_j(v^1_{\cdot j});$$

(d) *positively homogeneous of degree 1*; i.e., for any $\lambda > 0$,

(2.4) $$f_j(\lambda v_{\cdot j}) = \lambda f_j(v_{\cdot j}).$$

It will be convenient to introduce the $m \times n$ allocation matrix

(2.5) $$V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ v_{21} & v_{22} & \cdots & v_{2n} \end{bmatrix}.$$

The element v_{ij} is the input of factor i into the production of commodity j. The jth column of V will be denoted v_j; according to this notation, v_j is the transpose of $v_{\cdot j}$, denoted $v_j = v'_{\cdot j}$.

Let l_i denote the country's total endowment of factor i. Then for each i the following resource constraint holds:

(2.6) $$\sum_{j=1}^{n} v_{ij} \leq l_i \quad (i = 1, 2, \ldots, m).$$

Using (2.5) this can be written in matrix notation as

(2.7) $$\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{21} & v_{22} & \cdots & v_{2n} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \leq \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix},$$

or simply

$$V\iota \leq l, \tag{2.8}$$

where ι is the column vector of n ones and $l = (l_1, l_2, \ldots, l_m)'$ is the column vector of factor endowments.

In the absence of any additional restrictions, condition (2.6) expresses the *perfect mobility* of factors among industries.

The country's *production-possibility set* is the set of all possible output combinations $y = (y_1, y_2, \ldots, y_n)$ that can be produced with the production functions (2.1) under the resource constraints (2.6). Formally, it may be denoted

$$\begin{aligned} \mathcal{Y}(l) = \Big\{ y : \text{there exist allocations } & v_{ij} \geq 0 \text{ such that} \\ y_j &= f_j(v_{\cdot j}) \quad (j = 1, 2, \ldots, n), \\ \sum_{j=1}^{n} v_{ij} &\leq l_i \qquad (i = 1, 2, \ldots, m) \Big\}. \end{aligned} \tag{2.9}$$

For notational convenience we may define the function $f(V)$ as the vector-valued function

$$f(V) = (f_1(v_1'), f_2(v_2'), \ldots, f_n(v_n'))' \tag{2.10}$$

and write (2.9) in the more compact form

$$\mathcal{Y}(l) = \{y : (\exists V \geq 0)\ y = f(V) \ \&\ V\iota \leq l\}. \tag{2.11}$$

Note that with this notation, condition (2.3) can be written (for $t = t_j$) in the form

$$f(V^0(I - T) + V^1 T) \geq (I - T) f(V^0) + T f(V^1) \tag{2.12}$$

where $T = \operatorname{diag}(t_1, t_2, \ldots, t_n)$ is an $n \times n$ diagonal matrix with $0 < t_j < 1$. Likewise, (2.4) may be written (for $\lambda = \lambda_j$) in the form

$$f(V\Lambda) = \Lambda f(V), \tag{2.13}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an $n \times n$ diagonal matrix with $\lambda_j > 0$.

Theorem 6. If assumptions (a), (b), and (c) hold, the production-possibility set $\mathcal{Y}(l)$ is convex.

Proof. Let y^0, y^1 both belong to $\mathcal{Y}(l)$; we are to show that for any t in the interval $0 < t < 1$, the output combination $y^t = (1-t)y^0 + ty^1$ also belongs to $\mathcal{Y}(l)$ (see Figure 6).

Since y^0, $y^1 \in \mathcal{Y}(l)$, this means that there exist two allocation matrices V^0, V^1 each satisfying (2.8), such that $y^0 = f(V^0)$ and $y^1 = f(V^1)$. Denote $V^t = (1-t)V^0 + tV^0$. Then from (2.8),

$$V^t\iota = (1-t)V^0\iota + tV^1\iota \leq (1-t)l + tl = l, \tag{2.14}$$

so V^t is a feasible allocation, and by concavity,

$$f(V^t) \geq (1-t)f(V^0) + tf(V^1) = y^t, \tag{2.15}$$

i.e., for each $j = 1, 2, \ldots, n$, denoting $v^t_{\cdot j} = (1-t)v^0_{\cdot j} + tv^1_{\cdot j}$,

$$f_j(v^t_{\cdot j}) \geq (1-t)f_j(v^0_{\cdot j}) + tf_j(v^1_{\cdot j}) = y^t_j. \tag{2.15$'$}$$

By continuity and monotonicity of f_j, there exist $\lambda^t_j \leq 1$ such that

$$f_j(\lambda^t_j v^t_{\cdot j}) = y^t_j (j = 1, 2, \ldots, n). \tag{2.16$'$}$$

(In particular, (2.16′) follows if the stronger homogeneity condition (d) holds, by taking $\lambda^t_j = y^t_j / f_j(v^t_{\cdot j})$ if $y^t_t > 0$, and 0 otherwise.) Equivalently,

$$f(V^t\Lambda^t) = y^t. \tag{2.16}$$

It remains only to verify that the matrix $V^t\Lambda$ of allocations $\lambda^t_j v^t_{ij}$ satisfies the constraint (2.8). This is immediate from the fact that $0 \leq \lambda^t_j \leq 1$, whence from (2.14),

$$V^t\Lambda^t\iota = V^t\lambda^t \leq V^t\iota \leq l. \qquad \square \tag{2.17}$$

Note that homogeneity of production functions is not needed for the above result.

3 The domestic-product function

Let $p = (p_1, p_2, \ldots, p_n)'$ denote a vector of prices. The *domestic-product function* (cf. Samuelson, 1953; Chipman, 1972, 1974) is defined as the function

$$\Pi(p, l) = \max_{y \in \mathcal{Y}(l)} p \cdot y. \tag{3.1}$$

[See also Dixit and Norman (1980), who use the terminology "revenue function." In Chipman (1972, 1974) this was called the "production function for foreign exchange" and the "national-product function" respectively.]

For any fixed p, this has all the properties of a production function, but with some special peculiar features. These are illustrated in Figure 7 to be explained shortly.

For each commodity, $j = 1, 2, \ldots, n$, define the upper-contour set

$$A_j(y_j) = \{l^j = (l_1^j, l_2^j, \ldots, l_m^j) : f_j(l^j) \geq y_j\}. \tag{3.2}$$

Then in particular,

$$A_j(Y/p_j) = \{l^j : p_j f_j(l^j) \geq Y\} \tag{3.3}$$

is the set of factor-input combinations that will yield, at the given price p_j, an amount of commodity j worth at least Y. Throughout this section it will be assumed that each f_j satisfies properties (a)–(d) of the preceding section.

Let us now introduce a stronger monotonicity condition that refers to the entire vector-valued function $f(V)$ of (2.10). It may be stated as follows: f is

(e) *strictly monotone*, i.e., for each $V = [v_{ij}]$ and each $i = 1, 2, \ldots, m$, there is a $j = 1, 2, \ldots, n$ such that $\delta > 0$ implies

$$f_j(v_{1j}, v_{2j}, \ldots, v_{ij} + \delta, \ldots, v_{mj}) > f_j(v_{1j}, v_{2j}, \ldots, v_{ij}, \ldots, v_{mj}). \tag{3.4}$$

In words, if there is an increase in the amount of any one of the m endowments, it is possible to find an industry where this additional input will lead to increased output.

For any family of sets $S_1, S_2, \ldots, S_n$, each a subset of m-dimensional Euclidean space E^m, the *arithmetic mean* of this family (which is, for convex S_j, also the *convex hull* of $\cup_{j=1}^n S_j$) is defined by and denoted

$$\mathop{\mathrm{M}}_{j=1}^{n} S_j = \{s \in E^m : (\exists s^j \in S_j, \lambda_j \geq 0, j = 1, 2, \ldots, n) \quad \sum_{j=1}^{n} \lambda_j = 1 \text{ and } s = \sum_{j=1}^{n} \lambda_j s^j\}. \tag{3.5}$$

Analogously to (3.2) we define the upper-contour set of the national-product function by

$$A(p, Y) = \{l \in E_+^m : \Pi(p, l) \geq Y\}. \tag{3.6}$$

The following theorem characterizes the isoquants of the function $\Pi(p, \cdot)$ (see Figure 7).

Theorem 7. Let all prices p_j be positive, $j = 1, 2, \ldots, n$, and let f satisfy conditions (a)–(d) of Section 2, as well as the strict montonicity condition (e). Then

$$A(p, Y) = \mathop{\mathrm{M}}_{j=1}^{n} A_j(Y/p_j), \tag{3.7}$$

i.e., the upper-contour set consisting of all factor combinations l that give rise to a national product of at least Y, is the arithmetic mean of the n upper-contour sets consisting, for each commodity j, of all factor combinations l^j that, when allocated entirely to industry j, give rise to a national product of at least Y.

Proof. (i) Let us first prove that

$$\mathop{\mathrm{M}}_{j=1}^{n} A_j(Y/p_j) \subseteq A(p, Y). \tag{3.8}$$

Let

$$l \in \mathop{\mathrm{M}}_{j=1}^{n} A_j(Y/p_j).$$

Then, by definition (3.5), there exist $l^j \in A(Y/p_j)$ and $\lambda_j \geq 0$ such that

$$\sum_{j=1}^{n} \lambda_j = 1 \quad \text{and} \quad \sum_{j=1}^{n} \lambda_j l^j = l.$$

By definition (3.3), each l^j satisfies $p_j f_j(l^j) \geq Y$, hence from the definition (3.1) of Π and the homogeneity of degree 1 of each f_j, we have

$$\Pi(p, l) \geq \sum_{j=1}^{n} p_j f_j(\lambda_j l^j) = \sum_{j=1}^{n} \lambda_j p_j f_j(l^j) \geq Y \sum_{j=1}^{n} \lambda_j = Y.$$

From definition (3.6) it follows that $l \in A(p, Y)$, and (3.8) follows.

(ii) We now show that

$$A(p, Y) \subseteq \mathop{\mathrm{M}}_{j=1}^{n} A_j(Y/p_j). \tag{3.9}$$

Let $l \in A(p, Y)$; then by definitions (3.6), (3.1) and (2.9), there exist allocations $v_{\cdot j} \in E^m_+$ such that

$$\sum_{j=1}^{n} v_{\cdot j} \leq l \quad \text{and} \quad \sum_{j=1}^{n} p_j f_j(v_{\cdot j}) = \Pi(p, l) \geq Y. \tag{3.10}$$

By the strict monotonicity of f, the first inequality of (3.10) must be an equality; for, if for some $i = i'$ we have $\sum_{j=1}^{n} v_{i'j} < l_{i'}$, then for some $j = j'$ and $0 < \delta \leq l_{i'} - \sum_{j=1}^{n} v_{i'j}$ the inequality (3.4) is satisfied, violating the definition (3.1) of $\Pi(p, l)$. Now define

$$\lambda_j = \frac{p_j f_j(v_{\cdot j})}{\Pi(p, l)}, \quad l^j = \frac{v_{\cdot j}}{\lambda_j} \quad (j = 1, 2, \ldots, n). \tag{3.11}$$

Then

$$\sum_{j=1}^{n} \lambda_j l^j = l \quad \text{where} \quad \sum_{j=1}^{n} \lambda_j = 1. \tag{3.12}$$

By homogeneity we have

$$p_j f_j(l^j) = p_j f_j(v_{\cdot j}/\lambda_j) = p_j f_j(v_{\cdot j})/\lambda_j = \Pi(p, l) \geq Y,$$

hence $l^j \in A_j(Y/p_j)$ from (3.3). Together with (3.10) this implies that (3.9) holds. □

Since for each fixed p the domestic-product function $\Pi(p, \cdot)$ has the properties of a production function (i.e., it is continuous, concave, monotone, and positively homogeneous of degree 1), we may associate with it a corresponding minimum-unit cost function $\Gamma(p, \cdot)$ defined by

$$\Gamma(p, w) = \min_{l}\{w \cdot l : \Pi(p, l) \geq 1\}. \tag{3.13}$$

This will be called the *domestic-cost function.* Letting $g_j(w) = \min_{v_{\cdot j}}\{w \cdot v_{\cdot j} : f_j(v_{\cdot j}) \geq 1\}$ denote the minimum-unit cost function dual to the production function $f_j(v_{\cdot j})$, we may define the upper-contour sets

$$A_j^*(p_j) = \{w : g_j(w) \geq p_j\} \tag{3.14}$$

and

$$A^*(p) = \{w : \Gamma(p, w) \geq 1\}. \tag{3.15}$$

The boundary of the intersection of all the sets (3.14) for $j = 1, 2, \ldots, n$ is known as the "factor-rental frontier" (or "factor-price frontier"—cf. Woodland, 1982, pp. 49–52). The following theorem shows that it is also the contour of the corresponding domestic-cost function. Its shape will be similar to that depicted in Figure 4.

Theorem 8. Let the prices p_j be positive for $j = 1, 2, \ldots, n$, and let f satisfy conditions (a) to (e) of §1.2. Then

$$A^*(p) = \bigcap_{j=1}^{n} A_j^*(p_j). \tag{3.16}$$

Proof. Let $w \in A^*(p)$; then $\Gamma(p, w) \geq 1$, i.e., $w \cdot l \geq 1$ for all $l \in A(p, l)$. Choose such an l and let V be the optimal resource-allocation matrix; then

$$\Pi(p, l) = \sum_{j=1}^{n} p_j f_j(v_{\cdot j}) \geq 1. \tag{3.17}$$

Defining λ_j and l^j as in (3.11), this gives (by homogeneity)

$$\sum_{j=1}^{n} p_j f_j(\lambda_j l^j) = \sum_{j=1}^{n} \lambda_j p_j f_j(l^j) \geq 1, \tag{3.18}$$

and since $\lambda_j > 0$ and $\sum_{j=1}^{n} \lambda_j = 1$ this implies $p_j f_j(l^j) \geq 1$, i.e., $l^j \in A_j(1/p_j)$, for each j. Now by hypothesis, (3.17) implies $w \cdot l \geq 1$ hence

$$\sum_{j=1}^{n} \lambda_j w \cdot l^j \geq 1, \tag{3.19}$$

and by the same reasoning as above this implies $w \cdot l^j \geq 1$ for all j, i.e.,

$$g_j(w)/p_j = \min_{l^j}\{w \cdot l^j : l^j \in A_j(1/p_j)\} \geq 1 \tag{3.20}$$

or $g_j(w) \geq p_j$. From the definition (3.14) this shows that $w \in A_j^*(p_j)$ for $j = 1, 2, \ldots, n$.

Conversely, let $w \in \bigcap_{j=1}^{n} A_j^*(p_j)$; then $g_j(w) \geq p_j$ for $j = 1, 2, \ldots, n$. From the definition of g_j, this implies $w \cdot l^j \geq 1$ for all $l^j \in A_j(l/p_j)$, $j = 1, 2, \ldots, n$. Choosing $l^j \in A_j(1/p_j)$ such that

$$\sum_{j=1}^{n} \lambda_j l^j = l,$$

we have

$$\Pi(p, l) \geq \sum_{j=1}^{n} p_j f_j(\lambda_j l^j) = \sum_{j=1}^{n} \lambda_j p_j f_j(l^j) \geq \sum_{j=1}^{n} \lambda_j = 1, \tag{3.21}$$

hence

$$w \cdot l = w \cdot \sum_{j=1}^{n} \lambda_j l^j = \sum_{j=1}^{n} \lambda_j w \cdot l^j \geq 1. \tag{3.22}$$

From the definition (3.13) this implies $\Gamma(p, w) \geq 1$, and thus by (3.15) it follows that $w \in A^*(p)$. □

Let us introduce a further assumption, that each f_j is (f) *differentiable.*

Then from Theorem 7 it follows that $\Pi(p, \cdot)$ is differentiable. Its partial derivative with respect to l_i is defined as the *Stolper-Samuelson function*

$$(3.23) \qquad \hat{w}_i(p, l) \equiv \frac{\partial}{\partial l_i} \Pi(p, l) \quad (i = 1, 2, \ldots, m),$$

and the corresponding vector-valued function $\hat{w}(p, l) = \partial \Pi(p, l)/\partial l$ is called the *Stolper-Samuelson mapping.* The values of this function are the shadow or implicit factor rentals of the respective factors.

Setting up the Lagrangean function

$$(3.24) \qquad L(V, w; p, l) = \sum_{j=1}^{n} p_j f_j(v_{\cdot j}) - \sum_{i=1}^{m} w_i \left(\sum_{j=1}^{n} v_{ij} - l_i \right)$$

corresponding to the definition of the domestic-product function, we obtain the Kuhn-Tucker conditions

$$(3.25a) \qquad \frac{\partial L}{\partial v_{ij}} = p_j \frac{\partial f_j}{\partial v_{ij}} - w_i \leq 0, \quad \left(p_j \frac{\partial f_j}{\partial v_{ij}} - w_i \right) v_{ij} = 0;$$

$$(3.25b) \qquad \frac{\partial L}{\partial w_i} = l_i - \sum_{j=1}^{n} v_{ij} \ \geq 0, \qquad \left(l_i - \sum_{j=1}^{n} v_{ij} \right) w_i = 0.$$

It will be observed that conditions (3.25a) constitute, for each $j = 1, 2, \ldots, n$, precisely the Kuhn-Tucker conditions for cost-minimization in industry j, where w_i is the ith factor rental. The rentals defined by the Stolper-Samuelson mapping are therefore the market rentals that will obtain in competitive equilibrium.

Let us now explore the consequences of assuming that the function Π is differentiable with respect to p as well as to l. Given p^0 and l, let y^0 maximize $p^0 \cdot y$ over $calY(l)$. Define the function

$$H(p, l) = \Pi(p, l) - p \cdot y^0.$$

Then $H(p^0, l) = 0$ and $H(p, l) \geq 0$ for $p \neq p^0$ (by the definition of Π), hence H reaches a minimum with respect to p at $p = p^0$. Since differentiability of Π implies differentiability of H, we have

$$\frac{\partial H(p^0, l)}{\partial p_j} = \frac{\partial \Pi(p^0, l)}{\partial p_j} - y_j^0 = 0.$$

This shows that y^0 is the *unique* y which maximizes $p^0 \cdot y$ subject to $y \in \mathcal{Y}(l)$. This is equivalent to saying that the *production-possibility frontier* $\hat{\mathcal{Y}}(l)$—i.e., the set of all $y \in \mathcal{Y}(l)$ which maximize $p \cdot y$ for some $p > 0$—is *strictly concave to the origin*. The apparently innocuous assumption that Π is differentiable with respect to p has thus led to an important substantive conclusion.

When Π is differentiable with respect to p, the function

$$(3.26) \qquad \hat{y}_j(p,l) = \frac{\partial}{\partial p_j}\Pi(p,l) \quad (j = 1,2,\ldots,n)$$

is called the *Rybczynski function* for commodity j. The corresponding vector-valued function $\hat{y}(p,l)$ is called the *Rybczynski mapping*.

In general, we may define the *Rybczynski correspondence* by

$$(3.27) \qquad \hat{y}(p,l) = \{y \in \mathcal{Y}(l) : p \cdot y = \Pi(p,l)\}.$$

The above result shows that if Π is differentiable with respect to p, this correspondence is a singleton-valued mapping. We shall now obtain a necessary and sufficient condition for this single-valuedness, i.e., for the strict concavity to the origin of $\mathcal{Y}(l)$.

Let the factor-output coefficients be denoted

$$(3.28) \qquad b_{ij}(w) = \frac{\partial g_j(w)}{\partial w_i} \quad (i = 1,2,\ldots,m; j = 1,2,\ldots,n)$$

where g_j is the minimum-unit-cost function dual to the production function f_j. The following result was obtained by Khang (1971) and Chipman (1972).

Theorem 9. Let p^0, l^0 be such that there exists a $y^0 > 0$ which maximizes $p^0 \cdot y$ subject to $y \in \mathcal{Y}(l^0)$, and let $w^0 = \hat{w}(p^0,l^0) = \partial\Pi(p^0,l^0)/\partial l$. Let f satisfy the strict monotonicity condition (e). Then in order that y^0 should be the unique maximizer of $p^0 \cdot y$ subject to $y \in \mathcal{Y}(l^0)$, it is necessary and sufficient that the n columns of the factor-output matrix

$$B(w^0) = \begin{bmatrix} b_{11}(w^0) & b_{12}(w^0) & \cdots & b_{1n}(w^0) \\ b_{21}(w^0) & b_{22}(w^0) & \cdots & b_{2n}(w^0) \\ \cdot & \cdot & \cdot & \cdot \\ b_{m1}(w^0) & b_{m2}(w^0) & \cdots & b_{mn}(w^0) \end{bmatrix}$$

be linearly independent.

Proof. For convenience, denote $B^0 = B(w^0)$. Then from strict monotonicity of f we have

$$B^0 y^0 = l^0. \tag{3.29}$$

First we show that if rank $B^0 < n$ then y^0 is not unique. Since rank $B^0 < n$ there exists a vector $z^0 \neq 0$ such that

$$B^0 z^0 = 0. \tag{3.30}$$

Choose $\varepsilon^0 > 0$ such that

$$y^0 \pm \varepsilon^0 z^0 > 0;$$

then $y^0 \pm \varepsilon z^0 > 0$ for $0 < \varepsilon < \varepsilon^0$. From (3.29) and (3.30) we have $B^0(y^0 \pm \varepsilon^0 z^0) = l^0$ whence $y^0 \pm \varepsilon^0 z^0 \in \mathcal{Y}(l^0)$. Since y^0 maximizes $p^0 \cdot y$ over $\mathcal{Y}(l^0)$,

$$p^0 \cdot y^0 \geq p^0 \cdot (y^0 \pm \varepsilon^0 z^0),$$

i.e., $0 \geq \varepsilon^0 p^0 \cdot z^0 \geq 0$. This implies $p^0 \cdot z^0 = 0$, hence $p^0 \cdot (y^0 \pm \varepsilon z^0) = p^0 \cdot y^0$ for $0 < \varepsilon < \varepsilon^0$, i.e.,

$$y^0 \pm \varepsilon z^0 \in \mathcal{Y}(l^0) \quad \text{for } 0 < \varepsilon < \varepsilon^0.$$

This shows that y^0 is not unique.

Conversely we show that if y^0 is not unique then rank $B^0 < n$. Suppose $y^0, y^1 > 0$ both maximize $p^0 \cdot y$ subject to $y \in \mathcal{Y}(l^0)$, where $y^1 \neq y^0$. Then $B^0 y^0 = B^0 y^1 = l$, hence $B^0(y^0 - y^1) = 0$; since $y^0 - y^1 \neq 0$, this implies that rank $B^0 < n$. □

From this result it follows that a necessary condition for the production-possiblity frontier to be strictly concave to the origin is that $m \geq n$. If $m < n$, it is a ruled surface. However, the condition $m \geq n$ is certainly not sufficient; one example is the case $m = n = 2$ when two isoquants for a dollar's worth of output are mutually tangential at a point along the endowment ray (cf. Lerner, 1933, p. 13). For further discussion of these points see Kemp, Khang, and Uekawa (1978), and for an interesting characterization see Inoue (1986) and Inoue and Wegge (1986).

To gain an intuitive understanding of the meaning of the differentiability of $\Pi(\cdot, l)$, let us assume that the f_j are differentiable and that the functions $\hat{v}_{ij}(p, l)$, obtained with the $\hat{w}_i(p, l)$ by solving the above constrained-maximum problem, are also single-valued and differentiable. Then from the identity

$$\Pi(p,l) = \sum_{j=1}^{n} p_j f_j(\hat{v}_{\cdot j}(p,l)) \tag{3.31}$$

we have

$$\frac{\partial \Pi}{\partial p_k} = y_k + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[p_j \frac{\partial f_j}{\partial v_{ij}} - w_i\right]\frac{\partial \hat{v}_{ij}}{\partial p_k} + \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \frac{\partial \hat{v}_{ij}}{\partial p_k}. \tag{3.32}$$

If $w_i > 0$ then

$$\sum_{j=1}^{n} \hat{v}_{ij}(p,l) = l_i$$

and thus

$$\sum_{j=1}^{n} \partial \hat{v}_{ij}/\partial p_k = 0,$$

hence the last term of (3.32) must vanish. If $v_{ij} > 0$ then the bracketed term in (3.32) vanishes (by the Kuhn-Tucker conditions). If the bracketed term is negative then $\hat{v}_{ij} = 0$ by the Kuhn-Tucker conditions, and thus $\partial \hat{v}_{ij}/\partial p_k = 0$. In either case, the second term on the right in (3.32) vanishes. The trouble occurs in the intermediate case in which factor i is on the verge of being employed in industry j, hence $p_j \partial f_j/\partial v_{ij} - w_i = 0$ and $v_{ij} = 0$; it is precisely in this case that $\hat{v}_{ij}(\cdot, l)$ will not be differentiable at that point. Formula (3.26) therefore fails at switching points where factors are on the verge of being employed in particular industries; a small price change in one direction will lead to their continued unemployment, but in the other direction to their being employed. Thus, $\Pi(\cdot, l)$ is nondifferentiable at such switching points. Likewise, it is nondifferentiable when the conditions of Theorem 9 fail, in which case a small price change may lead to a country's switching from specialization in one commodity to specialization in another. All this would become clearer if the theory were to be recast in terms of subdifferentials (cf. Rockafellar, 1970).

Since $\Pi(p, \cdot)$ has the properties of a production function, from Theorem 7, it is concave; and since, as was seen above, $H(p, l^0) = \Pi(p, l^0) - p \cdot y^0$ is a minimum at $p = p^0$, where $\Pi(p^0, l^0) = p^0 \cdot y^0$, $H(\cdot, l)$ is convex, hence $\Pi(\cdot, l)$ is convex. That is, $\Pi(p, l)$ is convex in p and concave in l. If it is twice continuously differentiable then Samuelson's (1953) "reciprocity theorem" holds:

$$\frac{\partial \hat{y}_j}{\partial l_i} = \frac{\partial^2 \Pi}{\partial p_j \partial l_i} = \frac{\partial^2 \Pi}{\partial l_i \partial p_j} = \frac{\partial \hat{w}_i}{\partial p_j}. \tag{3.33}$$

4 The Stolper-Samuelson and Rybczynski mappings

When a country diversifies its production, by which we shall mean that it produces all n consumable commodities, as long as it is not on the verge of specializing, its factor-endowment vector will lie in the interior of a diversification cone—the convex cone whose extreme rays pass through the factor-input vectors in the n industries which minimize costs at the given factor rentals (cf. McKenzie, 1955; Chipman, 1966). As is clear from Figure 7, the factor rentals will remain unchanged as the factor endowment vector varies within the interior of this cone; i.e., the function $\hat{w}(p, l)$ is independent of l for endowments l in this cone. Now if all n commodities are to be producted, costs cannot exceed prices; and competitive equilibrium requires that prices not exceed costs. Hence, from the homogeneity of degree 1 of the minimum-unit-cost functions, and by Theorem 5, we have

$$p_j = g_j(w) = \sum_{i=1}^{m} \frac{\partial g_j(w)}{\partial w_i} w_i = \sum_{i=1}^{m} b_{ij}(w) w_i, \tag{4.1}$$

or in matrix notation, where w and p denote column vectors of m factor rentals and n commodity prices respectively,

$$p = g(w) = B(w)'w \tag{4.2}$$

and thus

$$g(\hat{w}(p, l)) = p, \tag{4.3}$$

i.e., $\hat{w}(\cdot, l)$ is a local inverse of the mapping g. Since the Jacobian matrix of g, $B(w)'$, must have rank m if the diversification cone has a nonempty interior (hence $n \geq m$), the range of g is an m-dimensional manifold, hence (4.2) implies that the vector p of world prices cannot be varied arbitrarily (if the country is to continue to diversify) unless $n = m$.

Even when $n = m$, the mapping g is in general not globally univalent. Gale and Nikaido (1965) obtained strong sufficient conditions for global univalence, namely that the principal minors of $B(w)$ be positive (this condition can be slightly weakened). Inada (1971) obtained some alternative conditions. In a controversy with Pearce (1967), McKenzie (1967) showed that it did not suffice to assume that $B(w)$ had a non-vanishing determinant for all positive w. The condition that $|B(w)| \neq 0$ for some $w = w^0$ is of course sufficient for local invertibility of g, but this inverse mapping depends on l. If two countries with identical technologies have their endowment vectors l in the same diversification cone, their factor rentals will be equalized even if g is not globally univalent. Nikaido (1972) showed that a modification of conditions originally suggested by Samuelson (1953) is sufficient for global univalence of g.

Of particular interest is the nature of the Stolper-Samuelson mapping in regions where it is locally independent of l, i.e., the nature of the local inverses of g. For the reasons given above, discussion of this is effectively limited to the case $n = m$. Defining the diagonal matrices $W = \text{diag}\, w$ and $P = \text{diag}\, p$, and the matrix $\mathrm{B} = W B P^{-1}$, by dividing (4.1) through by p_j one sees that B is column-stochastic (i.e., has unit column sums in addition to having nonnegative elements); denoting its elements by $\beta_{ij} = w_i b_{ij}/p_j = \partial \log g_j / \partial \log w_i$, these satisfy $\sum_{i=1}^m \beta_{ij} = 1$. Denoting the elements of B^{-1} by b^{ij} and those of $\mathrm{B}^{-1} = P \mathrm{B}^{-1} W^{-1}$ by $\beta^{ij} = p_i b^{ij}/w_j$, these are equal to $\beta^{ij} = \partial \log \hat{w}_j / \partial \log p_i$. Denoting by ι_m the column vector of m 1s, from $\iota_m' \mathrm{B} = \iota_m'$ we have $\iota_m' \mathrm{B}^{-1} = \iota_m' \mathrm{B}\mathrm{B}^{-1} = \iota_m'$, hence B^{-1} also has unit column sums (cf. Chipman, 1969, p. 402).

In the case $m = n = 2$, if we follow the convention of numbering commodities and factors in such a way that, at the initial equilibrium, $|B(w)| > 0$, i.e.,

$$\text{(4.4)} \quad \begin{vmatrix} b_{11}(w) & b_{12}(w) \\ b_{21}(w) & b_{22}(w) \end{vmatrix} = b_{11}(w) b_{12}(w) \left[\frac{b_{22}(w)}{b_{12}(w)} - \frac{b_{21}(w)}{b_{11}(w)} \right] > 0$$

(which means that industry 2 uses a higher ratio of factor 2 to factor 1 than industry 1), then B^{-1}, which has nonpositive diagonal elements and unit column sums, must have diagonal elements greater than or equal to unity. If B has its elements all positive, then the off-diagonal elements of B^{-1} are negative and the diagonal elements greater than unity. This, in substance, is the Stolper-Samuelson (1941) theorem. In words, for some association of commodities and factors, a rise in one commodity price will lead to a more than proportionate rise in the corresponding factor rental.

A simple proof is illustrated in Figure 8, in the space of factor rentals. An initial equilibrium pair $w^0 = (w_1^0, w_2^0)$ of factor rentals is shown as the intersection point of the isoquant $g_1(w_1, w_2) = p_1^0$ with the isoquant $g_2(w_1, w_2) = p_2$. A rise in p_1 from p_1^0 to p_1^1 is shown by an upward shift in the isoquant $g_1(w_1, w_2) = p_1$ from p_1^0 to p_1^1, resulting in a new intersection point $w^1 = (w_1^1, w_2^1)$ with the isoquant $g_2(w_i, w_2) = p_2$, with a lower w_2 and a rise in w_1 proportionately higher than that of p_1 (as long as the elasticities of substitution of the cost functions are positive, i.e., as long as the elasticities of substitution of the production functions are finite).

The Stolper-Samuelson theorem clearly does not generalize to higher dimensions. Either much stronger assumptions or much weaker conclusions are required. See Chipman (1969), Kuhn (1968), Inada (1971), Uekawa (1971), Ethier (1974), Jones and Scheinkman (1977) and Neary (1985).

The Rybczynski functions $\hat{y}_j(p, l)$ exist as single-valued functions only for the case $m \geq n$. If all n commodities are produced, and all m factors are fully employed, they satisfy the resource-allocation equation

$$B(\hat{w}(p, l))\hat{y}(p, l) = l. \tag{4.5}$$

When $m = n$, since then $\hat{w}$ is locally independent of l, $\hat{y}(p, l)$ is locally linear in l for any fixed p and may be written as

$$\hat{y}(p, l) = B(g^{-1}(p))l \tag{4.6}$$

(cf. Chipman, 1971, p. 214). The curious shapes of the Rybczynski functions are illustrated in Figure 9 (cf. Chipman 1972, p. 216).

As in the case of the Stolper-Samuelson mapping, one can consider the elasticities of outputs with respect to factor endowments when $m = n$. Denoting $L = \text{diag } l$ and $Y = \text{diag } y$ (not to be confused with the national-income variable Y of Section 3 above), we may define the matrix $\Lambda = L^{-1}BY$ with elements of $\lambda_{ij} = b_{ij}y_j/l_i = \partial \log l_i / \partial \log y_j$ (interpreting l_i in this relationship as requirements or demand for factor i, rather than supply). Its inverse $\Lambda^{-1} = Y^{-1}B^{-1}L$

has elements $\lambda^{ij} = b^{ij} l_j / y_i = \partial \log \hat{y}_i / \partial \log l_j$. From the resource-allocation constraint

$$\sum_{j=1}^{n} b_{ij} y_j = l_i \tag{4.7}$$

it follows that

$$\sum_{j=1}^{n} \lambda_{ij} = 1,$$

i.e., that Λ is row-stochastic (its elements are nonnegative and its row sums are equal to unity). By the same reasoning as before, the row sums of Λ^{-1} are equal to unity. In the case $n = m = 2$, adhering to the convention (4.4) it follows that, when B has positive elements, the off-diagonal elements of Λ^{-1} are negative, and thus its diagonal elements are greater than unity. Thus, $\partial \log \hat{y}_i / \partial \log l_i > 1$ and $\partial \log \hat{y}_i / \partial \log l_j < 0$ for $j \neq i$; in words, a rise in the ith factor endowment will, at given world prices, lead to a more than proportionate rise in the output of the ith commodity, and a fall in the output of the jth commodity ($j \neq i$). As in the case of the Stolper-Samuelson theorem, this obviously does not generalize to higher dimensions unless stronger assumptions are made or weaker conclusions reached. A discussion of the nature of such generalized results will be found in Kemp and Wegge (1969), Wegge and Kemp (1969), and Ethier (1974).

5 Interindustrial relationships and other refinements

The formal model treated so far assumes that production is completely integrated, contrary to fact. Indeed, a large part of international trade is in intermediate products. The main justification for not allowing for intermediate inputs at the very beginning is that it may obscure the logic of the analysis with inessential details. However, in view of the importance of the phenomenon it is desirable at this point to see how the formal framework needs to be modified (cf. McKinnon, 1966; Melvin, 1969a, 1969b; Khang and Uekawa, 1973).

In place of (2.1) one needs to substitute the production function

$$q_j = f_j(u_{1j}, u_{2j}, \ldots, u_{nj}, v_{1j}, v_{2j}, \ldots, v_{mj}) = f_j(u_{\cdot j}, v_{\cdot j}) \tag{5.1}$$

(assumed homogeneous of degree 1) where q_j denotes gross output of commodity j, and u_{ij} denotes the amount of commodity i used as input to the production of commodity j. Its dual minimum-unit-cost function—equal to the price of commodity j when that commodity is produced—is denoted

$$p_j = g_j(p_1, p_2, \ldots, p_n, w_1, w_2, \ldots, w_m) = g_j(p, w), \tag{5.2}$$

and the input-output and factor-output coefficients are, in accordance with Theorem 5,

$$a_{ij}(p, w) = \partial g_j(p, w)/\partial p_i; b_{ij}(p, w) = \partial g_j(p, w)/\partial w_i. \tag{5.3}$$

The production-possibility set (2.9) is now replaced by the *net-output-possibility set* defined by

$$\mathcal{Y}(l) = \Big\{ y \in E^n : \text{ there exist allocations } u_{ij} \geq 0, v_{kj} \geq 0 \\ (i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m) \\ \text{such that } y_j = f_j(u_{\cdot j}, v_{\cdot j}) - \sum_{k=1}^{n} u_{jk} \text{ and } \sum_{j=1}^{n} v_{\cdot j} \leq l \Big\} \tag{5.4}$$

(cf. Khang and Uekawa, 1973). This set is convex. Khang and Uekawa (1973) developed a generalization of Theorem 9 (Section 3 above); see also Kemp, Khang and Uekawa (1978) and Färe (1979). The concept of a domestic-product function can also be generalized to this case (cf. Chipman, 1985a, pp. 405-6), allowing one to define net supply (Rybczynski) functions.

When all commodities are produced, the net outputs, gross outputs, and factor endowments are related by

$$y = [I - A(p, w)]q, \quad B(p, w)q \leq l \tag{5.5}$$

where $A = [a_{ij}], B = [b_{ij}]$ are the $n \times n$ and $m \times n$ input-output and factor-output matrices. Accordingly, the resource-allocation constraint may be expressed as

$$C(p, w)y \leq l \quad \text{where} \quad C = B(I - A)^{-1}. \tag{5.6}$$

When all commodities are produced and traded, the minimum-unit costs are equal to the prices hence

$$p = g(p, w) = A(p, w)'p + B(p, w)'w. \tag{5.7}$$

If the Jacobian matrix of this transformation, $I - A(p,w)'$, satisfies the Hawkins-Simon (1949) conditions of having positive principle minors, then by the results of Gale and Nikaido (1965), (5.7) defines a set of consolidated cost functions $\psi(w)$ which satisfy

$$p = \psi(w) = [I - A(\psi(w), w)]'^{-1} B(\psi(w), w)' w. \tag{5.8}$$

The set of production functions dual to these consolidated cost functions defines a set of integrated production functions $\phi_j(v_{\cdot j})$ corresponding to the unintegrated functions (5.1). These could be validly used in place of (5.1) provided net outputs were required to be nonnegative.

A concept which has proved very useful in analyzing trade in intermediate products is that of a *value-added production function* introduced by Khang (1971, 1973) and Bruno (1973). If one assumes that the production functions (5.1) are twice continuously differentiable and that all intermediate inputs are used in positive amounts, by setting the partial derivatives $\partial f_j/\partial u_{ij} = p_i/p_j$ for $i = 1, 2, \ldots, n$, one may define implicitly the functions

$$u_{ij} = \hat{u}_{ij}(v_{\cdot j}, p) \tag{5.9}$$

which, when substituted back into (5.1) yield the *value-added production functions*

$$V_j(v_{\cdot j}, p) = p_j f_j(\hat{u}_{\cdot j}(v_{\cdot j}, p), v_{\cdot j}) - \sum_{i=1}^{n} p_i \hat{u}_{ij}(v_{\cdot j}, p). \tag{5.10}$$

These are shown to inherit (for given p) the homogeneity and concavity properties of the original production functions (5.1), and in particular to satisfy the envelope conditions $\partial V_j/\partial v_{ij} = w_i$ and $\partial V_j/\partial p_i = \delta_{ij} q_j - u_{ij}$, where δ_{ij} is the Kronecker delta.

From the mn equations

$$\begin{aligned} \frac{\partial V_j}{\partial v_{ij}} - \frac{\partial V_{j+1}}{\partial v_{i,j+1}} &= 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n-1), \\ \sum_{j=1}^{n} v_{ij} &= l_i \quad (i = 1, 2, \ldots, m), \end{aligned} \tag{5.11}$$

provided $m \geq n$ one can in general solve for the functions $\hat{v}_{ij}(p, l)$.

The system of equations determining the $\partial\hat{v}_{ij}/\partial p_k$ has the form

$$\begin{bmatrix} J_1 & -J_2 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & -J_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & J_{n-1} & J_n \\ I & I & I & \cdots & I & I \end{bmatrix} \begin{bmatrix} \partial\hat{v}_{\cdot 1}/\partial p_k \\ \partial\hat{v}_{\cdot 2}/\partial p_k \\ \vdots \\ \partial\hat{v}_{\cdot n-1}/\partial p_k \\ \partial\hat{v}_{\cdot n}/\partial p_k \end{bmatrix} = - \begin{bmatrix} K_1 - K_2 \\ K_2 - K_3 \\ \vdots \\ K_{n-1} - K_n \\ 0 \end{bmatrix}$$

(5.12)

where

$$J_k = \left(\frac{\partial^2 V_k}{\partial v_{ik}\partial v_{jk}}\right)_{i,j=1,2,\ldots,m}, \quad K_j = \left(\frac{\partial^2 V_j}{\partial v_{ij}\partial p_k}\right)_{i=1,2,\ldots,m}$$

are $m \times n$ and $m \times 1$ (cf. Bruno, 1973, p. 215).

As an illustration one may take the case $m = n = 2$, and pose the question: will an increase in the price of commodity 1 result in a reallocation of resources to industry 1? This question is of interest from the point of view of the theory of "effective protection" (cf. Johnson, 1965; Corden, 1966, l971). In this case (5.12) reduces to

(5.13) $$\begin{bmatrix} J_1 & -J_2 \\ I & I \end{bmatrix} \begin{bmatrix} \partial\hat{v}_{\cdot 1}/\partial p_1 \\ \partial\hat{v}_{\cdot 2}/\partial p_1 \end{bmatrix} = - \begin{bmatrix} K_1 - K_2 \\ 0 \end{bmatrix}$$

and we find that, as long as the two value-added production functions use distinct factor ratios,

(5.14) $$\begin{bmatrix} J_1 & -J_2 \\ I & I \end{bmatrix}^{-1} = -(J_1 + J_2)^{-1} \begin{bmatrix} I & J_2 \\ -I & J_1 \end{bmatrix}$$

where the product on the right is interpreted as scalar multiplication (on the left) of the partitioned matrix on the right by the scalar $(J_1 + J_2)^{-1}$ (a 2×2 matrix). The matrix $J_1 + J_2$ is negative definite with positive off-diagonal elements, hence by the Hawkins-Simon (1949) theorem $(J_1 + J_2)^{-1} < 0$. The solution of (5.13) is then

(5.15) $$\begin{bmatrix} \partial\hat{v}_{\cdot 1}/\partial p_1 \\ \partial\hat{v}_{\cdot 2}/\partial p_1 \end{bmatrix} = -(J_1 + J_2)^{-}1 \begin{bmatrix} K_1 - K_2 \\ K_2 - K_1 \end{bmatrix}.$$

Thus, of course $\partial\hat{v}_{\cdot 2}/\partial p_1 = -\partial\hat{v}_{\cdot 1}/\partial p_1$.

Concentrating on industry 1 we obtain

$$\begin{bmatrix} \partial\hat{v}_{11}/\partial p_1 \\ \partial\hat{v}_{21}/\partial p_1 \end{bmatrix} = -(J_1 + J_2)^{-1} \begin{bmatrix} \partial^2 V_1/\partial v_{11}\partial p_1 - \partial^2 V_2/\partial v_{12}\partial p_1 \\ \partial^2 V_1/\partial v_{21}\partial p_1 - \partial^2 V_2/\partial v_{22}\partial p_1 \end{bmatrix}.$$

(5.16)

Note that in the case of integrated production the vector on the right in (5.16) reduces to $(\partial f_1/\partial v_{11}, \partial f_1/\partial v_{21})'$, and since these two terms are positive it follows that both factors will move into industry 1. Is the same true in the case of non-integrated production? For example, if the two industries are steel and autos, and an import quota is imposed on steel, is it possible that, owing to the decline in the auto industry's use of the costlier steel, one of the factors will move out of steel into autos? An example of this general kind was given by Ramaswami and Srinivasan (1971), but in relation to a model in which the sole intermediate input is an imported good not produced at home (see also Jones, 1971). It was shown by Bruno that a sufficient condition for both factors to move into industry 1 is that the production functions (5.1) be functionally separable, i.e., of the form $f_j(u_{\cdot j}, v_{\cdot j}) = f_j(\phi_j(u_{\cdot j}), \psi_j(v_{\cdot j}))$. For technical reasons (to assure invertibility of the individual J_k matrices) Bruno assumed decreasing returns to scale for some of his results. For extensions not requiring this see Uekawa (1979).

Other generalizations and refinements may be briefly mentioned. An important restriction that needs to be relaxed is the assumption of constant returns to scale. Variable returns to scale have been analyzed by Jones (1968), Herberg and Kemp (1969), Melvin (1971), Negishi (1972), and others. Inoue (1981) applied the concept of parametric external economies of scale introduced in Chipman (1970), allowing for economies of scale to be compatible with existence of competitive equilibrium, and obtained generalizations of the Samuelson reciprocity relations. In particular, if the production function facing a particular firm *nu*, among the N_j firms in industry j, is

$$y_{j\nu} = k_j f_j(v_{\cdot j\nu}) = k_j f_j(v_{1j\nu}, v_{2j\nu}, \ldots, v_{mj\nu}) \tag{5.17}$$

where f_j is concave and homogeneous of degree 1 and $k_j = \phi_j(y_j)$ where $y_j = \sum_{\nu=1}^{N_j} y_{j\nu}$ (i.e., the coefficient k_j depends on industrial output y_j but is treated as a parameter by each firm), then since each cost-minimizing firm will hire its factor inputs in the proportion $v_{\cdot j\nu} = \lambda_{j\nu} v_{\cdot j}$ to the industry input vector $v_{\cdot j}$, where $\lambda_{j\nu} > 0$ and $\sum_{\nu=1}^{N_j} \lambda_{j\nu} = 1$, one obtains from (5.17)

$$y_j = \sum_{\nu=1}^{N_j} y_{j\nu} = \phi_j(y_j) f_j(v_{\cdot j}). \tag{5.18}$$

Choosing ϕ_j to be of the form $\phi_j(y_j) = y_j^{1-1/\rho_j}$ this becomes

$$y_j = f_j(v_{\cdot j})^{\rho_j}. \tag{5.19}$$

Inoue showed that the reciprocity relation (3.33) is replaced by the condition $\partial\hat{y}_j/\partial l_i = \rho_j \partial\hat{w}_i/\partial p_j$ and that the symmetry condition $\partial\hat{w}_i/\partial p_j = \partial\hat{w}_j/\partial p_i$ is retained (as is to be expected from the parametric behavior of producers), but that $\rho_i\partial\hat{y}_j/\partial p_i = \rho_j\partial\hat{y}_i/\partial p_j$. He also obtained sufficient conditions for the Stolper-Samuelson and Rybczynski theorems to generalize to the case of external economies or diseconomies of scale.

Alternative approaches have been followed by Ethier (1979, 1982), Helpman (1981), and others who do not distinguish between internal and external economies and therefore leave open the question of existence of general equilibrium.

Other types of refinements that have been introduced include the attempt to allow for joint production; one may refer especially to Chang, Ethier, and Kemp (1980).

Finally, many authors have relaxed the assumption of international immobility of factors. The properties of the world production-possibility frontier in this case have been investigated by Chipman (1971), Uekawa (1972), and Otani (l973). The relation between capital mobility and technology transfer has been analyzed by McCulloch and Yellen (1982) and Chipman (1982). This is only a small part of a growing literature.

6 Trade-demand and trade-utility functions

A common simplifying assumption in international-trade theory is that consumer preferences can be aggregated. This assumption makes it possible to define a country's offer function in a simple way, as first indicated (in the case in which all goods are traded) by Meade (1953), and subsequently by Rader (1964), Chipman (1974, p. 34n; 1979) and Woodland (1980). An extension to the case in which some goods are nontradable was derived by Chipman (1981).

Let us assume that a country produces n_1 tradable and n_3 nontradable commodities and imports an additional n_2 commodities which it does not produce. Let $\mathbf{x}_r$ denote the $n_r \times 1$ vector of quantities consumed in the rth category, and $\mathbf{p}_r$ the corresponding price vector, and let $\mathcal{X} \subseteq E^{n_1+n_2+n_3}$ denote the consumption set. If aggregate consumption $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathcal{X}$ is generated by maximization of an aggregate utility function $U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ subject to a budget constraint

$$\sum_{r=1}^{3} \mathbf{p}_r \cdot \mathbf{x}_r \leq Y,$$

where Y is disposable national income, when the contours of U are strictly convex to the origin this yields a single-valued demand function $\mathbf{x} = \mathbf{h}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, Y)$. Let $\mathcal{Y}(l) \subseteq E^{n_1} \times \{0\}^{n_2} \times E_+^{n_3}$ denote the production-possibility set. The *trade set* may be defined as the set $\mathcal{Z}(l) = (\mathcal{X} - \mathcal{Y}(l)) \cap E^{n_1+n_2} \times \{0\}^{n_3}$. For $z \in \mathcal{Z}(l)$ the *trade-utility function* may be defined as

$$\hat{U}(\mathbf{z}_1, \mathbf{z}_2; l) = \max_{\mathbf{x}-\mathbf{z} \in \mathcal{Y}(l)} U(\mathbf{x}), \tag{6.1}$$

and the *trade-demand correspondence* by

$$\begin{aligned} \hat{\mathbf{h}}(\mathbf{p}_1, \mathbf{p}_2, D; l) = \{(\mathbf{z}_1, \mathbf{z}_2) : \mathbf{z} \in \mathcal{Z}(l) \text{ maximizes} \\ \hat{U}(\mathbf{z}_1, \mathbf{z}_2; l) \text{ subject to } \mathbf{p}_1 \cdot \mathbf{z}_1 + \mathbf{p}_2 \cdot \mathbf{z}_2 \leq D\}, \end{aligned} \tag{6.2}$$

where D is the deficit in the balance of payments on goods and services. When the production-possibility frontier $\hat{\mathcal{Y}}(l)$ is strictly concave to the origin, and the contours of $U(\mathbf{x})$ are strictly convex to the origin, the contours of $\hat{U}(\mathbf{z})$ are strictly convex to the origin, and $\hat{\mathbf{h}}$ becomes a single-valued function. When $n_3 = 0$,

$$\hat{\mathbf{h}}(\mathbf{p}, D; l) = \mathbf{h}(\mathbf{p}, \Pi(\mathbf{p}, l) + D) - \hat{\mathbf{y}}(\mathbf{p}, l). \tag{6.3}$$

When this function is twice differentiable it has the properties that $\partial \hat{h}_i / \partial D = \partial h_i / \partial Y$ and

$$\hat{s}_{ij} \equiv \frac{\partial \hat{h}_i}{\partial p_j} + \frac{\partial \hat{h}_i}{\partial D} \hat{h}_j = \frac{\partial h_i}{\partial p_j} + \frac{\partial h_i}{\partial Y} h_j - \frac{\partial \hat{y}_i}{\partial p_j} \equiv s_{ij} - t_{ij}, \tag{6.4}$$

where s_{ij} and $\hat{s}_{ij}$ denote the Slutsky substitution terms of the demand and trade-demand functions respectively, and t_{ij} is the transformation term.

If there are K countries, and a k superscript indicates the country, world equilibrium is defined by

$$\sum_{k=1}^{K} \mathbf{h}^k(\mathbf{p}_1, \mathbf{p}_2, D^k; l^k) = 0, \text{ where } \sum_{k=1}^{K} D^k = 0. \tag{6.5}$$

7 Marshallian offer functions and dynamic stability

The Marshallian offer functions may be derived in a straightforward fashion from the trade-demand functions. Suppose two countries are trading two commodities, and that country k is exporting commodity k and importing commodity $j \neq k$. Assuming constant factor endowments (and thus ignoring the dependence of the functions on these endowments), each country's exports may be expressed as a single-valued function of its imports, provided the import good is non-inferior. In the case of country 1, denoting $r_2 = p_2/p_1$ and $d^1 = D^1/p_1$, provided commodity 2 is non-inferior, so that $\partial\hat{h}_2^1/\partial p_2 < 0$, one may define its inverse trade-demand function $\hat{r}_2(z_2^1, d^1)$ implicitly by

$$\hat{h}_2^1(1, \hat{r}_2(z_2^1, d^1), d^1) = z_2^1, \tag{7.1}$$

and its *trade function* (for $d^1 = 0$, its Marshallian reciprocal demand function) by

$$-z_1^1 = F^1(z_2^1, d^1) = \hat{r}_2(z_2^1, d^1)z_2^1 - d^1. \tag{7.2}$$

Likewise for country 2, denoting $r_1 = p_1/p_2$ and $d^2 = D^2/p_2$, one defines in a similar way its inverse trade-demand function $\hat{r}_1$ by

$$\hat{h}_1^2(\hat{r}_1(z_1^2, d^2), 1, d^2) = z_1^2 \tag{7.3}$$

and its trade function by

$$-z_2^2 = F^2(z_1^2, d^2) = \hat{r}_1(z_1^2, d^2)z_1^2 - d^2. \tag{7.4}$$

Since $d^1 = -z_1^2 d^2/(d^2 + z_2^1)$ from the balance-of-payments contraint and the condition $D^1 = -D^2$, one may express the trade functions in the form

$$\begin{aligned} z_1^2 &= Z_1(z_2^1, d^2) = F^1\left(z_2^1, \frac{-z_1^2 d^2}{d^2 + z_2^1}\right) \\ z_2^1 &= Z_2(z_1^2, d^2) = F^2(z_1^2, d^2). \end{aligned} \tag{7.5}$$

Following Alexander (1951), the *elasticities of trade* may be defined by

$$\alpha^1 = \frac{z_2^1}{Z_1}\frac{\partial Z_1}{\partial z_2^1}, \alpha^2 = \frac{z_1^2}{Z_2}\frac{\partial Z_2}{\partial z_1^2}. \tag{7.6}$$

These may be related to the Marshallian elasticities of demand for imports

$$\eta^1 = -\frac{p_2}{\hat{h}_2^1}\frac{\partial \hat{h}_2^2}{\partial p_2}, \quad \eta^2 = -\frac{p_1}{\hat{h}_1^2}\frac{\partial \hat{h}_1^2}{\partial p_1} \tag{7.7}$$

in the following way. For country 2, (7.3) gives $\partial \hat{r}_1/\partial z_1^2 = 1/(\partial \hat{h}_1^2/\partial p_1)$, hence computing α_2 from (7.4) we find that

$$\alpha_2 = \left(\frac{1+d^2}{z_2^1}\right)\left(\frac{1-1}{\eta^2}\right) \tag{7.8}$$

hence

$$\eta^2 = \frac{1+d^2/z_2^1}{1+d^2/z_2^1-\alpha_2} = \frac{z_2^1+d^2}{z_2^1+d^2-z_1^2\partial F^2/\partial z_1^2}. \tag{7.9}$$

The third expression in (7.9) provides a convenient way to read off the value of the elasticity from a diagram of the displaced Marshallian offer curve (see Figure 10).

Marshall (1879), following the outlines sketched by Mill (1848, Vol. II, Book III, Ch. XXI, §1), introduced a dynamic process of adjustment which was formalized by Samuelson (1947, p. 266) as

$$\begin{aligned} \dot{z}_1^2 &= \phi_1(Z_1(z_2^1, d^2) - z_1^2) \equiv P_1(z_1^2, z_2^1, d^2) \\ \dot{z}_2^1 &= \phi_2(Z_2(z_1^2, d^2) - z_2^1) \equiv P_2(z_1^2, z_2^1, d^2) \end{aligned} \tag{7.10}$$

(for the case $d^2 = 0$), where the ϕ_i are sign-preserving functions. Intuitively, if at a given level of import volume a country offers more units of the export good than it is currently exporting, it will increase its exports. According to Marshall (1923, p. 341), an excess of exports offered over current exports is an indication that profits are being made in the export industry, whence it will expand. Marshall's description of the rationale for this process was very terse, but a good discussion will be found in Amano (1968).

A complete description and classification of the stability properties of (7.10) appears to be lacking in the literature, and one will be briefly supplied here, for the case $d^2 = 0$. The methods can be extended readily to the case $d^2 \neq 0$, but the conditions become considerably more complex. Another simplification in the ensuing development (which does not affect stability conditions but only the shapes of the trajectories) will be to assume that $\phi_1 = \phi_2$, and for notational simplicity they will be taken to be identity functions $\phi_i(u_i) = u_i$.

An equilibrium is defined as a pair of values $\bar{z}_1^2, \bar{z}_2^1$ for which $P_i(\bar{z}_1^2, \bar{z}_2^1) = 0, i = 1, 2$. Denoting deviations from these equilibrium values by $u_1 = z_1^2 - \bar{z}_1^2, u_2 = z_2^1 - \bar{z}_2^1$, and taking first-order Taylor approximations of the functions P_i around such an equilibrium, we obtain the system

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -1 & \gamma_1 \\ \gamma_2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7.11}$$

where

$$\gamma_1 = \frac{\partial F^1}{\partial z_2^1} \text{ and } \gamma_2 = \frac{\partial F^2}{\partial z_1^2} \tag{7.12}$$

and these derivatives are evaluated at the equilibrium point $(\bar{z}_1^2, \bar{z}_2^1)$. Writing (7.11) in matrix notation $\dot{u} = Au$, it is well known and easily shown that if the characteristic values of A are distinct, its characteristic vectors are linearly independent. Letting V be the matrix whose columns are these characteristic vectors, we have $V^{-1}AV = \Lambda$, where Λ is the diagonal matrix of characteristic values (roots) λ_i. Defining $u^* = V^{-1}u$, we have $\dot{u}^* = V^{-1}\dot{u} = V^{-1}Au = V^{-1}AVu^* = \Lambda u^*$, whence $\dot{u}_i^* = \lambda_i u_i^*$. Each of these differential equations is solved to obtain $u_i^* = b_i e^{\lambda_i t}$, hence the desired solution is $u = Vu^* = \sum_{i=1}^2 v^i b_i e^{\lambda_i t}$ where v^i is the ith column of V. Assuming (to obtain a simple normalization) that the top row of V has no zero elements, they may be chosen equal to unity and we have

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ v_1 \end{bmatrix} b_1 e^{\lambda_1 t} + \begin{bmatrix} 1 \\ v_2 \end{bmatrix} b_2 e^{\lambda_2 t}, \tag{7.13}$$

which substitutes a new coordinate system defined by the characteristic vectors (the columns of V as opposed to those of the identity matrix). These tilted coordinate axes spanned by the characteristic vectors are called “separatrices” and the v_is are called “distribution coefficients" (cf. Andronov et al., 1966, p. 258). Solving

$$\begin{bmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.14}$$

we obtain for the characteristic values

$$\lambda = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{21}a_{12}}}{2} = -1 \pm \sqrt{\gamma_1\gamma_2} \tag{7.15}$$

and for the distribution coefficients

$$v = \frac{a_{22} - a_{11} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{21}a_{12}}}{2a_{12}} = \frac{\pm\sqrt{\gamma_1\gamma_2}}{\gamma_1}. \tag{7.16}$$

First we may consider stability conditions. Asymptotic stability, in the sense $\lim_{t\to\infty} u_i(t) = 0$, requires that the real parts of both λ_i be negative. If the offer curves intersect with one positive and one negative slope, then $\gamma_1\gamma_2 < 0$ and the real parts of both roots are -1. Instability can therefore only occur when both roots are real. (The intermediate case of repeated roots, for which special methods are required, will not be taken up here.)

If the offer curves intersect with both positive or both negative slopes, instability can occur only if $\gamma_1\gamma_2 \geq 1$ hence the stability condition is, using (7.12), and (7.5) to (7.8),

$$1 > \gamma_1\gamma_2 = \alpha_1\alpha_2 = \left(\frac{1-1}{\eta^1}\right)\left(\frac{1-1}{\eta^2}\right) = \frac{(\eta^1 - 1)(\eta^2 - 1)}{\eta^1\eta^2}. \tag{7.17}$$

In the form $\gamma_1\gamma_2 < 1$, this stability condition was first obtained by Marshall (1923, p. 353). Since it has already been assumed that both goods are normal, the η^k are both positive and (7.17) reduces to the well-known condition

$$\eta^1 + \eta^2 - 1 > 0. \tag{7.18}$$

This condition was described by Hirschman (1949) as the "Marshall-Lerner condition," but this must be characterized as one of the great misnomers in the theory of international trade. A condition formally equivalent to (7.18) was described by Lerner (1944, p. 378), but it referred to a model originated by Bickerdike (1907, 1920) which may be interpreted as referring to economies which specialize in an exportable and a nontradable good which are produced with a single factor of production (cf. Chipman, 1978, p. 67). This is described as the case of "infinite elasticity of supply of exports." As it happens, for that particular model the Bickerdike-Lerner elasticities coincide with the Marshallian ones, but in any case formula (7.18) was, for this case, already derived by Bickerdike (1920, p. 121) in his own idiosyncratic notation. As for Marshall, the passage for which (7.18) is attributed to him considers a case of near-neutral equilibrium (in which the offer curves approximately coincide with negative slope) and states concerning such conditions (1923, p. 354):

> ...they assume the total elasticity of demand of each country to be less than unity, and on the average to be less than one half, throughout a large part of its schedule.

A statement of conditions of instability no less explicit than this was already in Mill (1848, Vol. II, Book III, Ch. XXI, §1, p. 158):

> ...until the increased cheapness of English goods induces foreign countries to take a greater pecuniary value, or until the increased dearness (positive or comparative) of foreign goods makes England take a less pecuniary value, the exports of England will be no nearer to paying for the imports than before ...

The term "Mill-Marshall condition" might be a more appropriate one to describe (7.18).

The various cases may now be classified. First, if $\gamma_1 > 0$ and $\gamma_2 > 0$, then taking $\lambda_1 = -1-\sqrt{\gamma_1\gamma_2}$ and $\lambda_2 = -1+\sqrt{\gamma_1\gamma_2}$, where $\gamma_1\gamma_2 < 1$ (i.e., $\gamma_2 < 1/\gamma_1$ —see Figure 11), we have $v_1 = -\sqrt{\gamma_2/\gamma_1}$ and $v_2 = \sqrt{\gamma_2/\gamma_1}$ and $\gamma_2 < v_2 < 1/\gamma_1$. Thus, country 1's offer curve is steeper than country 2's at the equilibrium point, and the separatrices pass between these two curves (this latter property no longer holds if $\phi_1' \neq \phi_2'$ in (7.10)). Figure 11 displays the new coordinate axes given by the characteristic vectors of (7.13). The trajectory of u is a linear combination of trajectories of points moving along these axes towards the equilibrium point. From the above differential equations $\dot{u}_i^* = \lambda_i u_i^*$ we obtain

$$(7.19) \qquad \frac{du_2^*}{du_1^*} = \frac{\lambda_2}{\lambda_1}\frac{u_2}{u_1} \quad \text{with solution} \quad u_2^* = C|u_1^*|^{\lambda_2/\lambda_1}$$

which gives the equation of family of a parabolas in the (u_1^*, u_2^*) plane. The transformation $u = Vu^*$ maps these into distorted parabolas as shown in Figure 11. This equilibrium is classified as a *stable node*.

The second case to be considered, corresponding to the northwest equilibrium point in Figure 12, has $\gamma_1 < 0, \gamma_2 < 0$ (both slopes negative), and $\gamma_1\gamma_2 < 1$, so that $-1/\gamma_1 > -\gamma_2$, i.e., country 1's offer curve is steeper (in absolute value) than country 2's. With the λ_i as before we now have $v_1 = \sqrt{\gamma_2/\gamma_1}$ and $v_2 = -\sqrt{\gamma_2/\gamma_1}$ so that $-\gamma_2 < -v_2 < -1/\gamma_1$. This case is also one of a *stable node*. The difference is that in the previous case the base of the parabolas was the positively-sloped separatrix whereas in this case it is the negatively-sloped separatrix.

The third case is the intermediate unstable equilibrium shown in Figure 12, where $\gamma_1 < 0, \gamma_2 < 0$, and $-\gamma_2 > -v_2 > -1/\gamma_1$, so that $\gamma_1\gamma_2 > 1$. The λ_i and v_i are as in the immediately preceding case, and the differential equation and solution (7.19) still hold, but this time it describes a family of hyperbolae. The separatrix with positive slope $v_1 = \sqrt{\gamma_2/\gamma_1}$ is stable, and the one with negative slope $v_2 = -\sqrt{\gamma_2/\gamma_1}$ is unstable. Its slope lies in between those of the two offer curves, i.e., $-\gamma_2 < -v_2 < -1/\gamma_1$ (as before, this relationship depends on the assumption that $\phi_1 = \phi_2$, hence $\phi_1' = \phi_2'$ in (7.10); this need not hold in general). This unstable equilibrium is a *saddle-point*.

The fourth and final case is that in which $\gamma_1 > 0$ and $\gamma_2 < 0$, as in the southeast equilibrium point in Figure 12. The roots are complex, and the equilibrium is a *stable focus* or spiral, with movement towards equilibrium in the clockwise direction. In the case (not shown) $\gamma_1 < 0$ and $\gamma_2 > 0$, the direction of movement would be contraclockwise.

Marshall (1923, p. 353) considered another possible case of unstable equilibrium, in which $\gamma_1 > 0, \gamma_2 > 0$, and $\gamma_1\gamma_2 > 1$ hence $\gamma_2 > 1/\gamma_1$ (country 2's offer curve is steeper than country 1's). Such a case, if it were possible, would be one of an *unstable node*.

The Marshallian dynamic-adjustment mechanism is what is described as a "non-tâtonnement process," in which trading takes place out of equilibrium. Alternative adjustment processes have been analyzed by Jones (1961, 1974b), Kemp (1964, pp. 66-9), Chipman (1978), and others.

Part 2. The Applied Theory

8 The explanation of trade flows

If there is any one thing that could legitimately be demanded of a theory of international trade, it is that it should be capable of explaining observed patterns and flows of trade among countries. The great strides in the subject have come with the development of new principles that provide such an explanation.

The principle of comparative advantage originated with Thornton (1802) whose problem was to explain why a country would lose gold reserves, i.e., why it would export gold. Gold outflows, he reasoned, would take place when (1802, p. 129; 1939, p. 150):

> ...goods, in comparison with gold coin, are made dear. The goods which are dear remain, therefore, in England; and the gold coin, which is cheap ..., goes abroad.

The importance of *comparative* cheapness of gold was reiterated in Thornton's summary of the basic principles of his work (1802, pp. 76–7; 1939, p. 247):

> I would be understood to say, that in a country in which *coin alone* circulates, if, through any accident the quantity should become greater in proportion to the goods which it has to transfer than it is in other countries, the coin becomes cheap as compared with goods, or, in other words, that goods become dear as compared with coin, and that a profit on the exportation of coin arises.

This principle was absorbed into Ricardo's early work (1811; 1951, pp. 56–7), so much so that it caused Malthus (1811a, p. 341) to praise it for

> the doctrine, that excess and deficiency of currency are only *relative* terms; that the circulation of a country can never be superabundant, except in relation to other countries.

The principle was extended by Torrens (1815) and Ricardo (1817) to the explanation of trade in commodities other than money. This theory held sway for over a century, but since it was combined with a labor theory of value implying that countries' cost ratios were fixed, it led to the uncomfortable conclusion that either countries would specialize, or some countries' cost ratios would dictate world price ratios (cf. Graham, 1923). Moreover, the cost ratios of the various countries were taken as data, not in need of further explanation.

The first attempt to probe into the reasons for differences in comparative costs was evidently that of Heckscher, who stated (1919; 1949, p. 278):

> A difference in the relative scarcity of the factors of production between one country and another is thus a necessary condition for a difference in comparative costs and consequently for international trade.

Heckscher did not say precisely what he meant by "comparative costs," though he stressed the criterion of different relative factor rentals (presumably under autarky); nor did he make any specific statement concerning how differences in relative factor endowments (which he assumed fixed) would determine the precise pattern of trade. But he did completely anticipate Lerner's (1933) and Samuelson's (1949) theorems that with identical productive techniques trade would equalize factor rentals among countries.

Haberler (1930, 1933) liberated the classical theory from the labor theory of value by introducing the concept of a strictly-concave-to-the-origin production-possibly frontier, based on the allocation of factors (assumed in fixed total supply) among industries. Most of the subsequent formal development of the theory by Lerner (1932, 1933), Leontief (1933), and Meade (1952) was built on Haberler's concept.

Ohlin (1933, p. 24) summarized his theory as follows:

> The first condition of trade is that some goods can be produced more cheaply in one region than another. In each of them the cheap goods are those containing relatively great quantities of the factors cheaper than in other regions. These cheap goods make up exports, whereas goods which can be more cheaply produced in the other regions are imported. We may say, therefore, that exports are in each region composed of articles into the production of which enter large quantities of cheap factors.

The criterion of "cheapness" was interpreted by Jones (1956) as referring to the pre-trade (autarkic) factor rentals in the respective countries. However, this appears to be directly contradicted by a statement of Ohlin's in the paragraph immediately following the above-cited passage:

> When reasoning like this we must, however, bear in mind one thing: whether a factor is cheaper or dearer in region A than in region B can be ascertained only when an exchange rate between the two countries has been established ...

This appears to be a confusion, since it would lead to the conclusion that if the Heckscher-Lerner-Samuelson factor-rental-equalization theorem (which Ohlin rejected) holds, then there will be no trade!

Fruitful progress in any field requires one to filter truth out of error in theories that are carelessly stated yet contain important insights. Jones (1956) filtered out two logically distinct propositions both of which were given the name "Heckscher-Ohlin theorem" and state that a country will export that commodity which is produced with relatively large amounts of its relatively abundant factor. The propositions differ in the definition or "relative abundance": the *physical definition* (differences in relative factor endowments) and the *price definition* (differences in ratios of autarkic factor rentals) (see also Bhagwati, 1957).

The trouble with the "price definition" of relative factor abundance is that it leads to a theory that is practically devoid of empirical content. In an internationally trading economy, autarkic prices are not observed. The theory would explain or predict observable trade patterns by data which cannot be observed. There is, moreover, a logical problem with the "price definition," pointed out by Inada (1967): since competitive equilibrium is in general not unique, pre-trade factor rentals are not unique, hence the proposition is both ambiguous and false unless conditions are postulated that guarantee uniqueness (such as identical homothetic preferences).

The proposition under the "physical definition" of factor abundance also holds under quite limited circumstances. A minimal set of sufficient conditions is the following: (1) there are two countries, two commodities—both tradable at zero transport costs—and two factors of production—both perfectly mobile between industries within countries but completely immobile between countries; (2) production functions are identical between countries, obey constant returns to scale and concavity, and use different ratios of factor inputs for all factor rentals; (3) preferences within and as between countries are identical and homothetic; and (4) trade is balanced. If any one of these conditions is omitted, the theorem can be shown to be false. It is not even clear how the theorem should be stated if condition (1) is generalized (but this will be discussed below). Only with (2) can one conclude that, at all price ratios, one country will produce a larger ratio of commodity 1 to commodity 2 than the other. Without (3), inhabitants of each country may have a strong relative preference for the commodity which that country produces in relative abundance, so that each country ends up importing rather than exporting that commodity. Without (4), a country that is borrowing or receiving a unilateral transfer from another may import both commodities from the other country.

A formal proof of the "Heckscher-Ohlin theorem" (physical version) proceeds as follows (cf. Riezman, 1974). Let $x_j^k, y_j^k, z_j^k = x_j^k - y_j^k$ denote consumption, production, and net import of commodity j in country k, and let l_i^k denote country k's endowment in factor i. Suppose $l_1^1/l_2^1 > l_1^2/l_2^2$, and assume that at all factor rentals w_1, w_2, the factor-output ratios satisfy $b_{11}(w)/b_{21}(w) > b_{12}(w)/b_{22}(w)$. Then from the Rybczynski theorem it follows that

$$\hat{y}_1(p_1, p_2, l_1^k, l_2^k)/\hat{y}_2(p_1, p_2, l_1^k, l_2^k)$$

is an increasing function of l_1^k/l_2^k, hence at all (including equilibrium) prices, $y_1^1/y_2^1 > y_1^2/y_2^2$. From identical homothetic preferences, at all (including equilibrium) prices, $x_1^1/x_2^1 = x_1^2/x_2^2$. Now suppose by way of contradiction that country 1 does not export commodity 1, i.e., $z_1^1 \geq 0$, or $x_1^1 \geq y_1^1$. Then from balanced trade, $z_2^1 \leq 0$, or $x_2^1 \leq y_2^1$, hence $x_1^1/x_2^1 \geq y_1^1/y_2^1$. Since $z_j^1 + z_j^2 = 0$ from free tradability of commodities at zero transport costs, a similar argument for country 2 shows that $y_1^2/y_2^2 \geq x_1^2/x_2^2$. So $x_1^1/x_2^1 \geq y_1^1/y_2^1 > y_1^2/y_2^2 \geq x_1^2/x_2^2$, violating the equality $x_1^1/x_2^1 = x_1^2/x_2^2$. This contradiction proves the result.

Apparently the first attempt to subject the Heckscher-Ohlin theory to empirical test was that of Leontief (1953). It is apparent from the above that as long as a precise statment of the theory for more than two commodities, factors, and countries is lacking, the problem of testing it empirically is elusive at best.

The most prominent attempt to generalize the theory is that of Vanek (1968), which has been followed up by Leamer (1980, 1985). The ensuing summary will follow Leamer's treatment, somewhat generalized. Let A and B be $n \times n$ and $m \times n$ matrices (which in general depend on prices and factor rentals) of input-output and factor-output coefficients (assumed identical among countries), and let $C = B(I-A)^{-1}$ denote the integrated factor-output matrix. It is assumed that all n goods are traded at zero transport costs, and that a world equilibrium exists with equalization of factor rentals (which requires one effectively to assume $n \geq m$), so that it makes sense to aggregate factor endowments over countries. The world consumption, production, and factor-endowment vectors are denoted

$$x = \sum_{k=1}^{K} x^k, \quad y = \sum_{k=1}^{K} y^k, \quad \text{and} \quad l = \sum_{k=1}^{K} l^k.$$

Finally, it is assumed that preferences are identical and homothetic within and as between countries; this assures that whatever be the world prices, commodities will be consumed in the same proportion in all countries, and therefore $x^k = \alpha_k x$ where

$$\alpha_k > 0, \quad \sum_{k=1}^{K} \alpha_k = 1.$$

Since world equilibrium requires $x = y$, we have $x^k = \alpha_k y$. The vector of "net imports of factor i by country k" is defined as

$$\hat{l}^k = Cz^k = C(x^k - y^k) = C(\alpha_k y - y^k) = \alpha_k l - l^k, \tag{8.1}$$

where use is made of the full-employment condition $Cy^k = l^k$. Note that $\hat{l}^k$ is unique, even though y^k is not unique when $n > m$. Country k is said to be relatively well endowed in factor i compared to factor j if $l_i^k/l_j^k > l_i/l_j$. Leamer (1980) establishes a number of propositions:

Proposition 1 (Leamer's Corollary 2). If country k is a net exporter of factor i and a net importer of factor j, then it is relatively well endowed in factor i relative to factor j, i.e., $\hat{l}_i^k < 0$ and $\hat{l}_j^k > 0$ imply $l_i^k/l_j^k > l_i/l_j$.

This is proved by noting from (8.1) that $\hat{l}_i^k = \alpha_k l_i - l_i^k < 0$ implies $l_i^k > \alpha_k l_i$ and similarly $\hat{l}_j^k = \alpha_k l_j - l_j^k > 0$ implies $l_j^k < \alpha_k l_j$ hence $l_i^k/l_i > \alpha_k > l_j^k/l_j$.

Note that this generalizes not the Heckscher-Ohlin theorem but its converse. One would like to be able to say that $l_i^k/l_j^k > l_i/l_j$ implies $\hat{l}_i^k < 0$ and $\hat{l}_j^k > 0$, but this is in fact not true. An example used by Leamer to establish a different point can be used to establish this one as well. Let

$$C = \begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 0.5 \\ 1 & 0 & 3 \end{bmatrix},\ y^k = \begin{bmatrix} 8 \\ 16 \\ 5 \end{bmatrix},\ y = \begin{bmatrix} 12 \\ 68 \\ 52 \end{bmatrix},\ p = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then country-k and world factor endowments are $l^k = (53, 58.5, 23)'$ and $l = (168, 198, 168)'$ respectively, hence $l_1^k/l_1 = 0.32 > 0.30 = l_2^k/l_2$. However, national and world income are $p \cdot y^k = 29$ and $p \cdot y = 132$ respectively, giving $\alpha_k = 0.22$, hence $x^k = (2.64, 14.94, 11.42)'$ and thus $z^k = (-5.36, -1.06, 6.42)'$ (country k exports commodities 1 and 2 and imports commodity 3). It follows from (8.1) that $\hat{l}^k = (-16.1, -15.0, 13.9)'$ hence country k is a net exporter of both factors 1 and 2 and a net importer of factor 3.

The Vanek-Leamer "generalization" of the Heckscher-Ohlin theorem is thus disappointing on two counts: first, it replaces the problem of explaining trade flows in actual commodities by that of explaining flows of abstract amounts of factors of production "embodied" in the trade flows; secondly, it reverses the problem and uses the abstract embodied trade flows to explain or predict (or to "reveal") the relative abundance of factors within countries. There is still a third cause for concern: the terminology "amounts of factor services embodied in goods traded" to describe the empirically measurable entities (8.1) is justifiable only under the very special assumptions of the model, especially those of identical homothetic preferences and international equalization of factor rentals. For the description of the variables of a model to be valid only under the special assumptions of the model appears to be poor scientific practice. This is just a terminological objection, but words can be treacherous and terminology can lead one astray.

Proposition 2 (Leamer's Corollary 1). Country k is revealed to be relatively well endowed in factor i relative to factor j (i.e., $l_i^k/l_j^k > l_i/l_j$) if and only if $l_i^k/l_j^k > (l_i^k + \hat{l}_i^k)/(l_j^k + \hat{l}_j^k)$.

This follows very simply from the fact that $l_i^k + \hat{l}_i^k = \alpha_k l_i$, from (8.1). Since $l^k + \hat{l}^k = Cx^k$, the entities $l_i^k + \hat{l}_i^k$ are interpreted as "the amount of factor i embodied in country k's consumption." The significance of this proposition lies in the fact that one can infer a country's relative abundance (compared to the world) in one factor relative to another from data on this country's endowments, technology matrix, and trade alone. It is rather remarkable that one should be able to do this without any data on factor endowments in the rest of the world. It shows the power of the assumptions of identical homothetic preferences and international equalization of factor rentals; but by the same token it leads one to be wary of assumptions that can provide so much information.

Leamer next considers Leontief's method of inferring a country's relative endowment ratios from trade, endowment, and technology data. Leontief considered imports and exports separately. Accordingly, one may define

$$z_j^k(+) = \max(z_j^k, 0), \quad z_j^k(-) = -\min(z_j^k, 0) \tag{8.2}$$

and

$$z^k(+) = (z_l^k(+), \ldots, z_n^k(+))', \quad z^k(-) = (z_1^k(-), \ldots, z_n^k(-))'. \tag{8.3}$$

Then $z^k(+)$ is the vector of absolute values of imports, and $z^k(-)$ the vector of absolute values of exports. Then one may define

$$\hat{l}^k(+) = Cz^k(+), \quad \hat{l}^k(-) = Cz^k(-), \tag{8.4}$$

where $\hat{l}^k(+)$ and $\hat{l}^k(-)$ are the vectors of factor services embodied in gross imports and gross exports respectively. These vectors satisfy

$$z^k = z^k(+) - z^k(-), \quad \hat{l}^k = \hat{l}^k(+) - \hat{l}^k(-). \tag{8.5}$$

Leontief's criterion (1953, p. 343) for country k to be relatively well endowed in factor i compared to factor j was, in effect (for $i \neq j$),

$$\frac{\hat{l}_i^k(-)}{\hat{l}_j^k(-)} > \frac{\hat{l}_i^k(+)}{\hat{l}_j^k(+)}. \tag{8.6}$$

This states that the ratio of amounts of factor i and factor j embodied in gross exports should be greater than the ratio of amounts of factor i and factor j embodied in gross imports.

Leamer's criticism is that the appropriate criterion is instead

$$\frac{l_i^k}{l_j^k} > \frac{l_i^k + \hat{l}_i^k}{l_j^k + \hat{l}_j^k}, \tag{8.7}$$

i.e., that the endowment of factor i relative to that of factor j should exceed the ratio of the amounts of factors i and j embodied in consumption, since by Proposition 2 the fraction on the right in (8.7) is equal to the ratio of world endowments l_i/l_j. The numerical illustration referred to above was used by Leamer to show that these inequalities can conflict when, as in the case of Leontief's data, the amounts $\hat{l}_i^k, \hat{l}_j^k$ of factors i and j embodied in net imports are both negative. Leamer showed that Leontief's procedure would be correct if $\hat{l}_i^k$ and $\hat{l}_j^k$ had opposite sign.

Proposition 3 (Leamer's Corollary 3). If $\hat{l}_i^k, \hat{l}_j^k$ have opposite sign then inequalities (8.6) and (8.7) are equivalent.

The method of proof is to show, first, that if $\hat{l}_i^k < 0$ and $\hat{l}_j^k > 0$ then both (8.6) and (8.7) hold, whereas if $\hat{l}_i^k > 0$ and $\hat{l}_j^k < 0$ then the inequalities opposite to both (8.6) and (8.7) hold. If $\hat{l}_i^k < 0$ and $\hat{l}_j^k > 0$ then $\hat{l}_i^k l_j^k < 0 < \hat{l}_j^k l_i^k$ hence $l_j^k \alpha_k l_i = l_j^k(l_j^k + \hat{l}_j^k) > l_i^k(l_j^k + \hat{l}_j^k) = l_i^k \alpha_k l_j$, establishing (8.7). Further, $\hat{l}_i^k = \hat{l}_i^k(+) - \hat{l}_i^k(-) < 0$ implies $\hat{l}_j^k(-)/\hat{l}_j^k(+) > 1$ and $\hat{l}_j^k = \hat{l}_j^k(+) - \hat{l}_j^k(-) > 0$ implies $\hat{l}_j^k(-)/\hat{l}_j^k(+) < 1$, whence (8.6) follows. The case $\hat{l}_i^k > 0, \hat{l}_j^k < 0$ is proved in exactly the same way, by reversing all the inequalities.

The following proposition provides a limited converse to Proposition 1, and thus a limited generalization of the Heckscher-Ohlin theorem:

Proposition 4. If there are two factors, both fully employed, and country k's trade is balanced, then $l_i^k/l_j^k > l_i/l_j$ implies $\hat{l}_i^k < 0$ and $\hat{l}_j^k > 0$.

This is proved by noting that the vector w of factor rentals must satisfy $w'C = p'$, hence from (8.1) $w'\hat{l}^k = w'Cz^k = p'z^k = 0$, the last equality being the condition of balanced trade. The full-employment condition $Cy^k = l^k$ implies that w has positive components, hence if $\hat{l}^k$ has only two components they must be of opposite sign. If $l_i^k/l_j^k > l_i/l_j$ then it follows from Proposition 1 that $\hat{l}_i^k < 0$ and $\hat{l}_j^k > 0$.

Leamer remarked (1980, p. 501) that in the unlikely world of two commodities Leontief's method was a correct method (when trade is balanced). But this was because Leamer assumed the number of products to be equal to the number of factors. However, it follows from Proposition 4 that Leontief's method is correct as long as there are two factors and trade is balanced.

Leamer showed that, on his method (which does not require any assumptions concerning balanced trade), the United States was relatively well endowed in capital relative to labor on the basis of Leontief's 1947 data—the opposite of Leontief's conclusion. However, this deduction—correct and ingenious though it is—constitutes a Pyrrhic victory; for if the Heckscher-Ohlin theory were truly generalizable, it should follow from this conclusion that the United States exported embodied capital services and imported embodied labor services, whereas it exported both. If one cannot draw such a conclusion, why is it useful to know that the United States is relatively well endowed in capital relative to labor?

Leamer's conclusion appears to be that the culprit was natural resources—a factor left out of account by Leontief. This could well be true. But there is another possible explanation. In 1947 the United States had an export surplus of $11.6 billion—roughly 5% of the gross national product. Even in the simple 2×2 case, such a violation of the conditions of the Heckscher-Ohlin theorem would be sufficient to invalidate the conclusion. A theory that purports to explain trade flows cannot afford to ignore trade imbalances.

Leontief's 1953 paper had an enormous impact on the theory of international trade. The foundations of the successor to the law of comparative advantage were questioned. Further empirical investigations were carried out and new hypotheses formulated. Some of the more notable of these will be discussed.

Grubel and Lloyd (1975) drew attention to the large proportion of international trade flows which constituted what they called "intra-industry trade," that is, two-way trade in products belonging to the same industrial category. They also constructed an index to measure the intensity of intra-industry trade. To analyze this, one of course must examine the way statistics of international trade are aggregated. Suppose the n commodities entering international trade are partitioned into $\bar{n}$ groups, and let G be an $\bar{n} \times n$ "grouping matrix," i.e., a matrix of ones and zeros, containing exactly one unit element in each column. Let $t^k = Pz^k$ be an $n \times 1$ vector of values of country k's net imports (= gross imports if positive, gross exports if negative—since cross-haulage may be ruled out at the finest levels of disaggregation), where $P = \text{diag}\, p$. Then from (8.2) and (8.3) above we may define the vectors

$$t^k(+) = Pz^k(+), \quad t^k(-) = Pz^k(-). \tag{8.8}$$

The components of the vector $t^k(+)$ are positive import values for commodities imported, and zeros otherwise; and those of $t^k(-)$ are positive export values for commodities exported and zeros otherwise. These vectors satisfy

$$t^k = t^k(+) - t^k(-); \quad |t^k| = t^k(+) + t^k(-), \tag{8.9}$$

where $|t^k|$ is the vector of absolute trade values.

Now from published trade statistics one will only observe the aggregate $\bar{n} \times 1$ vectors

$$\bar{t}^k(+) = Gt^k(+), \quad \bar{t}^k(-) = Gt^k(-). \tag{8.10}$$

The components of $\bar{t}^k(+)$ are aggregate imports, and those of $\bar{t}^k(-)$ aggregate exports, in the respective categories. Unlike the case with $t^k(+)$ and $t^k(-)$, whose inner product is zero, the inner product of $\bar{t}^k(+)$ and $\bar{t}^k(-)$ is in general not zero, that is, one will find imports and exports in both categories. From the data (8.10) one can obtain

$$\bar{t}^k(+) - \bar{t}^k(-) = G[t^k(+) - t^k(-)] = Gt^k \tag{8.11}$$

which gives the net imports in each category, and

$$\bar{t}^k(+) + \bar{t}^k(-) = G[t^k(+) + t^k(-)] = G|t^k| \tag{8.12}$$

which gives the total trade in each category. The Grubel-Lloyd index of intra-industry trade is essentially (except for multiplication by 100)

$$Q_{\text{GL}} = 1 - \frac{\iota'|Gt^k|}{\iota' G|t^k|} \tag{8.13}$$

(cf. Grubel and Lloyd, 1975, p. 22). Related indices were earlier introduced by Hirschman (1945), Verdoorn (1960), Michaely (1962), Kojima (1964), and Balassa (1966).

The Grubel-Lloyd index satisfies $0 \leq Q_{\mathrm{GL}} \leq 1$. The second inequality is immediate and the first follows from the fact that $|Gt^k| \leq G|t^k|$. The lower bound can be attained when aggregation is perfect in the sense that $\bar{t}^k(+)$ and $\bar{t}^k(-)$ are orthogonal, given that they are nonnegative; for it follows then from (8.10)–(8.12) that $|Gt^k| = |Gt^k(+) - Gt^k(-)| = Gt^k(+) + Gt^k(-) = G|t^k|$. On the other hand for the upper bound to be attained requires $Gt^k = 0$ which implies $\iota' Gt^k = \iota' t^k = 0$, i.e., balanced trade. An alternative index that surmounts this problem was suggested by Aquino (1978):

$$Q_{Aq} = 1 - \tfrac{1}{2}\iota' \left| \frac{Gt^k(+)}{\iota' Gt^k(+)} - \frac{Gt^k(-)}{\iota' Gt^k(-)} \right| . \tag{8.14}$$

The lower bound of 0 is attained when aggregation is perfect, since then

$$\left| \frac{Gt^k(+)}{\iota' Gt^k(+)} - \frac{Gt^k(-)}{\iota' Gt^k(-)} \right| = \frac{Gt^k(+)}{\iota' Gt^k(+)} + \frac{Gt^k(-)}{\iota' Gt^k(-)} \tag{8.15}$$

and the column sum of this vector is 2. The upper bound of 1 in (8.14) is attained when the two vectors on the right in (8.15) are equal to one another, and this does not require balanced trade. When trade is balanced, i.e., $\iota' t^k = 0$, then $\iota' Gt^k = \iota' t^k = 0$ hence from (8.11) $\iota' Gt^k(+) = \iota' Gt^k(-)$ and from (8.12) each of these is equal to $\frac{1}{2}G|t^k|$, hence the Aquino index (8.14) reduces to the Grubel-Lloyd index (8.13).

Grubel and Lloyd (1975, pp. 86–7) distinguished two criteria for aggregation of commodities into groups: substitutability of products in consumption (e.g., wooden versus metal furniture, or natural versus artificial yarn), and similarity of input coefficients in production (e.g., petroleum tar and gasoline, or steel bars and steel sheets). They argued that intra-industry trade was quite compatible with the Hecksher-Ohlin theory in the case of goods in the former category, but not in the case of the latter. They presented the thesis that only increasing returns to scale and product differentiation could account for such trade, as well as trade in a third category of goods (such as automobiles and cigarettes) which were characterized by both substitutability in consumption and similarity of input coefficients.

An outcome of this work was the development by Krugman (1979, 1980, 1982) of a model incorporating product differentiation and (internal) economies of scale, based on that of Dixit and Stiglitz (1977). A drawback of this model, however, is that it requires the imposition of strong “symmetry” conditions in order to allow for existence of equilibrium, and as pointed out in Chipman (1982b) this in effect requires one to assume that preferences are functionally dependent on the technology. It is a good question, then, as to whether models which are logically overdetermined can play a useful role as descriptors of the real world.

An alternative and very interesting approach was introduced by Lancaster (1980) who recognizes the need for a model to be internally consistent and discussed the Nash equilibria of a trade model in which products, conceived as bundles of characteristics, are economic variables in the Chamberlinian sense. He concluded that intra-industry trade could be expected to occur even between countries which are identical in all respects.

One thing that has been overlooked in the literature on monopolistic competition in international trade is the possibility that intra-industry trade could be explained by the standard Haberler-Lerner-Samuelson model. The argument used by Grubel and Lloyd (1975, p. 88) was that if production functions were identical as between industries (as well as between countries), production-possibility surfaces would be flat (in accordance with Theorem 9 above—and as noted by Lerner (1933) in an observation he attributed to Joan Robinson). They also argued (p. 89) that these constant rates of transformation would be the same between countries (which would be true only if endowment ratios were the same). They concluded that there could be no trade under such circumstances; but in fact, since the supply (Rybczynski) correspondences are multi-valued in this case, the correct conclusion is that the amount of trade is indeterminate. If endowment ratios differed only slightly, it is quite apparent from the geometry of the situation that there would be a great amount of trade.

The situation can be depicted in terms of the well-known “Lerner diagram” (Figure 13). Assume two countries to be mirror images of one another in their factor endowments, and to produce commodities whose production functions (identical as between countries) are mirror images of one another. Assume further that preferences are identical and homothetic, and symmetric as between the two commodities. Then owing to the symmetry, prices of the products will be equal in equilibrium. The upper contour sets $A_j(Y^k/p_j)$ for the two countries (with $Y^1 = Y^2$) are shown in the figure, as are the factor endowment vectors l^1 and l^2. The equilibrium resource allocations $v^k_{\cdot j}$ are shown as in Figure 7. Now suppose that the two production techniques become more similar, and so as to preserve the equilibrium prices suppose the production functions remain mirror images of one another. The new upper contour sets are denoted $A'_j(Y^k/p_j)$, and the new resource allocations $v^{k\prime}_{\cdot j}$. It is clear that after the production techniques have become closer, each country will allocate a larger proportion of its factors to its export industry. If it is further assumed that the countries have Mill-Cobb-Douglas preferences, hence spend one-half of their incomes on each commodity, since the prices are equal each country will devote one-half of its income to its exportable both before and after the change, but a larger proportion of its national product will be composed of exportables after than before the change. It follows that each country will export a larger proportion of its export good when production functions are more similar than when they are dissimilar. If only one of the production functions becomes more similar to the other (no matter which one) it can be shown by a more complicated argument that the conclusion still holds.

What bearing does this have on intra-industry trade? If the criterion for aggregation is similarity of production functions, then when production functions are dissimilar, the commodities will likely be classified in different industries, whereas if they are similar they will be grouped into the same industry. By the above reasoning, one would then expect to observe more intra-industry than inter-industry trade. While extension to many commodities and factors requires careful analysis, this heuristic reasoning suggests that the HLS model would predict intra-industry trade, contrary to received opinion (cf. Lancaster, 1980; Helpman and Krugman, 1985).

Quite a different criticism of received opinion is suggested by Finger (1975), who builds upon the n-commodity–2-factor model developed by Jones (1956), Melvin (1968), and Bhagwati (1972). Bhagwati pointed out, by way of correcting Jones, that because of the production indeterminancy when $n > 2$ (the non-strict concavity to the origin of production-possibility frontiers), when factor rentals are equalized between countries it is not necessarily the case that the ranking of commodities by factor ratios will correspond to the actual trade pattern for some dividing line within the ranking. Finger did not rely on this argument, but assumed non-equalization of factor rentals and thus a unique ranking of commodities by "exportability," to coin a term. Except for commodities near the dividing line, he noted that if commodities were grouped into industries according to factor intensities, there would be no intra-industry trade. His main point was that, in fact, commodities are not grouped into industries according to similarity of factor intensities.

Finger's argument does not carry over to the case of more than two factors. Here, then, there is a grey area since little is known concerning either (a) actual "closeness" of vectors of technical coefficients within as opposed to between industries or (b) what in fact the theory would predict in the general case of more than two commodities and factors.

Generalizations of the "Ricardian" (1-factor) and "Heckscher-Ohlin" (2-factor) theories to the case of a continuum of commodities were developed by Dornbusch, Fischer, and Samuelson (1977, 1980). While these contributions provide interesting results and promising new techniques of analysis, they contain no surprises concerning the predicted pattern of trade.

The other noteworthy approach towards generalizing the Heckscher-Ohlin theorem is that introduced independently by Dixit and Norman (1980) and Deardorff (1980, 1982), and further developed by Dixit and Woodland (1982). If $\mathbf{p}^k$ is the price vector of country k under autarky and $\mathbf{z}^k$ its net-import vector under free trade, then by applying the revealed-preference criterion (assuming identical homothetic preferences) to country k's trade-preferences, we have $\mathbf{p}^k \cdot \mathbf{z}^k \geq 0$. If there are two countries then from $\mathbf{z}^1 + \mathbf{z}^2 = 0$ we have $(\mathbf{p}^k - \mathbf{p}^i) \cdot \mathbf{z}^k \geq 0$ for $i \neq k$, i.e., there is a positive correlation between net imports and differences in autarky prices. Deardorff (1982) obtained an analogous relation between autarky factor rentals and factor content of net imports, assuming identical tastes and technologies among countries. These are elegant results; but their practical usefulness is limited by the fact that autarky prices are not observed, and that one is usually interested in explaining actual trade patterns. Dixit and Woodland (1981) and Woodland (1982, p. 205) derived sufficient conditions for an increase in one (physical) endowment to cause an increase in a country's export of the corresponding commodity.

The contributions of Linder (1961), Posner (1961), Hufbauer (1966) and Vernon (1966) offer some important new ideas concerning the role of dynamics in the determination of trade patterns. Common to these is the idea that a country must develop an internal market first for a product, then realize economies of scale in it, before it can export it. This type of idea of course can be traced back to List (1910). Vernon (1966) in his interesting Schumpeterian analysis developed the concept of a "product cycle": a country with a head-start can export a new product, but because of imitation by other countries must keep innovating to retain its technological lead and hence its comparative advantage in innovative products. For an interesting formalization see Jensen and Thursby (1986).

Finally one must note that there is a natural extension of laws of comparative advantage to the intertemporal sphere. Trade imbalances can be considered as intertemporal trades. In a model of one traded commodity and two periods, one may consider trade between present and future goods. A definite example has been presented in Chipman (1985b) in which each country has two factors (capital and labor) and produces two commodities (capital good and consumer good). The capital good is not traded, but used to augment the capital stock in the next period. If preferences as between the present and future good are identical and homothetic within and as between countries, and if production functions are identical between countries and have the property that the consumer-good industry uses a higher capital-labor ratio than the capital-good industry (a condition introduced by Uzawa, 1964a), then it follows that the country with the higher initial endowment of capital to labor will "export" the present good (i.e., lend) to the other country, and "import" the future good (i.e., be repaid by the other country in the next period).

Obviously such models do not allow for rescheduling of debts. But in the simplest case they remind us that balanced trade is not to be expected, and is in fact far from optimal. It would be far more efficient for a theory of international trade to deal simultaneously with the problem of predicting trade flows and that of predicting trade balances as well. The field of "international finance," or "balance-of-payments theory," may be thought of as the field of intertemporal international-trade theory, but usually simplified to allow for only one commodity (and no production). A goal for the future is the development of a theory that simultaneously accommodates many commodities, factors, countries, and periods.

9 The transfer problem

The term "transfer problem" originated in the 1924 report of the Committee of Experts on Reparations chaired by Gen. Charles G. Dawes (later that year elected Vice President of the United States), in charge of recommending the reparations payments of 1 billion gold marks to be paid by Germany to the Allies. The term initially referred to the problem of converting the German funds into foreign currency, as distinguished from the "budgetary problem" of first raising the funds by taxation. Following Keynes's (1929) formulation, the term has come to encompass all the structural dislocations involved in carrying out the transfer, including changes in the exchange rate and the terms of trade.

It was observed by Smith (1776, Book IV, Ch. I; Vol. II, p. 21) that a country could finance foreign expenditures by an export surplus, without the need to lose bullion reserves. On the other hand it was held by Thornton (1802, p. 139; 1939, p. 156) that:

> the immediate cause ... of the exportation of our coin has been an unfavourable exchange, produced partly by our heavy [foreign] expenditure, though chiefly by the superadded circumstance of two successively bad harvests.

And that in the case of the latter (1802, p. 131; 1939, p. 151):

> In order ... to induce the [foreign] country ... to take all its payments in goods, and no part of it in gold, it would be

requisite not only to prevent goods from being excessively dear, but even to render them excessively cheap.

This could be interpreted as saying that not only would the paying country's exchange rate have to depreciate but its terms of trade would have to deteriorate. It should be noted that this reasoning was applied by Thornton to the effect of a harvest failure (a decline in British production of importables) rather than to the effect of a transfer; a deterioration of Britain's terms of trade is to be expected in the former case but not necessarily in the latter.

King (1804) analyzed the widely-held opinion that the depreciation of the Irish relative to the British pound could be attributed to the rent payments by Irish tenants to absentee landlords residing in England. He pointed out (p. 86; 1844, p. 108):

> The residence of Irish proprietors in England has the necessary effect of diminishing the Irish imports, because the expenditure of revenue is transferred to another country; and it also increases the export of that produce which is no longer consumed at home.

The same reasoning was used by Foster (1804)—who referred to King (1804)—to argue that Britain's foreign expenditures produced the required export surplus (p. 18):

> ...that part of the money lent, which was destined for foreign expenditure, was necessarily sent out either in specie or in bills of exchange, but, in each case, *necessarily forced* the exportation of British produce to that amount, to pay for these bills of exchange.

The same conclusion—but without the specific reasoning—was also reached by Ricardo (1811; 1951, III, p. 63), after a lengthy discussion of Thornton's case of the effect of a bad harvest:

> If, which is a much stronger case, we agreed to pay a subsidy to a foreign power, money would not be exported whilst there were any goods which could more cheaply discharge the payment. The interest of individuals would render the exportation of the money unnecessary.

The qualifying phrase suggests that Ricardo thought Thornton was on firmer ground in the case of a bad harvest.

Malthus (1811a, pp. 344–5) took up Thornton's example of "a bad harvest, or . . . a large subsidy to a foreign power" and argued against Ricardo that

> . . . if the debt for the corn or the subsidy . . . is paid by the transmission of bullion, . . . it is owing precisely to the cause mentioned by Mr. Thornton —the unwillingness of the creditor nation to receive a great additional quantity of goods not wanted for immediate consumption, without being bribed to it by excessive cheapness.

It should be noted that this would not be inconsistent with King's analysis—in the case of a bad harvest as opposed to that of a transfer. In Ricardo's later work (1817, Ch. VI, pp. 162–72; 1951, Ch. VII, pp. 137–42)—no doubt influenced by his discussions with Malthus—his views shifted somewhat in the direction of those of Thornton and Malthus, for he allowed for price divergences between countries—due necessarily to transport costs—and for movements of bullion in response to disturbances.

Although representing himself as a disciple of Ricardo, Mill (1844, p. 42)—see also Mill (1848, II, Book III, Ch. XXI, §4, pp. 166–7)—presented a doctrine that was much closer to those of Thornton and Malthus, or at least closer to late (1817) than to early (1811) Ricardo:

> When a nation has regular payments to make in a foreign country, for which it is not to receive any return, its exports must annually exceed its imports by the amount of the payments which it is bound so to make. In order to force a demand for its exports greater than its imports will suffice to pay for, it must offer them at a rate of interchange more favourable to the foreign country, and less so to itself, than if it had no payments to make beyond the value of its imports.

Mill continued with his doctrine of the secondary burden (p. 43):

> Thus the imposition of a tribute is a double burthen to the country paying it, and a double gain to that which receives

> it. The tributary country pays to the other, first, the tax, whatever be its amount, and next, something more, which the one country loses in the increased cost of its imports, the other gains in the diminished cost of its own.

Taussig (1917, 1927) in his writings attributed the above theory to Ricardo as well as Mill; this attribution was challenged by Viner (1924, p. 203), but Viner went too far in attributing the theory to Thornton. Mill held unequivocally that a transfer would worsen the paying country's terms of trade. Still, there is no question that Thornton, Malthus, and Mill were in the camp of those who believed that a transfer would entail changes in relative prices, whereas King, Foster, and Ricardo (1811) were in the camp of those who believed that it need not.

After a long hiatus, the topic of the transfer problem was taken up again by Bastable (1889, pp. 12–16) and Nicholson (1903, II, pp. 289–91), both of whom criticized Mill and essentially restated the arguments of King and Foster (but without reference to these authors). In view of these criticisms one could ask: was Mill's doctrine simply the result of a blunder? that of confusing the effect of a unilateral transfer—in which purchasing power is simply redistributed from one country to another—with that of a harvest failure—in which the world supply of our country's importable is diminished, resulting in its price rising relative to that of the exportable? In support of such an interpretation is the fact that Marshall (1923, p. 349) subsequently made precisely such a blunder—as was first noted by Robertson (1931, p. 179)—in depicting a transfer by a shift in the paying country's offer curve, forgetting to shift that of the receiving country. Against such an interpretation is the fact that the King-Foster-Bastable-Nicholson explanation tacitly assumes that all goods are tradable at zero transport costs, whereas Mill was not such a fool as to believe that, under that assumption, it would make sense to say of a transfer (made in money): "This lowers prices in the remitting country, and raises them in the receiving" (Mill, 1848, II, Book III, Ch. XXI, §4, p. 167).

It was the great accomplishment of Taussig (1917) to introduce an idealization—the distinction between tradable goods ("international commodities") having zero transport costs, and nontradables ("domestic commodities") with effectively infinite transport costs—that could provide a definite interpretation of Mill's doctrine. In discussing the effect of a capital movement from Britain to the United States, the following hint was thrown out (pp. 396–7):

> . . . exporting industries in the United States . . . decline; . . . Less commodities are exported. More domestic commodities are made . . .

Taussig's theory was more fully developed in his book (1927, p. 35, Ch. 26), and it led to an interesting response by Wicksell (1918) and to a number of empirical investigations and further conceptual developments including those of Williams (1920), Graham (1922, 1925), Viner (1924), Wilson (1931), and White (1933). Graham's analysis of the transfer problem (1925, pp. 213–14)—in terms of adjustment of relative prices of tradable and nontradable commodities, and consequent resource reallocations between these sectors, foreshadowed much of Ohlin's subsequent treatment. Taussig's account was further developed by Viner (1937, pp. 323-65).

The most important contribution following Taussig's was that of Ohlin (1928), who followed the King-Foster-Bastable-Nicholson line of argument but with the noteworthy addition of nontradable commodities (which he called "home-market goods," p. 6). He also assumed that each country produced importables as well as exportables and home-market goods, in contrast to Taussig whose analysis took account only of resource allocation between the export and domestic sectors. Ohlin showed that in this model a transfer would result in resource reallocation into home-market industries out of *both* export and import-competing industries in the receiving country, with the reverse movement in the paying country. He concluded, as had Graham (1925) before him, that there would be a tendency for the prices of home-market goods to rise relatively to those of international goods in the receiving country, and fall in the paying country (this would now be described as an appreciation of the receiving country's "real exchange rate"). Thus, there need not be any change in the terms of trade. He went on to suggest that in a "progressive country" the resource allocation would proceed smoothly so that in the long run no changes in relative prices of nontradables need take place (p. 10). A somewhat similar argument was presented by Cassel (1928, pp. 14–23) in the same year.

There followed the famous debate between Keynes (1929) and Ohlin (1929). Keynes noted (p. 2):

> If £1 is taken from you and given to me and I choose to increase my consumption of precisely the same goods as those of which you are compelled to diminish yours, there is no Transfer Problem.

Arguing that this was not the correct representation of the facts, he stated that the transfer problem consisted (for the paying country) in "the diversion of production out of other employments into the export trades (or to produce goods previously imported)," and he regarded the difficulties in accomplishing this as considerable. He expressed the opinion that this could be accomplished in the case of German reparations payments by a devaluation of the mark, but observed that this course of action was forbidden by the Dawes Committee. It followed that Germany would have to undergo a painful deflation. Keynes's view appeared to be closer to Thornton's than to Mill's, since the difficulty of resource reallocation between tradables and nontradables played a larger role than the terms of trade. In the *Treatise* (1930, I., p. 334) he stressed the importance of the "terms of trade," but this term was defined as the ratio of the "money-rate of earnings" in the two countries, which is not the same as the ratio of the paying country's export to import prices.

The first formal model of the transfer problem was that of Pigou (1932), who considered the case of two tradable commodities and two countries each with separable Jevonian utility functions. He derived a condition for a transfer to worsen the paying country's terms of trade, expressed in terms of demand elasticities. Except for Metzler's (1942) treatment in terms of the Keynesian multiplier model of underemployment equilibrium—which lies completely outside the traditional discussion of the transfer problem—no further formalization took place until Samuelson's (1952, 1954) classic paper on the subject. Dispensing with Pigou's assumption of separable utilities, but retaining Pigou's formulation in terms of indirect (inverse) demand functions, Samuelson generalized Pigou's criterion and simplified it to the following form: for the transfer to worsen the paying country's terms of trade, it is necessary and sufficient (provided dynamic stability holds) that the paying country's marginal propensity to consume its export good be greater than the receiving country's marginal propensity to consume this same good (1954, p. 285).

Samuelson's result was obtained much more simply by Mundell (1960) using direct rather than indirect demand functions. The analysis may be most simply set out in terms of trade-demand functions (Section 6 above). If two countries with a common currency are trading two commodities, and country 1 makes a transfer of T currency units to country 2, world equilibrium is defined by the equation

$$\hat{h}_2^1(p_1, p_2, -T; l^1) + \hat{h}_2^2(p_1, p_2, T; l^2) = 0, \tag{9.1}$$

which states that the world excess demand for commodity 2 is zero; a similar equation follows for commodity 1 by the balance-of-payments constraints $p_1^k \hat{h}_1^k + p_2^k \hat{h}_2^k = D^k$ $(k = 1, 2)$. Supposing country 1 to be exporting commodity 1 to country 2, and choosing the price p_1 of commodity 1 as numéraire and setting it equal to $\bar{p}_1$, the above equation (for fixed endowment vectors l^1 and l^2) defines implicitly the function $p_2 = \bar{p}_2(T)$. We find that, assuming $\hat{h}_2^1$ and $\hat{h}_2^2$ to be differentiable,

$$\frac{d\bar{p}_2}{dT} = \frac{\partial \hat{h}_2^1/\partial D^1 - \partial \hat{h}_2^2/\partial D^2}{\partial \hat{h}_2^1/\partial p_2 + \partial \hat{h}_2^2/\partial p_2}. \tag{9.2}$$

Dynamic stability of equilibrium requires that if p_2 is above (resp. below) its equilibrium value $\bar{p}_2(T)$, it should fall (resp. rise); i.e., $\dot{p}_2 \equiv dp_2/dt$ should have the opposite sign to $p_2 - \bar{p}_2(T)$. If we assume that $\dot{p}_2$ has the same sign as the world excess demand for commodity 2, then stability will hold provided world excess demand is negative for $p_2 > \bar{p}_2(T)$ and positive for $p_2 < \bar{p}_2(T)$. In the neighborhood of $\bar{p}_2(T)$ this requires that the world excess demand for commodity 2 be a decreasing function of p_2, i.e., assuming differentiability,

$$\frac{\partial}{\partial p_2}(\hat{h}_2^1 + \hat{h}_2^2) = \frac{\partial \hat{h}_2^1}{\partial p_2} + \frac{\partial \hat{h}_2^2}{\partial p_2} < 0. \tag{9.3}$$

In conjunction with (9.2) this implies that a transfer will worsen the paying country's terms of trade if and only if

$$\frac{\partial \hat{h}_2^2}{\partial D^2} > \frac{\partial \hat{h}_2^1}{\partial D^1}, \tag{9.4}$$

i.e., if and only if country 2 will spend a larger amount of externally-received funds on its own export good than will country 1 on the same (its import) good. When there are no nontradables, hence $\partial\hat{h}_2^k/\partial D^k = \partial h_2/\partial Y^k$, this reduces to Samuelson's criterion (1952, p. 286; 1954, p. 284).

The stability condition (9.3) is usually expressed in terms of elasticities. Defining

$$\eta^k = -\frac{p_j}{\hat{h}_j^k}\frac{\partial\hat{h}_j^k}{\partial p_j}\;(j \neq k), \quad \hat{m}_j^k = p_j^k\frac{\partial\hat{h}_j^k}{\partial D^k}, \quad \delta^k = -\frac{D^k}{\hat{h}_k^k}\frac{\partial\hat{h}_k^k}{\partial D^k} \quad (k = 1, 2) \tag{9.5}$$

and using (and differentiating) the balance-of-payments constraints and the homogeneity of degree 0 of the functions $\hat{h}_j^k$ in p_1, p_2, D^k, we may write (9.2) in the alternative form

$$\frac{d\bar{p}_2}{dT} = \frac{\hat{m}_2^2 - \hat{m}_2^1}{\hat{h}_2^1\Delta} \tag{9.6}$$

where

$$\Delta = -\frac{p_2}{\hat{h}_2^1}\frac{\partial(\hat{h}_2^1 + \hat{h}_2^2)}{\partial p_2} = \eta^1 + \frac{p_1\hat{h}_1^2}{-p_2\hat{h}_2^2}(\eta^2 - 1) + \delta^2. \tag{9.7}$$

Condition (9.3) then translates into the stability condition $\Delta > 0$. Formula (9.7) generalizes the well-known formula $\Delta = \eta^1 + \eta^2 - 1 > 0$ obtained in (7.18) above , which Hirschman (1949) called the "Marshall-Lerner condition." For $T \neq 0$, (9.7) corrects Hirschman's expression, which omitted the term δ^2.

If there are no nontradables then, as observed in Section 6, $\partial\hat{h}_j^k/\partial D^k = \partial h_j^k/\partial Y^k$, and the condition that a transfer leave the terms of trade unchanged reduces to the condition that both countries have the same marginal propensities to consume each commodity. As Keynes noted, in this case there is no transfer problem.

Let us consider the general case in which we distingush three categories of commodities in country k: n_1^k tradables and n_3^k nontradables produced and consumed in country k by means of m^k primary factors, and n_2^k tradables imported but not produced by country k. The country's trade-demand function may be obtained as follows. Let $\mathbf{p}_r^k = (p_{r1}^k, p_{r2}^k, \ldots, p_{rn_r^k}^k)'$ denote the vector of n_r^k prices of commodities in category r $(r = 1, 2, 3)$ and let $\mathbf{y}_r^k = (y_{r1}^k, y_{r2}^k, \ldots, y_{rn_r^k}^k)'$ denote the vector of n_r^k outputs of commodities in category r $(r = 1, 3)$, in country k. Let $\mathbf{h}_r^k(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3^k, Y^k)$ denote country k's aggregate demand function for the n_r^k commodities in category r, as a function of the three sets of prices and disposable income Y^k (including any transfers). Finally, let $\mathbf{w}^k$ denote the vector of country k's m^k factor rentals. Then the first set of n_3^k equations states that the demand for nontradables equals the supply:

$$(9.8a) \qquad \mathbf{h}_3^k(\mathbf{p}_1^k, \mathbf{p}_2^k, \mathbf{p}_3^k, \Pi^k(\mathbf{p}_1^k, \mathbf{p}_3^k, l^k) + D^k) = \mathbf{y}_3^k.$$

The second set of $n_1^k + n_3^k$ equations states that prices are equal to minimum unit costs for all produced commodities:

$$(9.8b) \qquad \mathbf{g}_k^k(\mathbf{w}^k) = \mathbf{p}_k^k, \quad \mathbf{g}_3^k(\mathbf{w}^k) = \mathbf{p}_3^k.$$

The third set of m^k equations states that the demand for factors of production is equal to the supply:

$$(9.8c) \qquad \mathbf{B}_1^k(\mathbf{w}^k)\mathbf{y}_1^k + \mathbf{B}_3^k(\mathbf{w}^k)\mathbf{y}_3^k = l^k.$$

Here, $\mathbf{B}_r^k(\mathbf{w}^k)$ denotes the $m^k \times n_r^k$ matrix of factor-output ratios $b_{r,ij}^k = \partial g_{rj}^k / \partial w_i^k$, where $g_{rj}^k(\mathbf{w}^k)$ is the minimum-unit-cost function for the jth commodity in category r $(r = 1, 3)$.

Treating $\mathbf{p}_1, \mathbf{p}_2, D^k$, and l^k as parameters, the remaining variables $\mathbf{p}_3^k, \mathbf{w}^k, \mathbf{y}_1^k, \mathbf{y}_3^k$ constitute $n_3^k + m^k + n_1^k + n_3^k$ unknowns, which is equal to the number of equations in (9.8). This determines four "reduced-form" functions $\tilde{\mathbf{p}}_3^k(\cdot), \tilde{\mathbf{w}}^k(\cdot), \tilde{\mathbf{y}}_2^k(\cdot), \tilde{\mathbf{y}}_3^k(\cdot)$ whose arguments are $(\mathbf{p}_1, \mathbf{p}_2, D^k, l^k)$. The trade-demand functions are then defined for $r = 1, 2$ by

$$(9.9) \quad \hat{\mathbf{h}}_r^k(\cdot) = \mathbf{h}_r^k(\mathbf{p}_1, \mathbf{p}_2, \tilde{\mathbf{p}}_3^k(\cdot), \Pi^k(\mathbf{p}_1, \tilde{\mathbf{p}}_3^k(\cdot), l^k) + D^k) - \tilde{\mathbf{y}}_r^k(\cdot)$$

where $\tilde{\mathbf{y}}_2^k(\cdot) = 0$. It was shown in Chipman (1981) that these are generated by a maximizing trade-utility function $\hat{U}^k(\mathbf{z}_1^k, \mathbf{z}_2^k)$ subject to the balance-of-payments constraint $\mathbf{p}_1 \cdot \mathbf{z}_1^k + \mathbf{p}_2\mathbf{z}_2^k \leq D^k$, where $\mathbf{z}_j^k = \mathbf{x}_j^k - \mathbf{y}_j^k$. Defining $\hat{\mathbf{h}}^k(\cdot) = (\hat{\mathbf{h}}_1^k(\cdot), \mathbf{h}_2^k(\cdot))', \mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)'$, and $n = n_1^k + n_2^k$, a general solution of the transfer problem is obtained by solving $n - 1$ of the n equations $\hat{\mathbf{h}}^1(\cdot) + \hat{\mathbf{h}}^2(\cdot) = 0$, where $D^1 = -T, D^2 = T$, to obtain (subject to a normalization, e.g., setting the price of one tradable equal to unity) the function $\bar{\mathbf{p}}(T)$.

Some general remarks may be made about the solution of (9.8). In the case $m^k \geq n_1^k + n_3^k$ (there are at least as many primary factors as produced commodities), it follows from Theorem 9 that, save for exceptional cases, a single-valued Rybczynski function can be defined. Accordingly, substituting $\mathbf{y}_3^k = \hat{\mathbf{y}}_3^k(\mathbf{p}_1^k, \mathbf{p}_3^k, l^k)$ in (9.8a), one can solve immediately for the function $\tilde{\mathbf{p}}_3^k(\cdot)$ and then for the trade-demand function (9.9). At the other extreme, if $m^k = n_1^k$ then one can solve the first set of cost equations of (9.8b) for $\mathbf{w}^k = (\mathbf{g}_k^k)^{-1}(\mathbf{p}_k^k)$ and substitute into the second to obtain $\tilde{\mathbf{p}}_3^k(\cdot)$. These functions are then substituted in (9.8a) and (9.8c). The intermediate case $n_1^k < m^k < n_1^k + n_3^k$ presents much greater difficulties for explicit computation of solutions (cf. Chipman, 1980).

Some examples in the case of two tradables ($n_1^k + n_2^k = 2$) and one nontradable ($n_3^k = 1$) are of particular interest. To take the analytically simplest case first ($m^k \geq n_1^k + n_3^k$) we may distinguish two subcases: $(m^k, n_1^k) = (2, 1)$ and $(m^k, n_1^k) = (3, 2)$. These are illustrated in Figures 14 and 15 respectively. In Case $(2, 1)$ since each country specializes in its export good and its nontradable, and has two factors of production, each has a one-dimensional strictly-concave-to-the-origin production-possibility frontier as shown. In Case $(3, 2)$, each country produces all three commodities with three factors, and thus each has a two-dimensional strictly-concave-to-the-origin production-possibility frontier. In Case $(2, 1)$, when country 1 makes a transfer to country 2, resources are withdrawn from its nontradable sector and reallocated to its export sector, and the reverse movement takes place in country 2. World production of commodity 1 (country 1's export good) increases, and world production of commodity 2 (country 2's export good) decreases, so there is a general presumption that country 1's terms of trade will deteriorate, in accordance with the "orthodox" Mill-Taussig presumption. In Case $(3, 2)$, however, there is no reason to expect the terms of trade to change more in one direction than another; but there is a presumption that the price of the nontradable will fall relative to that of the tradables in the paying country, and rise in the receiving country. This is in accordance with Graham's (1925) account, as well as Ohlin's (1928)—at least in the short run. It also appears fully consistent with Keynes's analysis (1929, 1930).

Before proceeding to the analytics, it is worth considering two other cases, in which $m^k = n_1^k$. Again, we may distinguish two subcases: $(m^k, n_1^k) = (1,1)$ and $(m^k, n_1^k) = (2,2)$. These are illustrated in Figures 16 and 17 respectively. In Case $(1,1)$, each country has a linear production-possibility frontier as between the exportable and the nontradable. As in Case $(2,1)$, there is a strong presumption in favor of the "orthodox" doctrine; in fact, this case may be thought of as the one Mill had in the back of his mind. Case $(2,2)$, however, is quite different. Each country's production-possibility surface is ruled. As resources move between the tradables and nontradables sectors, it is possible for the movement to take place along the ruled rather than curved segments of the surface; in this case, there is no reason for any relative prices to change. This corresponds to Ohlin's view of the long-run outcome of a transfer, as well as with Cassel's (1928) view that capital movements will not disturb relative prices, and therefore purchasing-power parities. Contrasting Figures 15 and 17, we may say that the dispute between Keynes and Ohlin was really a dispute (if they had only realized it!) as to whether the production-possibility surface was or was not strictly concave to the origin.

Let us proceed to the analytics. In both the cases (2,1) and (3,2), the function $\tilde{p}_3^k(p_1, p_2, D^k, l^k)$ is defined implicitly by

$$(9.10)\quad h_3^k(p_1, p_2, \tilde{p}_3^k(\cdot), \Pi^k(p_1, p_2, \tilde{p}_3^k(\cdot), l^k) + D^k) = \hat{y}_3^k(p_1, p_2, \tilde{p}_3^k(\cdot), l^k).$$

Defining the Slutsky terms, transformation terms, and consumption coefficients by

$$(9.11)\qquad s_{ij}^k = \frac{\partial h_i^k}{\partial p_j^k} + \frac{\partial h_i^k}{\partial Y^k} h_j^k, \quad t_{ij}^k = \frac{\partial \hat{y}_i^k}{\partial p_j^k}, \quad c_i^k = \frac{\partial h_i^k}{\partial Y^k},$$

we find that $\partial \tilde{p}_3^k / \partial D^k = -(s_{33}^k - t_{33}^k)^{-1} c_3^k$ (hence an inward transfer raises the price of the nontradable provided it is a superior good), hence defining the trade-demand functions as in (9.9) we obtain

$$(9.12)\quad \hat{c}_i^k \equiv \partial \hat{h}_i^k / \partial D^k = c_i^k - (s_{i3}^k - t_{i3}^k)(s_{33}^k - t_{33}^k)^{-1} c_3^k \quad (i = 1, 2)$$

where, in the case $(m^k, n_1^k) = (2,1)$, $t_{i3}^k = 0$ for $i \neq k$. The condition for the "orthodox" presumption (9.4) can therefore be stated as $\hat{c}_2^1 - \hat{c}_2^2 < 0$.

A set of sufficient conditions is readily obtained (cf. Chipman, 1974). Suppose it is assumed that in the case of fixed production, hence pure exchange, a transfer will leave the terms of trade unchanged. In Chipman (1974a, p. 45) this was called the Hypothesis of Neutral Tastes. It states that $\hat{c}_2^1 = \hat{c}_2^2$ when $t_ij^k = 0$ for $i, j, k = 1, 2$, or

$$c_2^1 - s_{23}^1(s_{33}^1)^{-1}c_3^1 = c_2^2 - s_{23}^2(s_{33}^2)^{-1}c_3^2. \tag{9.13}$$

(An example of preferences satisfying (9.13) is identical Mill-Cobb-Douglas preferences as between the two countries.) Then sufficient conditions for $\hat{c}_2^1 < \hat{c}_2^2$ are

$$c_2^1 - (s_{23}^1 - t_{23}^1)(s_{33}^1 - t_{33}^1)^{-1}c_3^1 < c_2^1 - s_{23}^1(s_{33}^1)^{-1}c_3^1 \tag{9.14}$$

and

$$c_2^2 - s_{23}^2(s_{33}^2)^-1c_3^2 < c_2^2 - (s_{23}^2 - t_{23}^2)(s_{33}^2 - t_{33}^2)^{-1}c_3^2. \tag{9.15}$$

If $c_3^1 > 0$ and $c_3^2 > 0$, inequalities (9.14) and (9.15) are respectively equivalent to

$$\frac{s_{23}^1}{s_{33}^1} < \frac{t_{23}^1}{t_{33}^1} \quad \text{and} \quad \frac{s_{23}^2}{s_{33}^2} > \frac{t_{23}^2}{t_{33}^2}. \tag{9.16}$$

In the case $(m^k, n_1^k) = (2, 1)$ (corresponding to Figure 14), in which $t_{23}^1 = t_{13}^2 = 0$, these inequalities follow if $s_{23}^1 > 0$ and $s_{13}^2 > 0$, i.e., if in each country the importable and nontradable are Hicksian substitutes. In the case $(m^k, n_1^k) = (3, 2)$, conditions for (9.16) are more delicate, but under some special but symmetric assumptions it was found in Chipman (1974a, p. 68) that the orthodox presumption holds. A special case of case (2,1) was treated by Jones (1974b); see also Jones (1975).

Returning to the case $m^k = n_1^k$ let us first consider the case $(m^k, n_1^k) = (1, 1)$, corresponding to Figure 16. From the homogeneity of $g_i^k, g_i^k(w_1^k) = b_{ij}^k w_1^k$ where $b_{1i}^k = g_1^k(1)$, hence (9.8b) and (9.8c) yield

$$p_3^k = p_k b_{13}^k / b_{1k}^k \text{ and } b_{1k}^k y_k^k + b_{13}^k y_3^k = l_1^k \tag{9.17}$$

hence from (9.8a)

$$\tilde{y}_k^k(\cdot) = l_1^k/b_{1k}^k - (b_{13}^k/b_{1k}^k)h_3^k(p_1, p_2, p_k b_{13}^k/b_{1k}^k, l_1^k p_k/b_{lk}^k + D^k). \tag{9.18}$$

The trade-demand function (9.9) then becomes, for $i = 1, 2$,

$$\begin{aligned} \hat{h}_i^k(p_1, p_2, D^k, l_1^k) &= h_i^k(p_1, p_2, p_k b_{13}^k/b_{1k}^k, l_1^k p_k/b_{1k}^k + D^k) \\ &\quad -\delta_{ik}\tilde{y}_i^k(p_1, p_2, D^k, l_1^k) \end{aligned} \tag{9.19}$$

where $\delta_{ik} = 1$ if $i = k$ and 0 if $i \neq k$. From this we derive

$$(9.20) \qquad \frac{\partial \hat{h}_2^1}{\partial D^1} = \frac{\partial h_2^1}{\partial Y^1}, \quad \frac{\partial \hat{h}_2^1}{\partial D^2} = \frac{\partial h_2^2}{\partial Y^2} + \frac{b_{13}^2}{b_{12}^2}\frac{\partial h_3^2}{\partial Y^2}.$$

In words (multiplying all these expressions by p_2): In country 1, which does not produce commodity 2, an additional dollar given or loaned to it has the same effect on the consumption of the importable as an equivalent increase in disposable national income. In country 2, however, which produces and exports commodity 2, an additional dollar received from country 1 has not only the direct effect on demand for exportables, but also an indirect effect brought about by the diversion of resources to the nontradable and the need to compensate for the fall in production of the exportable.

Of course, it is not enough to assume identical and homothetic preferences to assure $\partial h_2^1/\partial Y^1 = \partial h_2^2/\partial Y^2$, since these functions depend on prices of nontradables, which will in general be different in the two countries. However, if preferences are generated by utility functions of the separable form $U^k(x_1^k, x_2^k, x_3^k) = F(\bar{U}^k(x_1^k, x_2^k), x_3^k)$ and if the $\bar{U}^k$ are identical and homogeneous as between the two countries, then $\partial h_2^1/\partial Y^1 = \partial h_2^2/Y^2$ (cf. Chipman, 1974, pp. 47–9), and provided the nontradables are superior goods in both countries, (9.20) yields the "orthodox" presumption (9.4). This may be identified with Mill's doctrine, particularly since it is consistent with the labor theory of value and with Mill's assumption of constant expenditure shares.

The case $(m^k, n_1^k) = (2, 2)$ remains to be considered. Solving the first set of equations of (9.8b) for $\mathbf{w}^k$ and substituting in the second, we obtain $\tilde{p}_3^k(p_1, p_2)$; likewise, substituting in the $b_{ij}^k(w_1, w_2)$ we obtain the functions $\hat{b}_{ij}^k(p_1, p_2)$. From (9.8a) we then have

$$\tilde{y}_3^k(p_1, p_2, D^k, l^k) = h_3^k(p_1, p_2, \tilde{p}_3^k(p_1, p_2), \Pi^k(p_1, p_2, \tilde{p}_3^k(p_1, p_2), l^k) + D^k) \qquad (9.21)$$

whence $\partial \tilde{y}_3^k/\partial D^k = \partial h_3^k/\partial Y^k$. From (9.8c) we have

$$(9.22) \quad \begin{bmatrix} \tilde{y}_1^k(\cdot) \\ \tilde{y}_2^k(\cdot) \end{bmatrix} = \begin{bmatrix} \hat{b}_{11}^k(\cdot) & \hat{b}_{12}^k(\cdot) \\ \hat{b}_{21}^k(\cdot) & \hat{b}_{22}^k(\cdot) \end{bmatrix}^{-1} \left[\begin{pmatrix} l_1^k \\ l_2^k \end{pmatrix} - \begin{pmatrix} \hat{b}_{13}^k(\cdot) \\ \hat{b}_{23}^k(\cdot) \end{pmatrix} \tilde{y}_3^k(\cdot) \right]$$

whence, using (9.9), we find that

$$\frac{\partial \hat{h}_2^k}{\partial D^k} = \frac{\partial h_2^k}{\partial Y^k} - \frac{\partial \tilde{y}_2^k}{\partial D^k} = \frac{\partial h_2^k}{\partial Y^k}$$

$$+ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_{11}^k(\cdot) & \hat{b}_{12}^k(\cdot) \\ \hat{b}_{21}^k(\cdot) & \hat{b}_{22}^k(\cdot) \end{bmatrix}^{-1} \begin{bmatrix} \hat{b}_{13}^k(\cdot) \\ \hat{b}_{23}^k(\cdot) \end{bmatrix} \frac{\partial h_3^k}{\partial Y^k}. \tag{9.23}$$

From this we may easily derive a sufficient condition for a transfer to leave all relative prices unaffected. Suppose production functions are identical in the two countries, and that factor rentals are initially equalized; then the $\hat{b}_{ij}^k(\cdot)$ are the same for $k = 1, 2$, and the prices of the nontradables are equal, i.e., $p_3^1 = p_3^2$. Accordingly,

$$\frac{\partial \hat{h}_2^1}{\partial D^1} - \frac{\partial \hat{h}_2^2}{\partial D^2} = \frac{\partial h_2^1}{\partial Y^1} - \frac{\partial h_2^2}{\partial Y^2}$$

$$+ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{b}_{11}(\cdot) & \hat{b}_{12}(\cdot) \\ \hat{b}_{21}(\cdot) & \hat{b}_{22}(\cdot) \end{bmatrix}^{-1} \begin{bmatrix} \hat{b}_{13}(\cdot) \\ \hat{b}_{23}(\cdot) \end{bmatrix} \left(\frac{\partial h_3^1}{\partial Y^1} - \frac{\partial h_3^2}{\partial Y^2} \right). \tag{9.24}$$

If it is assumed that preferences as between the two countries are identical and homothetic, given that the prices are the same the expression (9.24) vanishes. This may be identified with Ohlin's (1928) case—for the long run. In Figure 17, the direction of movement on the ruled production-possibility surfaces is along the ruled line segments.

Of course, if either production functions or preferences differ as between the two countries, the directions of movement in Figure 17 will be along the curved segments of the production-possibility surfaces, similar to those of Figure 15 except that the curvature and hence the price changes will not be so great. In this case there appears to be no presumption one way or the other with regard to the terms of trade (as opposed to the "real exchange rate"). The outcome depends on the ranking of factor intensities by industry (cf. Chipman, 1974, p. 61).

When preferences can be aggregated, the utility function which generates these preferences can usually be taken as a welfare indicator for some welfare criterion; for example, if preferences are identical and homothetic, the aggregate utility function is an indicator of potential welfare. In terms of the indirect trade-utility function

$$\hat{V}^k(p_1, p_2, D^k, l^k) = \hat{U}(\hat{\mathbf{h}}^k(p_1, p_2, D^k, l^k)) \tag{9.25}$$

we may define country 1's welfare as a function of the transfer by

$$W^1(T) = \hat{V}^1(\bar{p}_1, \bar{p}_2(T), -T, l^1). \tag{9.26}$$

Then a simple computation shows that

$$\frac{\partial W^1}{\partial T} = -\frac{\partial \hat{V}^1}{\partial D^1}\left[1 + z_2^1 \frac{d\bar{p}_2}{dT}\right]. \tag{9.27}$$

The bracketed term indicates the primary and secondary burden of the transfer on the paying country. Note that even if there are great dislocations involving a change in the real exchange rate—of the kind Keynes envisaged—there is no secondary burden unless the terms of trade deteriorate.

The general case of n tradable commodities can be treated similarly, yielding the expression (where one of the $\bar{p}_i$s is a constant and the rest are functions of T):

$$\frac{\partial W^1}{\partial T} = -\frac{\partial \hat{V}^1}{\partial D^1}\left[1 + \sum_{i=1}^{n} z_i^1 \frac{d\bar{p}_i}{dT}\right]. \tag{9.28}$$

Here, the "secondary burden" is measured by a change in the *difference* between an import and an export price index, rather than a *ratio* of these—showing incidentally that the usual procedure of measuring a country's terms of trade as the ratio of its export to its import prices is inappropriate (a difference between two variables is never a monotone function of their ratio unless the variables are equal to one another, in which case the difference and ratio are both constant).

Analysis of the transfer problem in the multi-commodity case is fairly straightforward and need not be taken up here; cf. Chipman (1980, 1981). Space does not allow discussion of the multi-country transfer problem and the associated "transfer paradoxes"; for a good summary of the state of the subject see Dixit (1983).

10 The theory of tariffs and quotas

The theory of the effect of tariffs and other trade barriers on the conditions of trading countries goes back to Torrens (1844, pp. 331–356) and Mill (1844, pp. 21–32) who showed that a country can improve its terms of trade by imposing a tariff or an export tax. Mill distinguished between a protecting and a non-protecting duty, the former being one sufficiently large to induce the country imposing it to start producing the import-competing good. He asserted (pp. 26–7) that there would be no advantage from a protecting duty; this followed from his assumption of constant costs, according to which a protective tariff would be equivalent to a prohibitive one.

The theory was briefly developed by Edgeworth (1894) in terms of Marshall's (1879) offer curves, but his treatment contained a flaw later uncovered by Lerner (1936)—the allegation of lack of symmetry between import and export taxes.

The next important step was the contribution by Bickerdike (1907), who established the proposition that a country could gain from a sufficiently small tariff, and could optimize its gains by a suitable choice of tariff rate. Bickerdike's theory—presented with extreme terseness—was greatly clarified by Edgeworth (1908). Bickerdike also noted what came later to be known as "Lerner's symmetry theorem."

Marshall (1923) analyzed tariffs in terms of his offer curves, and noticed that if the foreign country's offer curve is inelastic a tariff will lead to an increase in amounts of both commodities available to consumers, and would thus constitute a clear gain to the country. In most essential respects this observation had already been made by Mill (1844, p. 22).

The modern theory of tariffs starts with Lerner's (1936) contribution, continues with Stolper and Samuelson's (1941) fundamental work, two important papers by Metzler (1949a, 1949b) and one by Bhagwati (1959), two major developments of Bickerdike's theory by Graaff (1949) and Johnson (1950), and an important contribution by Johnson (1960) further developed by Bhagwati and Johnson (1961) and Rao (1971). Also noteworthy are the expositions by Mundell (1960) and Jones (1969). The theory of quotas is less well developed—most of the theoretical analysis being of a partial-equilibrium nature; the primary reference is Bhagwati (1965). The traditional theory takes trade restrictions such as tariffs or quotas as exogenously-controlled instruments and examines their effects. In recent years there has been a great deal of interest in the opposite problem of explaining trade restrictions; it will not be possible to cover that literature here. However, mention should be made of the important model of retaliatory tariff behavior and equilibrium originated by Johnson (1954) and developed by Gorman (1958), Panchamukhi (1961), Kemp (1964, Ch. 15), Horwell (1966), Kuga (1973), Otani (1980), Mayer (1981), Riezman (1982) and Thursby and Jensen (1983), as well as the model of retaliatory quota behavior studied by Rodriguez (1974) and Tower (1975).

The treatment to follow will consist of a synthesis of contemporary theory of trade restrictions, for the case of two tradable commodities, two factors, and two countries, divided into the theory of tariffs and the theory of quotas. And each will itself be divided into the classical aggregative treatment—in which each country is perceived as acting as a single rational agent—and the disaggregative treatment introduced by Johnson (1960) in which each factor of production, as well as the government, is treated as a rational agent.

10.1 Aggregative tariff theory

It will be assumed that country 1 is to impose a tariff on its imports of commodity 2 from country 2, and that commodity 1 in country 1 uses a larger proportion of factor 1 to factor 2 than commodity 2 in the initial equilibrium.

Country 1's excess demand for its import good is defined by

$$(10.1)\quad \hat{z}_2^1(p_1^2, p_2^2, T_2; l^1) = \hat{h}_2^1(p_1^2, T_2 p_2^2, (T_2 - 1)p_2^2 \hat{z}_2^1(p_1^2, p_2 2, T_2; l^1); l^1)$$

where $\hat{h}_2^1(p_1^1, p_2^1, D^1; l^1)$ is country 1's trade-demand function (as in (6.3) above), and $T_2 = 1 + \tau_2$ is the tariff factor and τ_2 the tariff rate. Superscripts denote countries. It will be noted that the function $\hat{z}_2^1(\cdot)$ appears on both sides of the above equation, so it needs to be shown that a function $\hat{z}_2^1(\cdot)$ satisfying (10.1) exists and is unique. The existence of such a function defined locally (in a neighborhood of initial equilibrium prices and tariff factor) follows from the implicit-function theorem provided $(T_2 - 1)p_2^2 \partial \hat{h}_2^1 / \partial D^1 - 1 \neq 0$ at the initial equilibrium. More is needed for $\hat{z}_2^1(\cdot)$ to be defined globally, however. For each fixed p_1^2, p_2^2, T_2, l^1, (10.1) defines a mapping from the space of values z_2^1 of excess demand into itself, and from the principle of contraction mappings (cf. Kolmogorov and Fomin, 1957, p. 43), if $(1-1/T_2)\hat{m}_2^1$ lies in the interval $[0,1)$, where $\hat{m}_2^1 = p_2^1 \partial \hat{h}_2^1/\partial D^1$ is country 1's marginal trade-propensity to consume commodity 2, then the mapping is a contraction and has a unique fixed point z_2^1. Since this argument has to be carried out for all prices and tariff factors, it is necessary to assume that $(1-1/T_2)\hat{m}_2^1$ is bounded below 1. Since $1 \leq T_2 < \infty, 0 \leq 1 - 1/T_2 < 1$; it is thus sufficient to assume that $\hat{m}_2^1$ is bounded below 1. Since $\hat{m}_1^1 + \hat{m}_2^1 = 1$ this is equivalent to assuming that $\hat{m}_1^1$ is bounded above zero, i.e., the export good must be strongly superior.

An iterative procedure explained by Kolmogorov and Fomin (1957, p. 44), consisting of starting with a value of z_2^1 on the right to obtain a value on the left, and using this as the new value on the right, corresponds precisely to the procedure originally used by Metzler (1949b, p. 347) to define the tariff-inclusive offer function—as Johnson (1960) has described it. See also Bhagwati and Johnson (1961, p. 230). Thus, Metzler used the principle of contraction mappings without apparently being aware of it. Of course, the above argument is still not entirely rigorous since the function (10.1) is not meaningful unless $z_2^1 > 0$; hence a more subtle argument is required. In the n-commodity case, the situation becomes still more complex. For some techniques of proof of existence of equilibrium under tariff distortions see Sontheiemer (1971).

World equilibrium is defined by the condition

$$\hat{z}_2^1(p_1^2, p_2^2, T_2; l^1) + \hat{z}_2^2(p_1^2, p_2^2; l^2) = 0, \tag{10.2}$$

where $\hat{z}_2^2$ coincides with country 2's trade-demand function with $D^2 = 0$, and if commodity 1 is taken as numéraire and its price $\bar{p}_1^2$ held constant, for constant factor endowments (10.2) defines implicitly the function $\bar{p}_2^2(T_2)$. It yields the condition

$$\frac{d\bar{p}_2^2}{dT_2} = -\frac{\partial \hat{z}_2^1/\partial T_2}{\partial \hat{z}_2^1/\partial p_2^2 + \partial \hat{z}_2^2/\partial p_2^2}. \tag{10.3}$$

As in (9.3) above, dynamic stability requires that the denominator expression in (10.3) be negative, consequently the sign of $d\bar{p}_2^2/dT_2$ is the same as that of $\partial \hat{z}_2^1/\partial T_2$. This illustrates the general principle of comparative statics: to find the effect of a tariff on the external price of the import good, one must ascertain the effect of the tariff on the import of that good when the external price is held constant. A tariff will improve country 1's terms of trade, then, if and only if it reduces country 1's demand for the import good when the terms of trade are held constant. It remains only to compute $\partial \hat{z}_2^1/\partial T_2$.

Differentiating both sides of (10.1) with respect to T_2 one obtains after collecting terms

$$\frac{\partial \hat{z}_2^1}{\partial T_2} = \frac{p_2^2 \hat{s}_{22}^2}{1-(1-1/T_2)\hat{m}_2^1}. \tag{10.4}$$

This is negative, since the denominator has been assumed positive and the trade-Slutsky term $\hat{s}_{22}^2$ is necessarily negative—unless there is a kink in the trade-indifference curve at the initial equilibrium. This is the simple proof of the classical proposition, which can be traced back to Torrens and Mill, that a tariff will improve a country's terms of trade. The proof also suggests why the result need not be true in general—if preferences are not aggregable, for example if the government does not distribute the revenues and has preferences that differ from those of the public (cf. Lerner, 1936, and §10.2 below).

In accordance with the above-mentioned principle of comparative statics, if one wishes to ascertain the effect of a tariff on the *internal* price of the import good, one must ask the question: What would be the effect of a tariff on demand for imports if the *internal* price were held constant?

To bring out an interesting aspect of the problem the formulation will be generalized (as in fact was done by Lerner, 1936) to leave open the question of what fraction of the tariff proceeds is collected by each of the two countries. Let ρ be the proportion collected by country 1 and $1-\rho$ that collected by country 2, where $0 \leq \rho \leq 1$. Then the two countries'excess-demand functions $\tilde{z}_2^k(p_1^1, p_2^1, T_2, \rho; l^k)$ are defined implicitly by

$$\begin{aligned} \tilde{z}_2^1(\cdot) &= \hat{h}_2^1(p_1^1, p_2^1, \rho(1-1/T_2)p_2^1\tilde{z}_2^1(\cdot); l^1) \\ \tilde{z}_2^2(\cdot) &= \hat{h}_2^2(p_1^1, p_2^1/T_2, -(1-\rho)(1-1/T_2)p_2^1\tilde{z}_2^2(\cdot); l^2) \end{aligned} \tag{10.5}$$

provided the $\hat{m}_k^k = p_k^k \partial \hat{h}_k^k/\partial D^k$ are both bounded above zero. (Note that the countries' respective balance-of-trade deficits, denominated in their own prices, satisfy

$$
\begin{aligned}
(10.6)\qquad 0 &= p_1^1 z_1^1 + p_2^1 z_2^1 - D^1 = p_1^1 z_1^1 + p_2^1 z_2^1 - \rho(1 - 1/T_2) p_2^1 z_2^1 \\
&= -p_1^1 z_1^2 - p_2^2 z_2^2 - (1-\rho)(1 - 1/T_2) p_2^1 z_2^1 \\
&= -p_1^2 z_1^2 - p_2^2 z_2^2 + D^2
\end{aligned}
$$

whence $D^1 + D^2 = \tau_2 p_2^2 z_2^1 \neq 0$ when $\tau_2 > 0$). The condition $\tilde{z}_2^1(\cdot) + \tilde{z}_2^2(\cdot) = 0$ of world equilibrium implicitly defines the function $\bar{p}_2(T_2, \rho)$ which satisfies

$$
(10.7)\qquad
\begin{aligned}
\frac{\partial \bar{p}_2^1}{\partial T_2} &= -\frac{\partial \tilde{z}_2^1/\partial T_2 + \partial \tilde{z}_2^2/\partial T_2}{\partial \tilde{z}_2^1/\partial p_2^1 + \partial \tilde{z}_2^2/\partial p_2^1}; \\
\frac{\partial \bar{p}_2^1}{\partial \rho} &= -\frac{\partial \tilde{z}_2^1/\partial \rho + \partial \tilde{z}_2^2/\partial \rho}{\partial \tilde{z}_2^1/\partial p_2^1 + \partial \tilde{z}_2^2/\partial p_2^1}.
\end{aligned}
$$

Consider first the effect of a tariff on the internal price of country 1's import good. From the previous stability argument, this has the sign of the effect of the tariff on demand for the import good when this internal price is held constant. When p_2^1 is held constant, it is apparent from (10.5) that there is only an income effect in country 1. Writing the derivatives as elasticities one finds that

$$
(10.8)\qquad \frac{T_2}{\tilde{z}_2^1}\frac{\partial \tilde{z}_2^1}{\partial T_2} = \rho \hat{m}_2^{1\prime}; \qquad \frac{T_2}{\tilde{z}_2^2}\frac{\partial \tilde{z}_2^2}{\partial T_2} = \rho \hat{m}_2^{2\prime} - \hat{\sigma}_{22}^{2}{}'
$$

where $\hat{\sigma}_{ij}^k = p_j^k \hat{s}_{ij}^k / z_i^k$ denotes the trade-Slutsky elasticity and where $\hat{m}_2^{1\prime}/\hat{m}_2^1 = \hat{\sigma}_{22}^1{}'/\hat{\sigma}_{22}^1 = T_2 - \rho(T_2 - 1)\hat{m}_2^1$ and $\hat{m}_2^{2\prime}/\hat{m}_2^2 = \hat{\sigma}_{22}^2{}'/\hat{\sigma}_{22}^2 = 1 + (1-\rho)(T_2 - 1)\hat{m}_2^2$. Accordingly, the first equation of (10.7) can be expressed in elasticity form as

$$
(10.9)\qquad
\begin{aligned}
\frac{T_2}{\bar{p}_2^1}\frac{\partial \bar{p}_2^1}{\partial T_2} &= \frac{\rho \hat{m}_2^{1\prime} + (1-\rho) T_2 \hat{m}_2^{2\prime} + \eta^2 - 1}{\eta^1 + \eta^2 - 1} \\
&= \frac{\rho(\hat{m}_2^{1\prime} - \hat{m}_2^{2\prime}) + \hat{\sigma}_{22}^2{}'}{\eta^1 + \eta^2 - 1}.
\end{aligned}
$$

The first equation of (10.9) is a generalization of Metzler's formula. There is a joint income effect from the two countries given by the average of their adjusted marginal trade-propensities to consume commodity 2. For the tariff to raise the domestic price of the import good, the sum of this term and the elasticity of country 2's demand for imports must exceed unity. The reason for this is simple. If p_2^1 is held constant, in country 1 only the tariff revenues can be a source of its increased demand for imports. But the tariff lowers the price of commodity 2 in country 2, so there is both a general effect on country 2's demand from this source and the effect of the tariff revenues.

The second formula of (10.9) is perhaps even more instructive. It is obtained from the first from the decompositions

$$\begin{aligned} \eta^1 &= \frac{[\rho + (1-\rho)T_2]\hat{m}_2^1 - T_2\hat{\sigma}_{22}^1}{T_2 - \rho(T_2-1)\hat{m}_2^1}, \\ \eta^2 &= \frac{1 - \hat{m}_2^2 + \hat{\sigma}_{22}^2}{1 + (1-\rho)(T_2-1)\hat{m}_2^2}. \end{aligned} \tag{10.10}$$

Since the trade-Slutsky elasticity $\hat{\sigma}_{22}^2 = p_2^2 \hat{s}_{22}^2 / z_2^2$ is positive (because $z_2^2 < 0$), a sufficient condition for a tariff to raise the internal price of country 1's import good is that $\hat{m}_2^{1\prime} \geq \hat{m}_2^{2\prime}$, and this of course will be recognized as Samuelson's (1952) condition for a transfer to a country to worsen or leave unchanged its terms of trade.

This can be explained by looking at the elasticities

$$\frac{1}{\tilde{z}_2^1}\frac{\partial \tilde{z}_2^1}{\partial \rho} = (T_2 - 1)\hat{m}_2^{1\prime}; \quad \frac{1}{\tilde{z}_2^2}\frac{\partial \tilde{z}_2^2}{\partial \rho} = (T_2 - 1)\hat{m}_2^{2\prime}, \tag{10.11}$$

yielding for the second equation of (10.7) the expression

$$\frac{1}{\hat{p}_2^1}\frac{\partial \hat{p}_2^1}{\partial \rho} = \frac{(T_2 - 1)(\hat{m}_2^{1\prime} - \hat{m}_2^{2\prime})}{\eta^1 + \eta^2 - 1}. \tag{10.12}$$

The effect of a tariff imposed and collected by a country can be broken into two stages: In stage 1, the tariff revenues are allocated to the foreign country; consequently, this is equivalent to an export tax imposed by country 2. In accordance with Lerner's (1936) symmetry theorem, this is equivalent to an import tariff imposed by country 2, hence it improves country 2's terms of trade and thus (since p_1 is held constant) raises the domestic price of country 1's import good. In stage 2, the revenues from country 2's export tax are transferred to country 1, so that it becomes in effect an import tariff imposed by country 1. If the transfer has the "orthodox" effect of improving country 1's terms of trade, it will lower the previously raised price of its import good. The net result will then be uncertain. If the transfer has the "anti-orthodox" effect, the domestic price of country 1's import good will rise further.

The possibility that a tariff might lower the domestic price of the good on which it is imposed has come to be generally known as the "Metzler case" (Johnson, 1960) or the "Metzler paradox" (cf. Jones, 1974), although the possibility was briefly noted by Lerner (1936) in a footnote that also noted the other possibility that a tariff could worsen a country's terms of trade if the government (which purchases at external prices) has a sufficiently strong preference for importables. To avoid confusion between these two "paradoxes," and because of the prominence of this effect in Metzler's work, the term "Metzler paradox" will prove convenient.

In comparing an import tariff imposed by country 1 to an export tax imposed by country 2, one need only note that a small transfer of country 2's tax revenues to country 1 will raise country 1's potential welfare and lower country 2's. Integrating over a path from $\rho = 0$ to $\rho = 1$, since the integrand is positive so will be the integral. Hence an import tariff is preferable from country 1's point of view. Since for any quota equilibrium there is a corresponding tariff equilibrium (but not conversely—see §10.3 below), this reasoning establishes the superiority of an import quota to a "voluntary export restraint," i.e., an export quota imposed by country 2.

10.2 Disaggregative tariff theory

Consideration of separate preferences as between the government and the public was introduced by Lerner (1936). Johnson (1960) went further and considered the separate preferences of the two factors of production. The following treatment encompasses both, so that three separate classes of consumers are considered. The government will be called class 0, and factors 1 and 2 constitute classes 1 and 2. If preferences are homothetic and the proportional distribution of income within a class remains unchanged, then the class's aggregate demand is generated by an aggregate preference relation (cf. Chipman, 1974b); if preferences within a class are identical and homothetic, then the class's preferences can be aggregated egardless of how income is distributed within the class. In either case, the utility function of the class can be considered only as an indicator of its *potential* welfare (i.e., a rise in utility means that gainers could compensate losers).

In both Lerner's (1936) and Bhagwati and Johnson's (1961) treatment it is assumed that the government does not itself pay the tariff on imported goods. Of course, employees and many departments of government generally will purchase imported goods in domestic markets, but the Lerner assumption will be followed here, leading to the specification of the government's demand function for the jth commodity in country 1 by $x^1_{0j} = h^1_{0j}(p^2_1, p^2_2, Y^1_0)$, where Y^1_0 is government revenue, assumed equal to the tariff revenues retained by the government. The demand on the part of factor i will instead be $x^1_{ij} = h^1_{ij}(p^1_1, p^1_2, Y^1_i)$ for $i = 1, 2$, where factor i's income consists of its earned income plus its share in the tariff revenues. Dutiable imports are defined by $z^1_{d2} = x^1_{12} + x^1_{22} - y^1_2$, where y^1_2 is the output of the importable in country 1. Letting δ^1_i denote the share of class i in the tariff revenues, country 1's demand for imports $z^1_2 = x^1_{02} + z^1_{d2}$ is given by

$$
\begin{aligned}
x^1_{02} &= h^1_{02}(p^2_1, p^2_2, \delta^1_0(T_2 - 1)p^2_2 z^1_{d2}) \\
z^1_{d2} &= \sum_{i=1}^{2} h^1_{i2}(p^2_1, T_2 p^2_2, l^1_i \hat{w}^1_i(p^2_1, T_2 p^2_2, l^1_1, l^1_2) + \delta^1_i(T_2 - 1)p^2_2 z^1_{d2}) \\
&\quad - \hat{y}^1_2(p^2_1, T_2 p^2_2, l^1_1, l^1_2)
\end{aligned}
\tag{10.13}
$$

where $\hat{w}^1_i$ is the Stolper-Samuelson function for the ith factor, which will generally (when both commodities are produced) depend only on the prices. The second equation of (10.13) implicitly defines the function $z^1_{d2} = \hat{z}^1_{d2}(p^2_1, p^2_2, T_2, \delta^1, l^1)$; when this is substituted in the first equation of (10.13), the two summed equations define the function $\hat{z}^2_1(p^2_1, p^2_2, T_2; \delta^1, l^1)$ which determines country 1's demand for imports. It should be noted that the formulation is meaningful only if dutiable imports are positive.

There are two main questions of interest in this model: (1) under what conditions will a tariff improve country 1's terms of trade? (2) what will the effect of a tariff be on the welfare of the separate classes?

To answer the first question, by virtue of (10.3) one needs only to compute the partial derivative of country 1's demand for imports with respect to the tariff factor. This is found to be

$$
\frac{\partial \hat{z}^1_2}{\partial T_2} = \delta^1_0 m^1_{02} z^1_{d2} + \frac{M^1_{02}}{T_2 M^1_{d2}} \left\{ p^1_2 \left(\sum_{i=1}^{2} s^1_{i,22} - t^1_{22} \right) + \sum_{i=1}^{2} m^1_{i2} a^1_i \right\} \tag{10.14}
$$

where

$$(10.15)\qquad \begin{aligned} M^1_{02} &= 1 + (T_2 - 1)\delta^1_0 m^1_{02}, \\ M^1_{d2} &= 1 - (1 - 1/T_2)\sum_{i=1}^{2} \delta^1_i m^1_{i2}, \quad \text{and} \\ a^1_i &= l^1_i \frac{\partial \hat{w}^1_i}{\partial p^1_2} + \delta^1_i z^1_{d2} - x^1_{i2}. \end{aligned}$$

The terms $s^1_{i,22}$ in (10.14) are the factors' Slutsky substitution terms, and the m^1_{i2} are the classes' marginal propensities to consume the import good, defined as $m^1_{02} = p^2_2 \partial h^1_{02}/\partial Y^1_0$ and $m^1_{i2} = p^1_2 \partial h^1_{i2}/\partial Y^1_i$ for $i = 1, 2$.

Using Samuelson's reciprocity relation (3.33) and the homogeneity of degree 1 in factor endowments of the Rybczynski function one sees that

$$(10.16)\qquad \begin{aligned} \sum_{i=1}^{2} a^1_i &= \sum_{i=1}^{2} l^1_i \frac{\partial \hat{y}^1_2}{\partial l^1_i} + (\delta^1_1 + \delta^1_2) z^1_{d2} - \sum_{i=1}^{2} x^1_{i2} \\ &= \hat{y}^1_2 - \sum_{i=1}^{2} x^1_{i2} + (1 - \delta^1_0) z^1_{d2} = -\delta^1_0 z^1_{d2} \leq 0. \end{aligned}$$

By convention, country 1 is assumed to be relatively well endowed in factor 1 compared to factor 2, and commodity 1 uses a higher ratio of factor 1 to factor 2 than commodity 2 at the initial factor rentals. Accordingly, from the Stolper-Samuelson relation $\partial \hat{w}^1_2/\partial p^1_2 > w^1_2/p^1_2$ and the inequalities $0 \leq 1 - 1/T_2 < 1$ one finds from (10.15) that

$$(10.17)\qquad \begin{aligned} p^1_2 a^1_2 &> l^1_2 w^1_2 + \delta^1_2 p^1_2 z^1_{d2} - p^1_2 x^1_{22} \\ &\geq l^1_2 w^1_2 + \delta^1_2 (T_2 - 1) p^2_2 z^1_{d2} - p^1_2 x^1_{22} \\ &= Y^1_2 - p^1_2 x^1_{22} \geq 0 \end{aligned}$$

hence from (10.16) $a^1_1 = -a^1_2 - \delta^1_0 z^1_{d2} < 0$. It follows that

$$(10.18)\qquad \sum_{i=1}^{2} m^1_{i2} a^1_i = (m^1_{22} - m^1_{12}) a^1_2 - m^1_{12} \delta^1_0 z^1_{d2}.$$

Substituting (10.18) in (10.14) one obtains the formula

$$(10.19)\qquad \begin{aligned} \frac{\partial \hat{z}^1_2}{\partial T_2} &= \delta^1_0 z^1_{d2} \left(m^1_{02} - \frac{M^1_{02}}{T_2 M^1_{d2}} m^1_{12} \right) \\ &\quad + \frac{M^1_{02}}{T_2 M^1_{d2}} \left\{ p^1_2 \left(\sum_{i=1}^{2} s^1_{i,22} - t^1_{22} \right) + (m^1_{22} - m^1_{12}) a^1_2 \right\}. \end{aligned}$$

When $T_2 = 1$ initially, the expressions $M^1_{02}/(T_2 M^1_{d2})$ reduce to 1.

From (10.19) we can obtain the main results of Lerner (1936) and Johnson (1960). In Lerner's case, the two factors have identical preferences hence $m_{12}^1 = m_{22}^1$ and the second term on the right in (10.19) is negative. When the tariff rate is initially zero, a necessary condition for a tariff increase to worsen the terms of trade is that $m_{02}^1 > m_{12}^1$, i.e., that the government's marginal propensity to consume the import good be greater than that of the public. This is what Johnson (1960) called the "Lerner case"; it may be called the "Lerner paradox." In Johnson's case, in which all tariff revenues are distributed to the factors, hence $\delta_0^1 = 0$, a necessary condition for a tariff to worsen the terms of trade is that $m_{22}^1 > m_{12}^1$, i.e., that factor 2 have a greater marginal propensity to consume the importable than factor 1. This may be called the "Johnson paradox." When $\delta_0^1 = 0$ and $m_{12}^1 = m_{22}^1$, formula (10.19) reduces as it should to (10.4).

These paradoxes are easily explained. In Lerner's case, since the government makes its purchases at external prices, there is no substitution effect, only an income effect, hence if its marginal propensity to consume importables is higher than factor 1's this income effect may outweigh the substitution effect of the higher internal price. In Johnson's case, since at constant external prices factor 2 gains and factor 1 loses earned income by virtue of the Stolper-Samuelson theorem, if factor 2 has a relatively strong preference for the product in which it is used relatively intensively (commodity 2), the distributional income effect might outweigh the substitution effect.

Question (2) is also of great interest. From the relation $p_2^2 = \bar{p}_2^2(T_2)$ between the equilibrium price of commodity 2 in country 2 and the tariff factor (the price of commodity 1, equal in both countries, being held constant) one obtains the equilibrium value of the internal price $\bar{p}_2^1(T_2) = T_2\bar{p}_2^2(T_2)$, the amount imported $\bar{z}_2^1(T_2) = \hat{z}_2^1(\bar{p}_1^2, \bar{p}_2^2(T_2), T_2)$, and factor i's income

$$\bar{Y}_i^1(T_2) = l_i^1\hat{w}_i^1(\bar{p}_1^2, \bar{p}_2^2(T_2)) + \delta_i^1(T_2 - 1)\bar{p}_2^2(T_2)\bar{z}_2^1(T_2).$$

Defining factor i's potential welfare

$$W_i^1(T_2) = V_i^1(\bar{p}_1^1, \bar{p}_2^1(T_2), \bar{Y}_i^1(T_2))$$

in terms of its indirect utility function, one has

$$\frac{\partial W_i^1}{\partial T_2} = \frac{\partial V_i^1}{\partial Y_i^1}\left[-h_{i2}^1\frac{d\bar{p}_2^1}{dT_2} + \frac{d\bar{Y}_i^1}{dT_2}\right]. \tag{10.20}$$

After a series of steps one finds that (cf. Rao, 1971)

$$\frac{dW_i^1}{dT_2} = \frac{\partial V_i^1}{\partial Y_i^1}\left\{a_i^1\frac{d\bar{p}_2^1}{dT_2} - \delta_i^1 z_2^1[1-\tau_2(\eta^2-1)]\frac{d\bar{p}_2^2}{dT_2}\right\}. \tag{10.21}$$

The term in brackets is positive so long as $\tau_2 < 1/(\eta^2 - 1)$, i.e., the initial tariff rate is less than the "optimal tariff" (Johnson, 1950). In the absence of either a "Johnson paradox" or a "Metzler paradox" under these circumstances, $d\bar{p}_2^2/dT_2 < 0$ and $d\bar{p}_2^1/dT_2 > 0$, so factor 2 is a clear gainer. Since $a_1^1 < 0$, for $i = 1$ formula (10.21) indicates the conditions required, say when all tariff proceeds are distributed to factor 1 ($\delta_1^1 = 1$), for factor 1 to be compensated for its loss of earnings.

Of great interest is the question of whether any factor in country 1 stands to gain by having the import tariff replaced by an export tax on commodity 2 imposed by country 2. An analysis similar to that of the previous subsection could be carried out, but it is enough to provide general indications. If country 1 makes a transfer to country 2, and if the "orthodox presumption" holds, both p_2^1 and p_2^2 will rise, i.e., country 1's terms of trade will deteriorate and country 2's will improve. By the Stolper-Samuelson theorem, factor 2 in country 1 will gain and so will factor 2 in country 2; and in both countries factor 1 will suffer a decline in real earnings. Assuming factor 1 previously collected all the tariff revenues in country 1, it will now lose doubly: its real rental will fall and its share in the tariff proceeds will disappear. In country 2, factor 1 might come out even, if it receives all the proceeds from the export tax. If this factor is neutral, it is then two to one in favor of the change. If and to the extent that quotas are equivalent to tariffs, this might provide an explanation for "voluntary export restraints," even though from the point of view of the aggregative model such a policy on the part of country 1 would amount to "shooting oneself in the foot." (For other aspects of the problem see the interesting discussion in K. Jones, 1984.)

10.3 The aggregative theory of quotas

If country 1 imposes a quota of q_2 on its imports of commodity 2, its demand for imports will be determined by

$$\hat{z}_2^1(p_1^2, p_2^2, q_2; l^1) = \min\{\hat{h}_2^1(p_1^2, p_2^2, 0; l^1), q_2\}. \tag{10.22}$$

This will be called the quota-constrained demand for imports. Owners of import licenses will make a profit of $(p_2^1 - p_2^2)z_2^1$, and aggregate excess demand will be determined by

$$z_2^1 = \hat{h}_2^1(p_1^2, p_2^1, (p_2^1 - p_2^2)z_2^1; l^1). \tag{10.23}$$

If the quota is ineffective [i.e., $q_2 > \hat{h}_2^1(p_1^2, p_2^2, 0; l^1)$] then $p_2^1 = p_2^2$; if it is effective, then setting $z_2^1 = q_2$ in (10.23) implicitly defines the function

$$p_2^1 = \hat{p}_2^1(p_2^2, q_2) \tag{10.24}$$

(where the arguments p_1^2, l^1 are suppressed, these being supposed constant). Defining $\hat{c}_j^k = \partial \hat{h}_j^k / \partial D^k, \hat{m}_j^k = p_j^k c_j^k$, and the implicit tariff factor $T_2 = p_2^1/p_2^2$, the derivatives of (10.24) when the quota is effective are found to be

$$\frac{\partial \hat{p}_2^1}{\partial p_2^2} = \frac{q_2 \hat{c}_2^1}{\hat{s}_{22}^1}, \quad \frac{\partial \hat{p}_2^1}{\partial q_2} = \frac{1 - (1 - 1/T_2)\hat{m}_2^1}{\hat{s}_{22}^1}. \tag{10.25}$$

When the quota is ineffective we have of course $\partial \hat{p}_2^1/\partial p_2^2 = 1$ and $\partial \hat{p}_2^1/\partial q_2 = 0$. Figure 18 depicts the shape of $\hat{p}_2^1$ as a function of p_2^2 and q_2 separately. It is interesting in particular to note that when the quota is effective, a change in the external price of the import good will lead to a change in the internal price *in the opposite direction.* This is easily explained: if the external price falls, profits of holders of import licenses will increase; as long as the import good is trade-superior (i.e., $\hat{c}_2^1 > 0$), this will lead to a rise in demand for imports which must be choked off by a price increase to maintain the level of the quota. This is one respect in which a quota is quite different from a tariff. It also makes it less than straightforward to extend the analysis of shared tariff revenues (§10.1 above) to the case of profits from quota licenses.

World equilibrium is defined by

$$\hat{z}_2^1(p_1^2, p_2^2, q_2; l^1) + \hat{z}_2^2(p_1^2, p_2^2; l^2) = 0 \tag{10.26}$$

where $\hat{z}_2^2(p_1^2, p_2^2; l^2) = \hat{h}_2^2(p_1^2, p_2^2, 0; l^2)$, and when the quota is effective this leads to (holding p_1^2, l^1, l^2 constant)

$$\frac{d\bar{p}_2^2}{dq_2} = -\frac{1}{\partial \hat{z}_2^2/\partial p_2^2}, \quad \text{i.e.,} \quad \frac{q_2}{p_2^2}\frac{d\bar{p}_2^2}{dq_2} = \frac{1}{\eta^2 - 1}. \tag{10.27}$$

Since $\eta^1 = 0$ when the quota is binding, dynamic stability requires $\eta^2 > 1$. Figure 19 displays the "catastrophic" effect of a quota when country 2's demand for imports is inelastic at the initial equilibrium (cf. Falvey, 1975). This is another respect in which a quota is different from a tariff.

From (10.27) and stability, a tightening of the quota (a decrease in q_2) will lead to a fall in the world price of commodity 2. One would expect it to lead to a rise in the domestic price of commodity 2 in country 1. From (10.24) we have

$$\text{(10.28)} \qquad \frac{d\bar{p}_2^1}{dq_2} = \frac{\partial \hat{p}_2^1}{\partial p_2^2}\frac{d\bar{p}_2^2}{dq_2} + \frac{\partial \hat{p}_2^1}{\partial q_2},$$

and from inspection of the expressions (10.25) and (10.27) it is clear that as long as both goods are superior $d\bar{p}_2^1/dq_2 < 0$ as expected.

10.4 The disaggregative theory of quotas

Suppose class 0 is the group of holders of import licenses and class i is factor i for $i = 1, 2$. A license-holder facing given internal and external prices of the import good, the former greater than the latter, will optimize by taking the profit on the license and consuming in domestic markets. Thus it is more appropriate to assume that all agents make their purchases at domestic prices. This is another difference between tariffs and quotas.

The disaggregative case differs from the aggregative one simply by replacing the aggregate trade-demand function by $\tilde{z}_2^1(p_1^1, p_2^1, p_2^2, l^1, \delta^1)$ which is defined implicitly by

$$\text{(10.29)} \qquad \begin{aligned} \tilde{z}_2^1(\cdot) \;=\; & \sum_{i=0}^{2} h_{i2}^1(p_1^1, p_2^1, l_i^1 \hat{w}_i^1(p_1^1, p_2^1, l^1) \\ & + \delta_i^1(p_2^1 - p_2^2)\tilde{z}_2^1(\cdot)) - \hat{y}_2^1(p_1^1, p_2^1, l^1), \end{aligned}$$

where by definition $l_0^1 = w_0^1 = 0$. (As in the disaggregative tariff case, this formulation is slightly less general than the aggregative one in that nontradables are excluded.) Country 1's excess-demand function is defined by

$$\text{(10.30)} \quad \hat{z}_2^1(p_1^2, p_2^1, p_2^2, q_2; l^1, \delta^1) = \min\{\tilde{z}_2^1(p_1^2, p_2^2, p_2^2; l^1, \delta^1), q_2\}.$$

and the function (10.24) is now defined implicitly by

$$\text{(10.31)} \qquad \tilde{z}_2^1(\bar{p}_1^2, \hat{p}_2^1, p_2^2; l^1, \delta^1) = \hat{z}_2^1(\bar{p}_1^2, \hat{p}_2^1, p_2^2, q_2; l^1, \delta^1).$$

When the quota is effective the right side of (10.31) is just q_2. Then the derivatives of $\hat{p}_2^1$ satisfy

$$\text{(10.32)} \qquad \frac{\partial \hat{p}_2^1}{\partial p_2^2} = -\frac{\partial \tilde{z}_2^1/\partial p_2^2}{\partial \tilde{z}_2^1/\partial p_2^1}, \quad \frac{\partial \hat{p}_2^1}{\partial q_2} = \frac{1}{\partial \tilde{z}_2^1/\partial p_2^1}$$

and from (10.29) we have

$$\text{(10.33)} \qquad \begin{aligned} \frac{\partial \tilde{z}_2^1}{\partial p_2^1} &= \frac{\sum_{i=0}^2 s_{i,22}^1 - t_{22}^1 + \sum_{i=0}^2 c_{i2}^1 a_i^1}{1 - \delta_i^1(1 - 1/T_2)\bar{m}_2^1}, \\ \frac{\partial \tilde{z}_2^1}{\partial p_2^2} &= \frac{-q_1 \bar{m}_2^1}{1 - \delta_i^1(1 - 1/T_2)\bar{m}_2^1} \end{aligned}$$

where

$$\bar{m}_2^1 = \sum_{i=0}^2 \delta_i^1 m_{i2}, \quad T_2 = \frac{p_2^1}{p_2^2},$$

and

$$a_i^1 = l_i^1 \partial \hat{w}_i^1/\partial p_2^1 + \delta_2^1 z_2^1 - x_{i2}^1$$

is as in (10.15). Thus, formulas (10.25) are replaced by formulas in which $\hat{m}_2^1$ is replaced by $\bar{m}_2^1$ and $\hat{s}_{22}^1$ is replaced by

$$\text{(10.34)} \qquad \sum_{i=0}^2 s_{i,22}^1 - t_{22}^1 + \sum_{i=0}^2 c_{i2}^1 a_i^1.$$

Because of possible differences in preferences this term need not be negative. However, a domestic stability argument could be used to show that it must be negative in the initial equilibrium, hence the disaggregative model does not introduce any essentially different features.

A great objective of the theory of commercial policy is to formulate the problem of international conflict in game-theoretic terms. Before one can do this one must have a payoff matrix. The types of computations that have been illustrated here constitute the raw material for such a strategic formulation. A beginning was developed by Scitovsky (1942), Johnson (1954) and Gorman (1958) in the case of tariffs and Rodriguez (1974) and Tower (1975) in the case of quotas. Many further promising developments have taken place (see the references cited at the beginning of this section, as well as the excellent survey by McMillan, 1986). These models treat countries as aggregates. But we have seen that models of this type are unable to accommodate, let alone predict, the phenomenon of voluntary export restraints. A reasonable conjecture is that this tool provides the mechanism for a side-payment to otherwise injured parties in the foreign country, so as to avoid retaliation. Thus a strategic formulation must also consider the transfer problem. It is clear, also, that a proper formulation would require consideration of at least a four-person game. One may look forward to rich developments in this area in the future.

References

Alexander, S. S. 1951. "Devaluation versus Import Restriction as an Instrument for Improving Trade Balance." International Monetary Fund *Staff Papers*, 1, April, 379–396.

Amano, A. 1968. "Stability Conditions in the Pure Theory of International Trade: a Rehabilitation of the Marshallian Approach." *Quarterly Journal of Economics*, 82, May, 326–339.

Andronov, A. A., Vitt, A. A. and Khaikin, S. E. 1966. *Theory of Oscillators*. Reading, Mass.: Addison-Wesley Publishing Co.

Aquino, A. 1978. "Intra-Industry Trade and Inter-Industry Specialization as Concurrent Sources of International Trade in Manufactures." *Weltwirtschaftliches Archiv*, 114, Heft 2, 275–296.

Arrow, K. J., Chenery, H., Minhas, B. S. and Solow, R. M. 1961. Capital-Labor Substitution and Economic Efficiency. *Review of Economics and Statistics*, 43, August, 225–250.

Balassa, B. 1966. "Tariff Reductions and Trade in Manufactures among the Industrial Countries." *American Economic Review*, 56, June, 466–473.

Baldwin, R. E. 1982. "Gottfried Haberler's Contributions to International Trade Theory." *Quarterly Journal of Economics*, 97, February, 141–159.

Bastable, C. F. 1889. "On some Applications of the Theory of International Trade." *Quarterly Journal of Economics*, 4, October, 1–17.

Bhagwati, J. N. 1959. "Protection, Real Wages and Real Incomes." *Economic Journal*, 69, December, 733–748.

Bhagwati, J. N. 1965. "On the Equivalence of Tariffs and Quotas." In *Trade, Growth, and the Balance of Payments*, ed. R. E. Caves, H. G. Johnson and P. B. Kenen, Amsterdam: North-Holland.

Bhagwati, J. N. 1967. "The Proofs of the Theorems on Comparative Advantage." *Economic Journal*, 77, March, 75–83.

Bhagwati, J. N. 1972. "The Heckscher-Ohlin Theorem in the Multi-Commodity Case." *Journal of Political Economy*, 80, September–October, 1052–1055.

Bhagwati, J. N. and Johnson, H. G. 1961. "A Generalized Theory of the Effects of Tariffs on the Terms of Trade." *Oxford Economic Papers*, N. S., 13, October, 225–253.

Bickerdike, C. F. 1907. Review of *Protective and Preferential Import Duties* by A. C. Pigou. *Economic Journal*, 17, March, 98–102.

Bickerdike, C. F. 1920. "The Instability of Foreign Exchange." *Economic Journal*, 30, March, 118–122.

Bruno, M. 1973. "Protection and Tariff Change under General Equilibrium." *Journal of International Economics*, 3, August, 205–225.

Cassel, G. 1928. "The International Movements of Capital." In G. Cassel, T. E. Gregory, R. E. Kuczynski and H. K. Norton, *Foreign Investments*. Chicago: University of Chicago Press, pp. 1–93.

Chang, W. W., Ethier, W. J. and Kemp, M. C. 1980. "The Theorems of International Trade with Joint Production." *Journal of International Economics*, 10, August, 377–394.

Chipman, J. S. 1965-66. "A Survey of the Theory of International Trade." *Econometrica*, 33, July, 477–519; 33, October, 685–760; 34, January, 18–76.

Chipman, J. S. 1969. “Factor-Price Equalization and the Stolper-Samuelson Theorem.” *International Economic Review*, 10, October, 399–406.

Chipman, J. S. 1970. “External Economies of Scale and Competitive Equilibrium.” *Quarterly Journal of Economics*, 84, August, 347–385.

Chipman, J. S. 1971. “International Trade with Capital mobility: A Substitution Theorem.” In *Trade, Balance of Payments, and Growth*, ed. J. N. Bhagwati, R. A. Mundell, R. W. Jones and J. Vanek. Amsterdam: North-Holland, pp. 201–237.

Chipman, J. S. 1972. “The Theory of Exploitative Trade and Investment Policies: A Reformulation and Synthesis.” In *International Economics and Development*, ed. L. E. Di Marco. New York: Academic Press, pp. 881–916.

Chipman, J. S. 1974a. “The Transfer Problem Once Again.” In *Trade, Stability, and Macroeconomics*, ed. G. Horwich and P. A. Samuelson. New York: Academic Press, pp. 19–78.

Chipman, J. S. 1974b. “Homothetic Preferences and Aggregation.” *Journal of Economic Theory*, 8, May, 26–38.

Chipman, J. S. 1978. “A Reconsideration of the ‘Elasticity Approach’ to Balance-of-Payments Adjustment Problems.” In *Breadth and Depth in Economics*, ed. J. S. Dreyer. Lexington, Mass.: Heath, pp. 49–95.

Chipman, J. S. 1979. “The Theory and Application of Trade Utility Functions.” In *General Equilibrium, Growth, and Trade*, ed. J. F. Green and J. A. Scheinkman. New York: Academic Press, pp. 277–296.

Chipman, J. S. 1980. “Exchange-Rate Flexibility and Resource Allocation.” In *Flexible Exchange Rates and the Balance of Payments*, ed. J. S. Chipman and C. P. Kindleberger. Amsterdam: North-Holland, pp. 159–209.

Chipman, J. S. 1981. “A General-Equilibrium Framework for Analyzing the Responses of Imports and Exports to External Price

Changes: An Aggregation Theorem." In *Methods of Operations Research*, Vol. 44, ed. G. Bamberg and O. Opitz. Königstein: Verlag Anton Hain, Meisenheim GmbH, pp. 43–56.

Chipman, J. S. 1982a. "Capital Movement as a Substitute for Technology Transfer: A Comment." *Journal of International Economics*, 12, February, 107–109.

Chipman, J. S. 1982b. "Comment." In *Import Competition and Response*, ed. J. N. Bhagwati, Chicago: University of Chicago Press, pp. 218–221.

Chipman, J. S. 1985a. "Relative Prices, Capital Movements, and Sectoral Technical Change: Theory and an Empirical Test." In *Structural Adjustment in Developed Open Economies*, ed. K. Jungenfelt and D. Hague, London: Macmillan, pp. 395–454.

Chipman, J. S. 1985b. "A Two-Period Model of International Trade and Payments" (MS).

Corden, W. M. 1966. "The Structure of a Tariff System and the Effective Protective Rate." *Journal of Political Economy*, 74, June, 221-237.

Corden, W. M. 1971. "The Substitution Problem in the Theory of Effective Protection." *Journal of International Economics*, 1, February, 37–57.

Deardorff, A. V. 1980. "The General Validity of the Law of Comparative Advantage." *Journal of Political Economy*, 88, October, 941–957.

Deardorff, A. V. 1982. "The General Validity of the Heckscher-Ohlin Theorem." *American Economic Review*, 72, September, 683–694.

Debreu, G. 1959. *Theory of Value*. New York: Wiley.

Dixit, A. 1983. "The Multi-Country Transfer Problem." *Economics Letters*, 13, 49–53.

Dixit, A. K. and Norman, V. 1980. *The Theory of International Trade*. Digswell Place, Welwyn: Cambridge University Press.

Dixit, A. K. and Stiglitz, J. E. 1977. "Monopolistic Competition and Optimum Product Diversity." *American Economic Review*, 67, June, 297–308.

Dixit, A. K. and Woodland, A. 1982. "The Relationship between Factor Endowments and Commodity Trade." *Journal of International Economics*, 13, November, 201–214.

Dornbusch, R., Fischer, S. and Samuelson, P. A. 1977. "Comparative Advantage, Trade and Payments in a Ricardian Model with a Continuum of Goods." *American Economic Review*, 67, December, 823–829.

Dornbusch, R., Fischer, S. and Samuelson, P. A. 1980. "Heckscher-Ohlin Trade Theory with a Continuum of Goods." *Quarterly Journal of Economics*, 95, September, 203–224.

Drabicki, J. Z. and Takayama, A. 1979. "An Antinomy in the Theory of Comparative Advantage." *Journal of International Economics*, 9, May, 211–223.

Edgeworth, F. Y. 1894. "The Theory of International Values." *Economic Journal*, 4, March, 35–50; 4, September, 424–443; 4, December 606–638.

Edgeworth, F. Y. 1908. "Appreciations of Mathematical Theories. III." *Economic Journal*, 18, September, 392–403; 18, December, 541–556.

Ethier, W. 1974. "Some of the Theorems of International Trade with Many Goods and Factors." *Journal of International Economics*, 4, May, 199–206.

Ethier, W. 1979. "Internationally Decreasing Costs and World Trade." *Journal of International Economics*, 9, February, 1–24.

Ethier, W. J. 1982. "National and International Returns to Scale in the Modern Theory of International Trade." *American Economic Review*, 72, June, 389–405.

Falvey, R. E. 1975. "A Note on the Distinction between Tariffs and Quotas." *Economica*, N. S., 42, 319–326.

Färe, R. 1979. "On the Flatness of the Transformation Surface, a Counterexample." Discussion Paper No. 79-17, Department of Economics, Southern Illinois University, Carbondale, Ill.

Fenchel, W. 1953. *Convex Cones, Sets and Functions.* Princeton University, Department of Mathematics (mimeo).

Finger, J. M. 1975. "Trade Overlap and Intra-Industry Trade." *Economic Inquiry*, 13, December, 581–589.

Foster, J. L. 1804. *An Essay on the Principle of Commercial Exchanges, and More Particularly of Exchange between Great Britain and Ireland.* London: J. Hatchard.

Gale, D. and Nikaido, H. 1965. "The Jacobian Matrix and Global Univalence of Mappings." *Mathematische Annalen*, 49, 81–93.

Gorman, W. M. 1958. "Tariffs, Retaliation, and the Elasticity of Demand for Imports." *Review of Economic Studies*, 25, June, 133–162.

Graaff, J. de V. 1959. "On Optimum Tariff Structures." *Review of Economic Studies*, 17, No. 1, 47–59.

Graham, F. D. 1922. "International Trade under Depreciated Paper. The United States, 1862-79." *Quarterly Journal of Economics*, 36, February, 220–273.

Graham, F. D. 1923. "The Theory of International Values Reexamined." *Quarterly Journal of Economics*, 38, November, 54–86.

Graham, F. D. 1925. "Germany's Capacity to Pay and the Reparation Plan." *American Economic Review*, 15, June, 209–227.

Grubel, H. G. and Lloyd, P. J. 1975. *Intra-Industry Trade.* London: Macmillan.

Haberler, G. 1930. "Die Theorie der komparativen Kosten und ihre Auswertung für die Begründung des Freihandels." *Weltwirtschaftliches Archiv*, 32, July, 350–370. English translation: "The Theory of Comparative Costs and Its Use in the Defense of Free Trade," in *Selected Essays of Gottfried Haberler*, ed. A. Y. C. Koo, Cambridge, Mass.: The MIT Press, pp. 3–19.

Haberler, G. 1933. *Die Internationale Handel.* Berlin: Julius Springer. English translation (revised by the author): *The Theory of International Trade with Its Applications to Commercial Policy.* London: William Hodge & Company, 1936.

Hawkins, D. and Simon, H. A. 1949. "Note: Some Conditions of Macroeconomic Stability." *Econometrica*, 17, July-October, 245–248.

Heckscher, E. 1919. "The Effect of Foreign Trade on the Distribution of Income" (in Swedish). *Ekonomisk Tidskrift* 21, II, 1-32. English translation in *Readings in the Theory of International Trade*, ed. H. S. Ellis and L. A. Metzler, Homewood, Ill.: Irwin, 1950, pp. 272–300.

Helpman, E. 1981. "International Trade in the Presence of Product Differentiation, Economies of Scale and Monopolistic Competition: A Chamberlin-Heckscher-Ohlin Approach." *Journal of International Economics*, 11, August, 305–340.

Helpman, E. and Krugman, P. R. 1985. *Market Structure and Foreign Trade.* Cambridge, Mass. The MIT Press.

Herberg, H. and Kemp, M. C. 1969. "Some Implications of Variable Returns to Scale." *Canadian Journal of Economics*, 2, August, 403–415.

Hirschman, A. O. 1945. *National Power and the Structure of Foreign Trade.* Berkeley and Los Angeles: University of California Press. Expanded edition, c. 1979.

Hirschman, A. O. 1949. "Devaluation and the Trade Balance: A Note." *Review of Economics and Statistics*, 31, February, 50–53.

Horwell, D.J. 1966. "Optimum Tariffs and Tariff Policy." *Review of Economic Studies*, 33, April, 147–158.

Hufbauer, G. C. 1966. *Synthetic Materials and the Theory of International Trade.* London: Gerald Duckworth & Co. Ltd.

Inada, K. 1967. "A Note on the Heckscher-Ohlin Theorem." *Economic Record*, 43, March, 88–96.

Inada, K. 1971. "The Production Coefficient Matrix and the Stolper-Samuelson Condition." *Econometrica*, 39, March. 219–239.

Inoue, T. 1981. "A Generalization of the Samuelson Reciprocity Relation, the Stolper-Samuelson Theorem, and the Rybczynski Theorem under Variable Returns to Scale." *Journal of International Economics*, 11, February, 79–98.

Inoue, T. 1986. "On the Shape of the World Production Possibility Frontier with Three Goods and Two Primary Factors with and without Capital Mobility." *International Economic Review*, 27, October, 707–726.

Inoue, T. and Wegge, L. L. 1986. "On the Geometry of the Production Possibility Frontier." *International Economic Review*, 27, October, 727-737.

Jensen, R. and Thursby, M. 1986. "A Strategic Approach to the Product Life Cycle." *Journal of International Economics*, 21, November, 269–284.

Johnson, H. G. 1950. "Optimum Welfare and Maximum Revenue Tariffs." *Review of Economic Studies*, 19, No. 1, 28–35.

Johnson, H. G. 1954. "Optimum Tariffs and Retaliation." *Review of Economic Studies*, 21, No. 2, 142–153.

Johnson, H. G. 1960. "Income Distribution, the Offer Curve, and the Effect of Tariffs." *Manchester School of Economic and Social Studies*, 28, September, 215–242.

Johnson, H. G. 1965. "The Theory of Tariff Structure, with Special Reference to World Trade and Development." In H. G. Johnson and P. B. Kenen, *Trade and Development*, Geneva: Librairie Droz, pp. 9–29.

Jones, K. 1984. "The Political Economy of Voluntary Export Restraints." *Kyklos*, 37, Fasc. 1, 82–101.

Jones, R. W. 1956. "Factor Proportions and the Heckscher-Ohlin Theorem." *Review of Economic Studies*, 24, No. 1, 1–10.

Jones, R. W. 1961. "Stability Conditions in International Trade: A General Equilibrium Analysis." *International Economic Review*, 2, May, 199–209.

Jones, R. W. 1965a. "Duality in International Trade: A Geometrical Note." *Canadian Journal of Economics and Political Science*, 31, August, 390–393.

Jones, R. W. 1965b. "The Structure of Simple General Equilibrium Models." *Journal of Political Economy*, 73, December, 557–572.

Jones, R. W. 1968. "Variable Returns to Scale in General Equilibrium Theory." *International Economic Review*, 9, October, 261-272.

Jones, R. W. 1969. "Tariffs and Trade in General Equilibrium: Comment." *American Economic Review*, 59, June, 418-424.

Jones, R. W. 1970. "The Transfer Problem Revisited." *Economica*, N. S., 37, May, 178–184.

Jones, R. W. 1971. "Effective Protection and Substitution." *Journal of International Economics*, 1, February, 59–82.

Jones, R. W. 1974a. "The Metzler Tariff Paradox: Extensions to Nontraded and Intermediate Commodities." In *Trade, Stability, and Macroeconomics*, ed. G. Horwich and P. A. Samuelson, New York: Academic Press, pp. 3–18.

Jones, R. W. 1974b. "Trade with Non-Traded Goods: The Anatomy of Interconnected Markets." *Economica*, N. S., 41, May, 121–138.

Jones, R. W. 1975. "Presumption and the Transfer Problem." *Journal of International Economics*, 5, August, 262–274.

Jones, R. W. and Scheinkman, J. A. 1977. "The Relevance of the Two-Sector Production Model in Trade Theory." *Journal of Political Economy*, 85, October, 909–935.

Katzner, D. W. 1970. *Static Demand Theory.* New York: Macmillan.

Kemp, M. C. 1964. *The Pure Theory of International Trade.* Englewood Cliffs, N. J.: Prentice-Hall.

Kemp, M. C., Khang, C., and Uekawa, Y. 1978. "On the Flatness of the Transformation Surface." *Journal of International Economics*, 8, November, 537-542.

Kemp, M. C. and Wegge, L. L. F. 1969. "On the Relation between Commodity Prices and Factor Rewards." *International Economic Review*, 10, October, 407–413.

Keynes, J. M. 1929. "The German Transfer Problem." *Economic Journal* 39, March, 1–7. "A Rejoinder," 39, June, 179–182.

Keynes, J. M. 1930. *A Treatise on Money.* 2 volumes. London: Macmillan.

Khang, C. 1971. "An Isovalue Locus Involving Intermediate Goods and Its Applications to the Pure Theory of International Trade." *Journal of International Economics*, 1, August, 315–325.

Khang, C. 1973. "Factor Substitution in the Theory of Effective Protection: A General Equilibrium Analysis." *Journal of International Economics*, 3, August, 227–243.

Khang, C. and Uekawa, Y. 1973. "The Production Possibility Set in a Model Allowing Interindustry Flows: The Necessary and Sufficient Conditions for Its Strict Convexity." *Journal of International Economics*, 3, August, 283–290.

King, P., 7th baron, 1804. *Thoughts on the Effects of the Bank Restrictions*, 2nd edition. London: T. Cadell and W. Davies. Reprinted in *A Selection from the Speeches and Writings of the Late Lord King*, London: Longman, Brown, Green, and Longmans, 1844, pp. 47–161.

Kojima, K. 1964. "The Pattern of International Trade among Advanced Countries." *Hitotsubashi Journal of Economics*, 5, June, 16–36.

Kolmogorov, A. N. and Fomin, S. V. 1957. *Functional Analysis*, Vol. 1. Translated from the 1st (1954) Russian edition. Rochester, N. Y.: Graylock Press.

Krugman, P. R. 1979. "Increasing Returns, Monopolistic Competition, and International Trade." *Journal of International Economics*, 9, November, 469–479.

Krugman, P. 1980. "Scale Economies, Product Differentiation, and the Pattern of Trade." *American Economic Review*, 70, December, 950–959.

Krugman, P. 1982. "Trade in Differentiated Products and the Political Economy of Trade Liberalization." In *Import Competition and Response*, ed. J. N. Bhagwati. Chicago: University of Chicago Press, pp. 197–208.

Kuga, K. 1973. "Tariff Retaliation and Policy Equilibrium." *Journal of International Economics*, 3, November, 351–366.

Kuhn, H. 1968. "Lectures on Mathematical Economics." In *Mathematics of the Decision Sciences*, Part 2, ed. G. B. Dantzig and A. F. Veinott, Providence, R. I.: American Mathematical Society, pp. 49–84.

Lancaster, K. 1980. "Intra-Industry Trade under Perfect Monopolistic Competition." *Journal of International Economics*, 10, May, 151–175.

Leamer, E. E. 1980. "The Leontief Paradox, Reconsidered." *Journal of Political Economy*, 88, June, 495–503.

Leamer, E. E. 1985. *Sources of International Comparative Advantage.* Cambridge, Mass.: The MIT Press.

Leontief, W. 1933. "The Use of Indifference Curves in the Analysis of Foreign Trade." *Quarterly Journal of Economics*, 47, May, 493–503.

Leontief, W. 1953. "Domestic Production and Foreign Trade; The American Capital Position Re-examined." *Proceedings of the American Philosophical Society*, 97, September, 332–349.

Lerner, A. P. 1932. "The Diagrammatical Representation of Cost Conditions in International Trade." *Economica*, 12, August, 346–356.

Lerner, A. P. 1933. "Factor Prices and International Trade." *Economica*, N. S., 19, February 1952, 1–15.

Lerner, A. P. 1934. "The Diagrammatical Representation of Demand Conditions in International Trade." *Economica*, N. S., 1, August, 319–334.

Lerner, A. P. 1936. "The Symmetry between Import and Export Taxes." *Economica*, N. S., 3, August, 306–313.

Lerner, A. P. 1944. *The Economics of Control.* New York: Macmillan.

Linder, S. B. 1961. *An Essay on Trade and Transformation.* Stockholm: Almqvist & Wiksell, and New York: Wiley.

List, F. 1841. *Das nationale System der politischen Œkonomie.* Jena: Gustav Fischer, 1904.

Malthus, T. R. 1811a. "Depreciation of Paper Currency." *Edinburgh Review*, 17, February, 339–372.

Malthus, T. R. 1811b. "Pamphlets on the Bullion Question." *Edinburgh Review*, 18, August, 448–470.

Marshall, A. 1879. *The Pure Theory of Foreign Trade.* London: London School of Economics and Political Science, 1930.

Marshall, A. 1923. *Money Credit and Commerce.* London: Macmillan.

Mayer, W. 1981. "Theoretical Considerations on Negotiated Tariff Adjustments." *Oxford Economic Papers*, N. S., 33, March, 135–153.

McCulloch, R. and Yellen, J. 1982. "Can Capital Movements Eliminate the Need for Technology Transfer?" *Journal of International Economics*, 12, February, 95–106.

McDougall, I. A. 1965. "Non-Traded Goods and the Transfer Problem." *Review of Economic Studies*, 32, January, 67–84.

McKenzie, L. W. 1955. "Equality of Factor Prices in World Trade." *Econometrica*, 23, July, 239–256.

McKenzie, L. W. 1967. "The Inversion of Cost Functions: A Counter-example." *International Economic Review*, 8, October, 271–278. "Theorem and Counter-example." 8, October, 279–285.

McKinnon, R. I. 1966. "Intermediate Products and Differential Tariffs: A Generalization of Lerner's Symmetry Theorem." *Quarterly Journal of Economics*, 80, November, 584–615.

McMillan, J. 1986. *Game Theory in International Economics*. London: Harwood Academic Publishers.

Meade, J. E. 1952. *A Geometry of International Trade*. London: George Allen and Unwin.

Melvin, J. R. 1968. "Production and Trade with Two Factors and Three Goods." *American Economic Review*, 58, December, 1249–1268.

Melvin, J. R. 1969a. "Intermediate Goods in Production Theory: The Differentiable Case." *Review of Economic Studies*, 36, January, 124–131.

Melvin, J. R. 1969b. "Intermediate Goods, the Production Possibility Curve, and Gains from Trade." *Quarterly Journal of Economics*, 83, February, 141–151.

Melvin, J. R. 1971. "International Trade Theory without Homogeneity." *Quarterly Journal of Economics*, 85, February, 66–76.

Metzler, L. A. 1942. "The Transfer Problem Reconsidered." *Journal of Political Economy*, 50, June, 397–414.

Metzler, L. A. 1949a. "Tariffs, the Terms of Trade, and the Distribution of National Income." *Journal of Political Economy*, 57, February, 1–29.

Metzler, L. A. 1949b. "Tariffs, International Demand, and Domestic Prices." *Journal of Political Economy*, 57, August, 345–351.

Michaely, M. 1962. *Concentration in International Trade*. Amsterdam: North-Holland.

Mill, J. S. 1844. *Essays on some Unsettled Questions of Political Economy*. London: John W. Parker.

Mill, J. S. 1948. *Principles of Political Economy with some of their Applications to Social Philosophy*. 2 vols, London: John W. Parker.

Mundell, R. A. 1960. "The Pure Theory of International Trade." *American Economic Review*, 50, March, 67–110.

Neary, J. P. 1985. "Two-by-Two International Trade Theory with Many Goods and Factors." *Econometrica*, 53, September, 1233–1247.

Negishi, T. 1972. *General Equilibrium Theory and International Trade*. Amsterdam: North-Holland.

Nicholson, J. S. 1903. *Principles of Political Economy*, 2nd edition. 3 volumes. London: Adam and Charles Black.

Nikaido, H. 1972. "Relative Shares and Factor Price Equalization." *Journal of International Economics*, 2, August, 257–264.

Ohlin, B. 1928. "The Reparations Problem." *Index* (Svenska Handelsbanken, Stockholm), No. 28, 2–33.

Ohlin, B. 1929. "Transfer Difficulties, Real and Imagined." *Economic Journal*, 39, June, 172–178.

Ohlin, B. 1933. *Interregional and International Trade*. Cambridge, Mass.: Harvard University Press.

Oniki, H. and Uzawa, H. 1965. "Patterns of Trade and Investment in a Dynamic Model of International Trade." *Review of Economic Studies*, 32, January, 15–38.

Otani, Y. 1973. "Neo-classical Technology Sets and Properties of Production Possibility Sets." *Econometrica*, 41, July, 667–682.

Otani, Y. 1980. “Strategic Equilibrium of Tariffs and General Equilibrium.” *Econometrica*, 48, April, 643–662.

Panchamukhi, V. R. 1961. “A Theory for Optimum Tariff Policy.” *Indian Economic Review*, 9, October, 178–198.

Pareto, V. 1896-97. *Cours d’Économie politique*. 2 volumes. Lausanne: F. Rouge.

Pearce, I. F. 1967. “More about Factor Price Equalization.” *International Economic Review*, 8, October, 255–270. “Rejoinder to Professor McKenzie.” 8, October, 296–299.

Pigou, A. C. 1932. “The Effect of Reparations on the Real Ratio of International Interchange.” *Economic Journal* 42, December, 532–543.

Posner, M. V. 1961. “International Trade and Technical Change.” *Oxford Economic Papers*, N. S., 13, October, 323–341.

Rader, J. T. 1964. “Edgeworth Exchange and General Economic Equilibrium.” *Yale Economic Essays*, 4, No. 1, 133–180.

Ramaswami, V. K. and Srinivasan, T. N. 1971. “Tariff Structure and Resource Allocation in the Presence of Factor Substitution.” In *Trade, Balance of Payments, and Growth*, ed. J. N. Bhagwati, R. W. Jones, R. A. Mundell, and J. Vanek, Amsterdam: North-Holland, pp. 291–299.

Rao, V. S. 1971. “Tariffs and Welfare of Factor Owners: A Normative Extension of the Stolper-Samuelson Theorem.” *Journal of International Economics*, 1, November, 401–415.

Ricardo, D. 1811. *The High Price of Bullion, A Proof of the Depreciation of Bank Notes*, 4th edition. London: John Murray. Reprinted in Sraffa (1951, Vol. III, pp. 47–127).

Ricardo, D. 1817. *On the Principles of Political Economy, and Taxation*. London: John Murray. Reprinted in Sraffa (1951, Vol. I).

Riezman, R. 1974. “A Note on the Heckscher-Ohlin Theorem.” *Tijdschrift voor Economie*, 19, No. 3, 339–343.

Riezman, R. 1982. Tariff Retaliation from a Strategic Viewpoint. *Southern Economic Journal*, 48, January, 583–593.

Robertson, D. H. 1931. "The Transfer Problem." In A. C. Pigou and D. H. Robertson, *Economic Essays and Addresses*, London: Macmillan, pp. 170–181.

Rockafellar, F. T. 1970. *Convex Analysis.* Princeton, N. J.: Princeton University Press.

Rodriguez, C. A. 1974. "The Non-Equivalence of Tariffs and Quotas under Retaliation." *Journal of International Economics*, 4, August, 295–298.

Rybczynski, T. M. 1955. "Factor Endowment and Relative Commodity Prices." *Economica*, N. S., 22, November, 336–341.

Samuelson, P. A. 1947. *Foundations of Economic Analysis.* Cambridge, Mass.: Harvard University Press.

Samuelson, P. A. 1949. "International Factor-Price Equalisation Once Again." *Economic Journal*, 59, June, 181–197.

Samuelson, P. A. 1952. "The Transfer Problem and Transport Costs: The Terms of Trade when Impediments are Absent." *Economic Journal*, 62, June, 278–304.

Samuelson, P. A. 1953. "Prices of Factors and Goods in General Equilibrium." *Review of Economic Studies*, 21, No. 1, 1–20.

Samuelson, P. A. 1954. "The Transfer Problem and Transport Costs, II: Analysis of Effects of Trade Impediments." *Economic Journal*, 64, June, 264–289.

Samuelson, P. A. 1971. "On the Trail of Conventional Beliefs about the Transfer Problem." In *Trade, Balance of Payments, and Growth*, ed. J. N. Bhagwati, R. W. Jones, R. A. Mundell and J. Vanek, Amsterdam: North-Holland, pp. 327–351.

Scitovsky, T. 1942. "A Reconsideration of the Theory of Tariffs." *Review of Economic Studies*, 9, No. 2, 89–110.

Shephard, R. W. 1953. *Cost and Production Functions.* Princeton: Princeton University Press. Reprinted, Berlin-Heidelberg-New York: Springer-Verlag, 1981.

Smith, A. 1776. *An Inquiry into the Nature and Causes of the Wealth of Nations.* 2 volumes. London: W. Strahan and T. Cadell.

Sontheimer, K. C. 1971. "The Existence of International Trade Equilibrium with Trade Tax-Subsidy Distortions." *Econometrica*, 39, November, 1015–1035.

Stolper, W. and Samuelson, P. A. 1941. "Protection and Real Wages." *Review of Economic Studies*, 9, November, 58–73.

Sraffa, P. 1951. *The Works and Correspondence of David Ricardo.* Vols. 1–4, Cambridge: University Press.

Taussig, F. W. 1917. "International Trade under Depreciated Paper. A Contribution to Theory." *Quarterly Journal of Economics*, 31, May, 380–403.

Taussig, F. W. 1927. *International Trade.* New York: Macmillan.

Thornton, H. 1802. *An Enquiry into the Nature and Effects of the Paper Credit of Great Britain.* London: J. Hatchard. Reprinted, ed. F. A. v. Hayek, London: George Allen & Unwin Ltd., 1939, and New York: Augustus M. Kelley, 1962.

Thursby, M. and Jensen, R. 1983. "A Conjectural Variation Approach to Strategic Tariff Equilibria." *Journal of International Economics*, 14, February, 145–161.

Torrens, R. 1815. *An Essay on the External Corn Trade.* London: J. Hatchard.

Torrens, R. 1844. *The Budget, On Commercial and Colonial Policy.* London: Smith, Elder, and Co.

Tower, E. 1975. "The Optimum Quota and Retaliation." *Review of Economic Studies*, 42, October, 623–630.

Uekawa, Y. 1971. "Generalization of the Stolper-Samuelson Theorem." *Econometrica*, 39, March, 197–217.

Uekawa, Y. 1972. "On the Existence of Incomplete Specialization in International Trade with Capital Mobility." *Journal of International Economics*, 2, February, 1–23.

Uekawa, Y. 1979. "The Theory of Effective Protection, Resource Allocation, and the Stolper-Samuelson Theorem: The Many-Industry Case." *Journal of International Economics*, 9, May, 151-171.

Uekawa, Y., Kemp, M. C., and Wegge, L. L. 1973. P- and PN-matrices, Minkowski- and Metzler-Matrices, and Generalizations of the Stolper-Samuelson and Samuelson-Rybczynski Theorems. *Journal of International Economics*, 3, February, 53-76.

Uzawa, H. 1962. "Production Functions with Constant Elasticity of Substitution." *Review of Economic Studies*, 29, October, 291–299.

Uzawa, H. 1964a. "Optimal Growth in a Two-Sector Model of Capital Accumulation." *Review of Economic Studies*, 31, January, 1–24.

Uzawa, H. 1964b. "Duality Principles in the Theory of Cost and Production." *International Economic Review*, 5, May, 216–220.

Vanek, J. 1968. "The Factor Proportions Theory: The N-Factor Case." *Kyklos*, 21, Fasc. 4, 749–754.

Verdoorn, P. J. 1960. "The Intra-Block Trade of Benelux." In *Economic Consequences of the Size of Nations*, ed. E. A. G. Robinson, London: Macmillan, pp. 291–329.

Vernon, R. 1966. "International Investment and International Trade in the Product Cycle." *Quarterly Journal of Economics*, 80, May, 190–207.

Viner, J. 1924. *Canada's Balance of International Indebtedness 1900-1913*. Cambridge, Mass.: Harvard University Press.

Viner, J. 1937. *Studies in the Theory of International Trade.* New York: Harper & Brothers.

Walras, L. 1874. *Élements d'Économie politique Pure.* Lausanne: L. Corbaz & C^{ie}.

Wan, H. Y. 1971. "A Simultaneous Variational Model for International Capital Movement." In *Trade, Balance of Payments, and Growth*, ed. J. N. Bhagwati, R. W. Jones, R. A. Mundell, and J. Vanek, Amsterdam: North-Holland, pp. 261–287.

Wegge, L. L. F. and Kemp, M. C. 1969. "Generalizations of the Stolper-Samuelson and Samuelson-Rybczynski Theorems in Terms of Conditional Input-Output Coefficients." *International Economic Review*, 10, October, 414–425.

White, H. D. 1933. *The French International Accounts, 1880-1993.* Cambridge, Mass.: Harvard University Press.

Wicksell, K. 1918. "International Freights and Prices." *Quarterly Journal of Economics*, 32, February, 404–410.

Williams, J. H. 1920. *Argentine International Trade under Inconvertible Paper Money.* Cambridge, Mass.: Harvard University Press.

Wilson, R. 1931. *Capital Imports and the Terms of Trade.* Melbourne: Melbourne University Press in association with Macmillan.

Woodland, A. D. 1980. "Direct and Indirect Trade Utility Functions." *Review of Economic Studies*, 47, October, 907–926.

Woodland, A. D. 1982. *International Trade and Resource Allocation.* Amsterdam: North-Holland.