

# Notes on International Trade under External Economies of Scale

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We shall consider some simple models of trade in which there is a single factor of production (labor) which produces some commodities under increasing, and some under constant or decreasing, returns to scale, where the economies of scale are external to individual firms and therefore compatible with competitive equilibrium. The model introduced by Tinbergen (1945) and Panagariya (1981) will be treated analytically and in detail. Since this model allows for multiple equilibria, the question of dynamic stability necessarily arises; this will have to be taken up in a separate set of notes.

## 1 External economies

This section will follow the closed-economy parametric formulation of Chipman (1970).

In industry  $i = 1, 2, \dots, n$  let the production function facing each of the  $N_i$  firms be given by

$$(1.1) \quad y_{i\nu} = k_i v_{i\nu} \quad (\nu = 1, 2, \dots, N_i),$$

where  $y_{i\nu}$  is the output of the  $\nu$ th firm in industry  $i$ , and  $v_{i\nu}$  is the amount of labor this firm employs. Then the output of, and the labor employed by, industry  $i$  are given by

$$(1.2) \quad y_i = \sum_{\nu=1}^{N_i} y_{i\nu} \quad \text{and} \quad v_i = \sum_{\nu=1}^{N_i} v_{i\nu}$$

respectively. Accordingly, from (1.2) and (1.1) we have

$$(1.3) \quad y_i = k_i v_i \quad (i = 1, 2, \dots, n).$$

Unknown to the individual firm, however, is the relation

$$(1.4) \quad k_i = \kappa_i v_i^{\rho_i - 1} \quad (0 < \rho_i < \infty).$$

If  $\rho_i > 1$ , this may be interpreted as the increased labor productivity resulting from the division of labor into various specialties as the scale of the industry expands.<sup>1</sup> The participation of firm  $\nu$  to this increase is so small that it is not taken into account in its actions. If  $\rho_i < 1$ , this may be interpreted as the reduced labor productivity resulting from pollution produced by the individual firms. Again, the contribution of the  $\nu$ th firm in industry  $i$  to this pollution is so small that it is not taken into account in its actions. Thus, economies are internal to the industry but external (parametric) to the individual firms.

Substituting (1.4) into (1.3) we obtain the *objective production function*

$$(1.5) \quad y_i = \kappa_i v_i^{\rho_i} \quad (i = 1, 2, \dots, n).$$

The relation (1.3) may be called the *subjective production function* of industry  $i$ .

Denoting the wage rate by  $w$ , the *total subjective<sup>2</sup> cost of production* in industry  $i$  may be denoted, from (1.3), by

$$(1.6) \quad C_i(w, y_i, k_i) = w v_i = w \frac{y_i}{k_i}.$$

Now, substituting the objective production function (1.5) in (1.4) so as to express the parameter  $k_i$  as a function of output, we obtain

$$(1.7) \quad k_i = \kappa_i v_i^{\rho_i - 1} = \kappa_i (\kappa_i^{-1/\rho_i} y_i^{1/\rho_i})^{\rho_i - 1} = \kappa_i^{1/\rho_i} y_i^{1 - 1/\rho_i} \equiv \hat{k}_i(y_i).$$

Substitution of (1.7) in (1.6) now defines the *objective<sup>2</sup> total cost function*

$$(1.8) \quad \hat{C}_i(w, y_i) = C_i(w, y_i, \hat{k}_i(y_i)) = \frac{w y_i}{\hat{k}_i(y_i)} = w \kappa_i^{-1/\rho_i} y_i^{1/\rho_i}.$$

The *objective (“social”) marginal cost* is then given by

$$(1.9) \quad \frac{\partial \hat{C}_i(w, y_i)}{\partial y_i} = \frac{w}{\rho_i} \kappa_i^{-1/\rho_i} y_i^{1/\rho_i - 1}.$$

On the other hand, the *subjective (“private”) marginal cost* is, from (1.6) and (1.7),

$$(1.10) \quad \frac{\partial C_i(w, y_i, k_i)}{\partial y_i} = \frac{w}{k_i} = w \kappa_i^{-1/\rho_i} y_i^{1/\rho_i - 1},$$

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<sup>1</sup>Ideally, this should be made an explicit part of the model. That is, the production technology should allow for labor to be divided into different specialized tasks as the level of output increases. Such a formulation would apply internationally as well as nationally, i.e., as scale increases the production process for a particular commodity could be carried out by specialized labor in different countries as well as within countries, thus providing a natural explanation for the phenomenon of intra-industry trade.

<sup>2</sup>The terms “private” and “social” cost were used by Pigou (1932) in a sense similar to the terms “subjective” and “objective” used here.

and this is clearly equal to both the subjective and objective *average cost*

$$(1.11) \quad c_i(w, y_i) \equiv C_i(w, y_i, k_i)/y_i = \hat{C}_i(w, y_i)/y_i = w\kappa^{-1/\rho_i}y_i^{1/\rho_i-1}.$$

Thus, prices of produced commodities in this model must be equal to their average costs, i.e.,

$$(1.12) \quad \frac{p_i}{p_j} = \frac{c_i(w, y_i)}{c_j(w, y_j)} = \kappa_i^{-1/\rho_i} \kappa_j^{1/\rho_j} y_i^{1/\rho_i-1} y_j^{1-1/\rho_j} \quad \text{if } y_i > 0.$$

To complete the model we assume that individuals have identical utility functions of the Mill-Graham form

$$(1.13) \quad U(x_1, x_2, \dots, x_n) = \prod_{i=1}^n x_i^{\theta_i} \quad (0 < \theta_i < 1, \sum_{i=1}^n \theta_i = 1).$$

Faced with given market prices  $p_i$  and income  $Y$ , consumers' demand functions will then be of the form

$$(1.14) \quad x_i = \frac{\theta_i Y}{p_i} \quad (i = 1, 2, \dots, n).$$

With a single factor of production, labor, of amount  $l$ , and a given wage rate,  $w$ , of course income is  $Y = wl$ .

## 2 Laissez-faire equilibrium under autarky

Under autarky, demand equals supply for each commodity, i.e.,  $x_i = y_i$  for  $i = 1, 2, \dots, n$ . Since total costs are equal to total revenues in each producing industry, i.e.,  $wv_i = p_i y_i$ , we have from (1.14)

$$(2.1) \quad v_i = \frac{p_i}{w} y_i = \frac{p_i}{w} x_i = \frac{p_i \theta_i w l}{w p_i} = \theta_i l.$$

Substituting (2.1) into the objective production function (1.5) we obtain

$$(2.2) \quad x_i = y_i = \kappa_i v_i^{\rho_i} = \kappa_i (\theta_i l)^{\rho_i} \quad (i = 1, 2, \dots, n).$$

Note that (2.1) automatically satisfies the resource-allocation constraint

$$(2.3) \quad \sum_{i=1}^n v_i = l.$$

From now on we shall specialize the assumptions to the case  $n = 2$ . For simplicity we shall also assume  $\kappa_i = 1$  in (1.4). This involves no real loss of generality, since the units in which quantities of commodities are measured may be chosen for convenience, and it will be convenient in what follows to adopt this assumption.

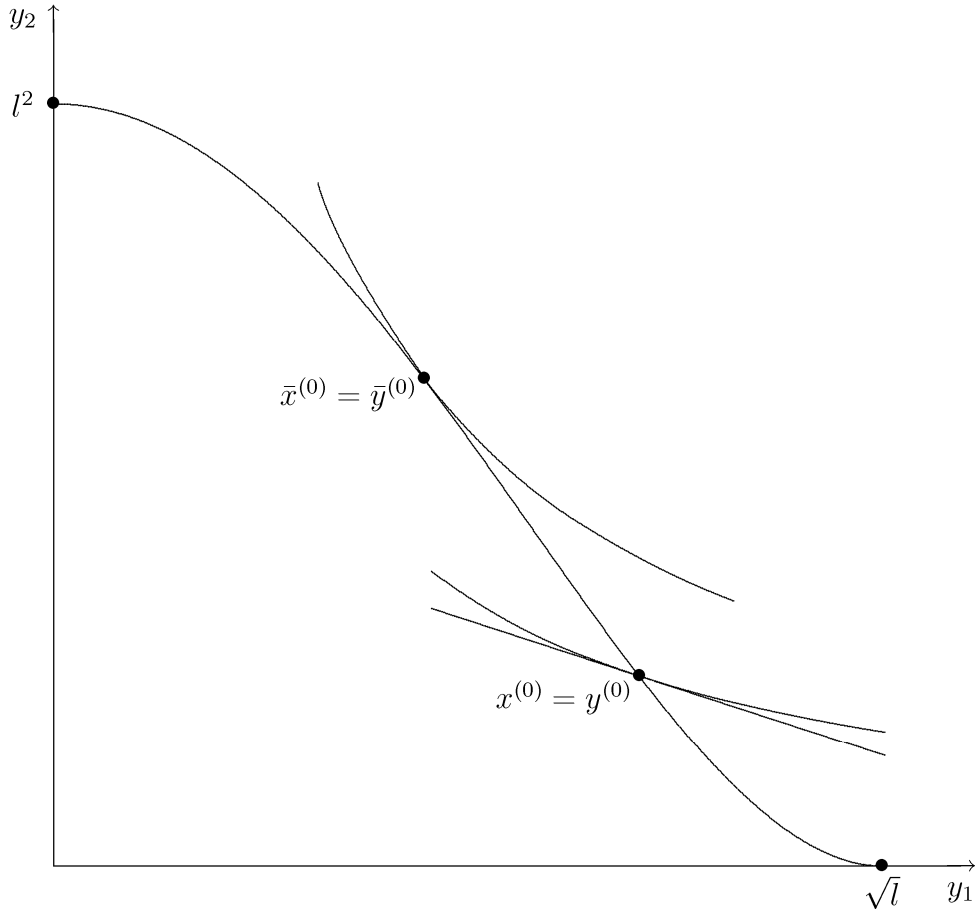


Figure 1  
 $\rho_1 = \frac{1}{2}, \rho_2 = 2, l = 1.2.$

The laissez-faire equilibrium outputs and quantities consumed under autarky have been given by (2.2). It remains to determine the laissez-faire equilibrium prices and the level of the country's utility.

First we consider the country's production-possibility frontier. Combining the objective production functions (1.5) with the resource-allocation constraint (2.3) for  $n = 2$  and  $\kappa_i = 1$  we obtain what will be called in section 4 the first basic equation for the open economy:

$$(2.4) \quad y_1^{1/\rho_1} + y_2^{1/\rho_2} = l.$$

This may be rewritten in the form

$$(2.5) \quad y_2 = (l - y_1^{1/\rho_1})^{\rho_2},$$

which is the equation of the production-possibility frontier. Its negative derivative (the marginal rate of transformation) is

$$(2.6) \quad -\frac{dy_2}{dy_1} = \frac{\rho_2}{\rho_1} y_1^{1/\rho_1-1} (l - y_1^{1/\rho_1})^{\rho_2-1} = \frac{\rho_2}{\rho_1} y_1^{1/\rho_1-1} y_2^{1-1/\rho_2} = \frac{\rho_2 p_1}{\rho_1 p_2},$$

the last equality following from (1.12) whenever  $y_1 > 0$  and  $y_2 > 0$ . If we assume that  $\rho_1 < 1$  and  $\rho_2 > 1$ , it follows immediately from (2.6) that the slope of this production-possibility frontier (the marginal rate of transformation) is zero if either  $y_1 = 0$  or  $y_2 = 0$ . The PPF thus has the shape indicated in Figure 1.

The equilibrium price ratio is equal to the marginal rate of substitution in consumption at the equilibrium levels of consumption, hence from (2.2):

$$(2.7) \quad \frac{p_1}{p_2} = -\frac{dx_2}{dx_1} = \frac{\partial U/\partial x_1}{\partial U/\partial x_2} \Big|_{x_i=(\theta_i l)^{\rho_i}} = \frac{x_2/\theta_2}{x_1/\theta_1} = \theta_1^{1-\rho_1} \theta_2^{\rho_2-1} l^{\rho_2-\rho_1}.$$

### 3 An example of autarky equilibrium (Figure 1)

Let us consider the case  $\rho_1 = \frac{1}{2}$  and  $\rho_2 = 2$ , with  $\theta_1 = \theta_2 = \frac{1}{2}$  and  $l = 1.2$ ; we shall take this case up again in section 5 below. The formula (2.5) for the production-possibility frontier then specializes to

$$(3.1) \quad y_2 = (l - y_1^2)^2$$

and this is the function drawn in Figure 1 for the case  $l = 1.2$ . From (2.2) we have the equilibrium (where the superscript (0) denotes autarky)

$$(3.2) \quad x_1^{(0)} = y_1^{(0)} = 0.6^{1/2} = 0.774596669241 \quad \text{and} \quad x_2^{(0)} = y_2^{(0)} = 0.6^2 = 0.36.$$

Since at this equilibrium utility is at a maximum at the equilibrium prices, the equilibrium price ratio must be

$$(3.3) \quad \frac{p_1^{(0)}}{p_2^{(0)}} = -\frac{dx_2}{dx_1} \Big|_{x_1=x_1^{(0)}, x_2=x_2^{(0)}} = \frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{x_2^{(0)}}{x_1^{(0)}} = 0.464758001545,$$

hence the equation of the budget line is

$$(3.4) \quad 0.464758001545x_1 + x_2 = 0.72$$

(see Figure 1). The maximum output of commodity 1 is  $l^{\rho_1} = \sqrt{l} = 1.095445$ , and the maximum output of commodity 2 is  $l^{\rho_2} = l^2 = 1.44$ . Utility is maximized subject to this production-possibility frontier when  $y_1 = \sqrt{0.24} = 0.489897948557$  and  $y_2 = 0.9216$  (shown by the point  $\bar{y}^{(0)}$  in Figure 1), where utility is  $\sqrt{y_1 y_2} = 0.671930018225$ , as compared with that of the laissez-faire equilibrium  $y^{(0)}$  of (3.2), where the utility is  $0.6^{5/4} = 0.528067042076$ .

## 4 Laissez-faire equilibrium for an open economy when $\rho_1 < 1$ and $\rho_2 > 1$

We consider the model presented by Panagariya (1981), in which  $\rho_1 < 1$  and  $\rho_2 > 1$ , and the world price ratio  $p_1/p_2$  is given.

As observed in section 1, if commodity  $i$  is produced and competition prevails (precluding positive profits), its price will be equal to its average cost (subjective marginal cost), which is given by (1.10). On the other hand, if the average cost exceeds the price, the commodity will not be produced. Thus,

$$(4.1) \quad p_i \leq c_i \equiv w y_i^{1/\rho_i - 1} \quad \text{with equality if } y_i > 0.$$

However, for  $i = 1$ , since  $\rho_1 < 1$ , it is clear that the right side of (4.1) is zero when  $y_1 = 0$ , so the strict inequality cannot hold. But since we may assume that  $p_1 > 0$ , this shows that  $y_1$  must be positive for  $\rho_1 < 1$ . In other words, *the country cannot specialize completely in commodity 2 (the increasing-return good)*. For  $i = 2$ , since  $\rho_2 > 1$ , the right side of (4.1) approaches infinity as  $y_2 \rightarrow 0$ , so it is possible to have  $c_2 > p_2$  and  $y_2 = 0$ ; i.e., *the country may specialize in its decreasing-return good*. We conclude, therefore, that the following basic inequality holds:

$$(4.2) \quad \frac{c_1}{c_2} = y_1^{1/\rho_1 - 1} y_2^{1 - 1/\rho_2} \leq \frac{p_1}{p_2},$$

equality holding whenever  $y_2 > 0$ . This is the second basic relation—supplementing the production relation (2.4)—that defines laissez-faire equilibrium in the open economy. In view of (2.6), the distortion between the average-cost ratio  $c_1/c_2$  and the marginal rate of transformation or marginal-cost ratio is given by

$$(4.3) \quad -\frac{dy_2}{dy_1} = \frac{\rho_2 c_1}{\rho_1 c_2}.$$

To solve for an equilibrium with diversification in the open economy, we may seek a solution to the simultaneous relations (2.4) and (4.2) when equality holds in the latter. If no solution can be found, we may conclude that strict inequality holds in (4.2), hence the country specializes—necessarily in commodity 1, the decreasing-return good. A solution of (4.2) with equality will then exist if and only if the world price ratio  $p_1/p_2$  does not exceed  $\rho_1/\rho_2$  times the maximum marginal rate of transformation  $\tau^*$ , which occurs at the point of inflection where, from (2.6) (cf. Panagariya 1981, p. 222),

$$(4.4) \quad \frac{d^2 y_1}{dy_1^2} = \frac{1}{\rho_2} y_1^{1/\rho_2 - 2} y_2^{1 - 2/\rho_2} [(\rho_2 - 1)y_1^{1/\rho_1} + (\rho_2 - 1)y_2^{1/\rho_2}] = 0.$$

We find readily that this occurs at

$$(4.5) \quad y_1^* = \left( \frac{1 - \rho_1}{\rho_2 - \rho_1} l \right)^{\rho_1} \quad \text{and} \quad y_2^* = \left( \frac{\rho_2 - 1}{\rho_2 - \rho_1} l \right)^{\rho_2}.$$

The maximum marginal rate of transformation is then

$$(4.6) \quad \tau^* \equiv - \left. \frac{dy_2}{dy_1} \right|_{y_1=y_1^*} = \frac{\rho_2 (1 - \rho_1)^{1-\rho_1} (\rho_2 - 1)^{\rho_2-1}}{\rho_1 (\rho_2 - \rho_1)^{\rho_2-\rho_1}} l^{\rho_2-\rho_1}.$$

For an equilibrium with diversification to exist, the marginal rate of transformation corresponding to the given international price ratio, which is given by (4.3) where equality holds in (4.2), must satisfy

$$(4.7) \quad - \frac{dy_2}{dy_1} = \frac{\rho_2 p_1}{\rho_1 p_2} \leq \tau^*, \quad \text{or} \quad \frac{p_1}{p_2} \leq \frac{\rho_1}{\rho_2} \tau^*.$$

Otherwise, the country will specialize in its decreasing-return good (commodity 1), and consumers will face a budget line whose slope is equal to the international price ratio. The second inequality of (4.7) is the basic condition that must be satisfied by the world price ratio in order for our country to produce both commodities in positive amounts in laissez-faire equilibrium.

## 5 Some illustrations

In this section I consider the special case  $\rho_1 = \frac{1}{2}$ ,  $\rho_2 = 2$ , and  $\theta_1 = \theta_2 = \frac{1}{2}$ , which is particularly tractable. Then by (4.4) the point of inflection  $y^* = (y_1^*, y_2^*)$  is given by

$$(5.1) \quad \frac{d^2 y_1}{dy_1^2} = 4(3y_1^2 - l) = 0, \quad \text{where } y_1^* = \sqrt{\frac{l}{3}} \text{ and } y_2^* = \left(\frac{2l}{3}\right)^2 \text{ from (4.5).}$$

The maximum marginal rate of transformation along the production-possibility frontier is then, from (4.6), given by

$$(5.2) \quad \tau^* = - \left. \frac{dy_2}{dy_1} \right|_{y_1=\sqrt{\frac{l}{3}}} = 8 \left(\frac{l}{3}\right)^{3/2}.$$

For an equilibrium with diversification to exist, the marginal rate of transformation must be less than or equal to this maximum, whence by (4.3) and (4.2) we must have, with world prices equated to average costs,

$$(5.3) \quad - \frac{dy_2}{dy_1} = \frac{\rho_2 p_1}{\rho_1 p_2} = 4 \frac{p_1}{p_2} \leq 8 \left(\frac{l}{3}\right)^{3/2}, \quad \text{or} \quad \frac{p_1}{p_2} \leq 2 \left(\frac{l}{3}\right)^{3/2}.$$

Now we consider the trade equilibrium for factor endowment  $l$  and world price ratio  $p_1/p_2$  satisfying (5.3). Our system (2.4) and (4.2) (with equality in the latter) becomes

$$(5.4) \quad \begin{aligned} y_1^2 + y_2^{1/2} &= l \\ y_1 y_2^{1/2} &= p_1/p_2. \end{aligned}$$

Substituting the first equation into the second gives  $y_1(l - y_1^2) = -p_1/p_2$ , yielding the cubic equation

$$(5.5) \quad y_1^3 - ly_1 + p_1/p_2 = 0.$$

For  $l$  and  $p_1/p_2$  satisfying (5.3), this has two positive roots and one negative root.<sup>3</sup>

### 5.1 The case $l = 1.2, p_1/p_2 = 0.4$ (Figure 2)

Let us consider the case  $l = 1.2$  and  $p_1/p_2 = 0.4 < 2(0.4)^{3/2}$  (the inequality  $p_1/p_2 < 2(0.4)^{3/2} = 0.505964425626$  being the condition for diversification (5.3). Solving (5.5) we obtain

$$(5.6) \quad \begin{array}{ll} y_1^{(1a)} = 0.378532150575 & \text{and} \quad y_2^{(1a)} = 1.116643232950, \\ y_1^{(1b)} = 0.855977943225 & \text{and} \quad y_2^{(1b)} = 0.218370935565, \end{array}$$

where the superscripts (1a) and (1b) indicate the free-trade solutions corresponding to the two positive roots of the cubic equation. This gives the two world price lines

$$(5.7) \quad \begin{array}{l} 0.4x_1 + x_2 = 1.268056093180 \\ 0.4x_1 + x_2 = 0.560762112855. \end{array}$$

(See Figure 2; also shown in the diagram, for comparison, is the equilibrium solution  $x^{(0)} = y^{(0)}$  under autarky previously displayed in Figure 1. Note that the production-possibility frontier has a different appearance in Figure 2 than in Figure 1; but this is simply because the horizontal axis had to be rescaled to allow for the inclusion of  $x^{(1a)}$ .) Utility is maximized subject to these respective budget constraints when

$$(5.8) \quad \begin{array}{ll} x_1^{(1a)} = 1.585070116480 & \text{and} \quad x_2^{(1a)} = 0.634028046592 \\ x_1^{(1b)} = 0.700952641069 & \text{and} \quad x_2^{(1b)} = 0.280381056428. \end{array}$$

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<sup>3</sup>It is known (cf. Hodgman (1931, p. 246; 1954, p. 295)) that the three roots are real if and only if

$$(5.3a) \quad \frac{(p_1/p_2)^2}{4} \leq \frac{l^3}{27},$$

and distinct if and only if the inequality is strict; otherwise there will be just one real root. This inequality (5.3a) is equivalent to (5.3) above.

Denoting the roots of (5.5) by  $\lambda_i$ , (5.5) may be expressed as

$$(5.5a) \quad y_1^3 - (\lambda_1 + \lambda_2 + \lambda_3)y_1^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)y_1 - \lambda_1\lambda_2\lambda_3.$$

If the first two roots are imaginary, say  $\lambda_1 = \mu + \nu\sqrt{-1}$  and  $\lambda_2 = \mu - \nu\sqrt{-1}$ , then  $\lambda_1\lambda_2\lambda_3 = (\mu^2 + \nu^2)\lambda_3$  is negative from (5.5); hence  $\lambda_3 < 0$ . If all three roots are real, then since there are two changes of sign in the sequence of coefficients in (5.5), then by Descartes's rule of signs (cf. Uspensky 1948, p. 121), there can be at most two positive roots; therefore one root must be negative, which again we may identify with  $\lambda_3$ . Thus, (5.5) must always have one negative root. Since  $\lambda_1 + \lambda_2 + \lambda_3 = 0$  from (5.5), and  $\lambda_3 < 0$ , it follows that  $\lambda_1 + \lambda_2 > 0$ ; but since  $\lambda_1\lambda_2\lambda_3 < 0$  from (5.5) and (5.5a), it follows that  $\lambda_1$  and  $\lambda_2$ , if real, must have the same sign; therefore if the roots are all real, both  $\lambda_1 > 0$  and  $\lambda_2 > 0$ .



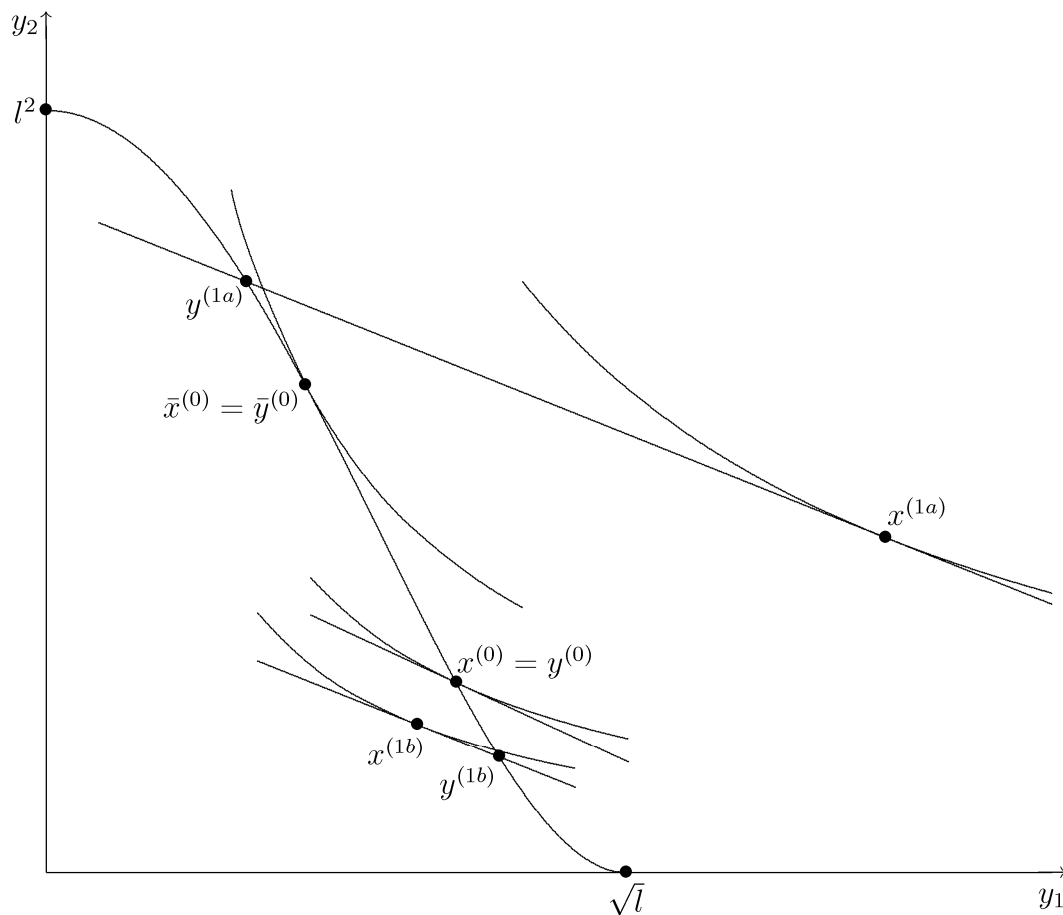


Figure 2

$$\rho_1 = \frac{1}{2}, \rho_2 = 2, l = 1.2, p_1/p_2 = 0.4$$

Thus, net imports are given by

$$\begin{aligned}
 z_1^{(1a)} &= x_1^{(1a)} - y_1^{(1a)} = +1.206537965900 & \text{and} \\
 z_2^{(1a)} &= x_2^{(1a)} - y_2^{(1a)} = -0.482615186358; \\
 z_1^{(1b)} &= x_1^{(1b)} - y_1^{(1b)} = -0.155025302156 & \text{and} \\
 z_2^{(1b)} &= x_2^{(1b)} - y_2^{(1b)} = +0.062010120863.
 \end{aligned}
 \tag{5.9}$$

Thus, in the first of the two possible trade equilibria, the country exports the increasing-return good, while in the second it exports the decreasing-return good. It is clear from the diagram that the country gains from trade in the first case, and loses in the

second case. This is confirmed by the computations

$$(5.10) \quad \begin{aligned} U(x_1^{(1a)}, x_2^{(1a)}) &= 1.002486363830 \\ U(x_1^{(0)}, x_2^{(0)}) &= 0.528067042076 \\ U(x_1^{(1b)}, x_2^{(1b)}) &= 0.443321375538. \end{aligned}$$

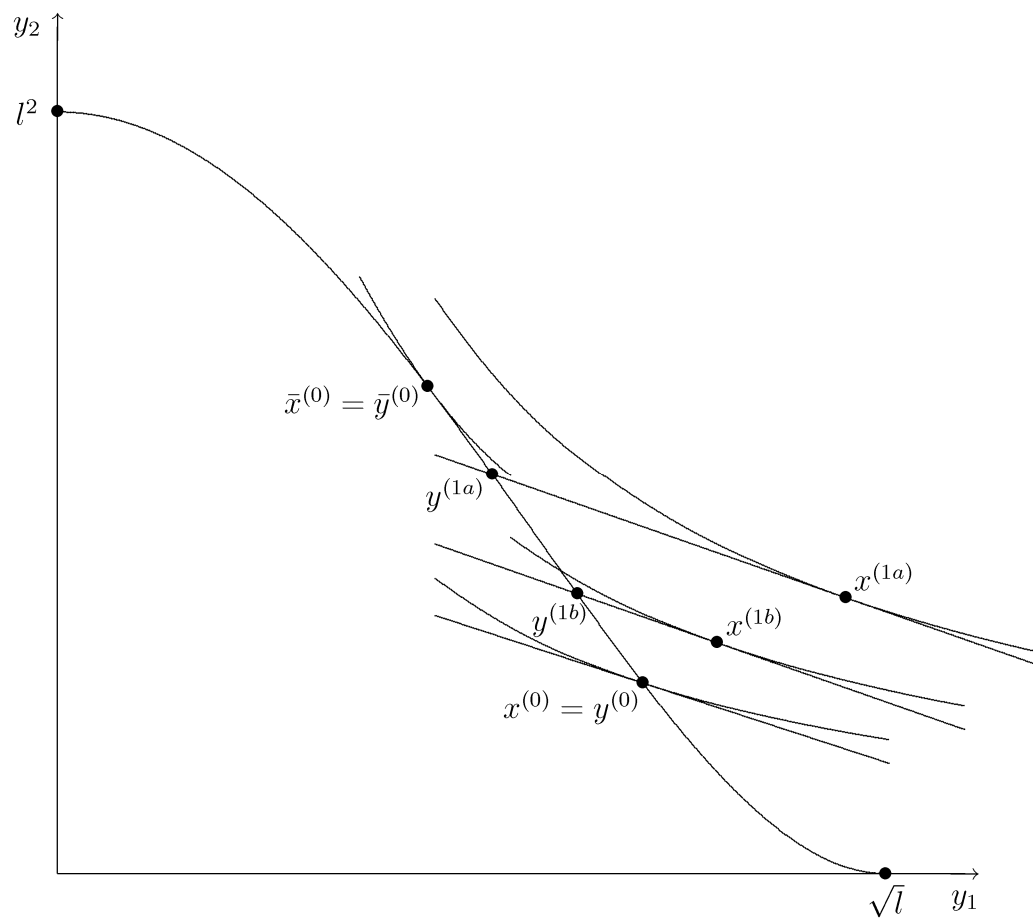


Figure 3  
 $\rho_1 = \frac{1}{2}, \rho_2 = 2, l = 1.2, p_1/p_2 = 0.5.$

## 5.2 The case $l = 1.2, p_1/p_2 = 0.5$ (Figure 3)

Now let us consider the case  $l = 1.2$  and  $p_1/p_2 = 0.5 < 2(0.4)^{3/2}$  (again see (4.7)). Solving (5.5) we obtain the solution

$$(5.11) \quad \begin{aligned} y_1^{(1a)} &= 0.575527835434 & \text{and} & & y_2^{(1a)} &= 0.754757335052, \\ y_1^{(1b)} &= 0.687723541444 & \text{and} & & y_2^{(1b)} &= 0.528581825931, \end{aligned}$$

This gives the two world price lines

$$(5.12) \quad \begin{aligned} 0.5x_1 + x_2 &= 1.042521252770 \\ 0.5x_1 + x_2 &= 0.872443596653 \end{aligned}$$

(see Figure 3). Utility is maximized subject to these respective budget constraints when

$$(5.13) \quad \begin{aligned} x_1^{(1a)} &= 1.042521252770 & \text{and} & & x_2^{(1a)} &= 0.521260626385 \\ x_1^{(1b)} &= 0.872443596653 & \text{and} & & x_2^{(1b)} &= 0.436221798326. \end{aligned}$$

Thus, net imports are given by

$$(5.14) \quad \begin{aligned} z_1^{(1a)} &= x_1^{(1a)} - y_1^{(1a)} = +0.466993417336 & \text{and} \\ z_2^{(1a)} &= x_2^{(1a)} - y_2^{(1a)} = -0.233496708667; \\ z_1^{(1b)} &= x_1^{(1b)} - y_1^{(1b)} = +0.184720055209 & \text{and} \\ z_2^{(1b)} &= x_2^{(1b)} - y_2^{(1b)} = -0.092360027605. \end{aligned}$$

Thus we see that in this case our country exports commodity 2 (the increasing-return good) and imports commodity 1 (the decreasing-return good) in both cases. The utility levels in this case are

$$(5.15) \quad \begin{aligned} U(x_1^{(1a)}, x_2^{(1a)}) &= 0.737173847365 \\ U(x_1^{(0)}, x_2^{(0)}) &= 0.528067042076 \\ U(x_1^{(1b)}, x_2^{(1b)}) &= 0.616910783396. \end{aligned}$$

Thus, the country gains from trade in both cases.

## 5.3 The borderline case in which $l = 1.2$ and the world price ratio $p_1/p_2$ equals the autarky price ratio 0.465 (Figure 4)

We recall from (3.3) that when  $l = 1.2$ ,  $\rho_1 = \frac{1}{2}$  and  $\rho_2 = 2$ , and  $\theta_1 = \theta_2 = \frac{1}{2}$ , the autarky price ratio is  $p_1/p_2 = 0.6^{3/2} = 0.464758001545$ . In subsections 5.1 and 5.2 we considered the cases in which the world price ratio  $p_1/p_2$  is 0.4 and 0.5 respectively.

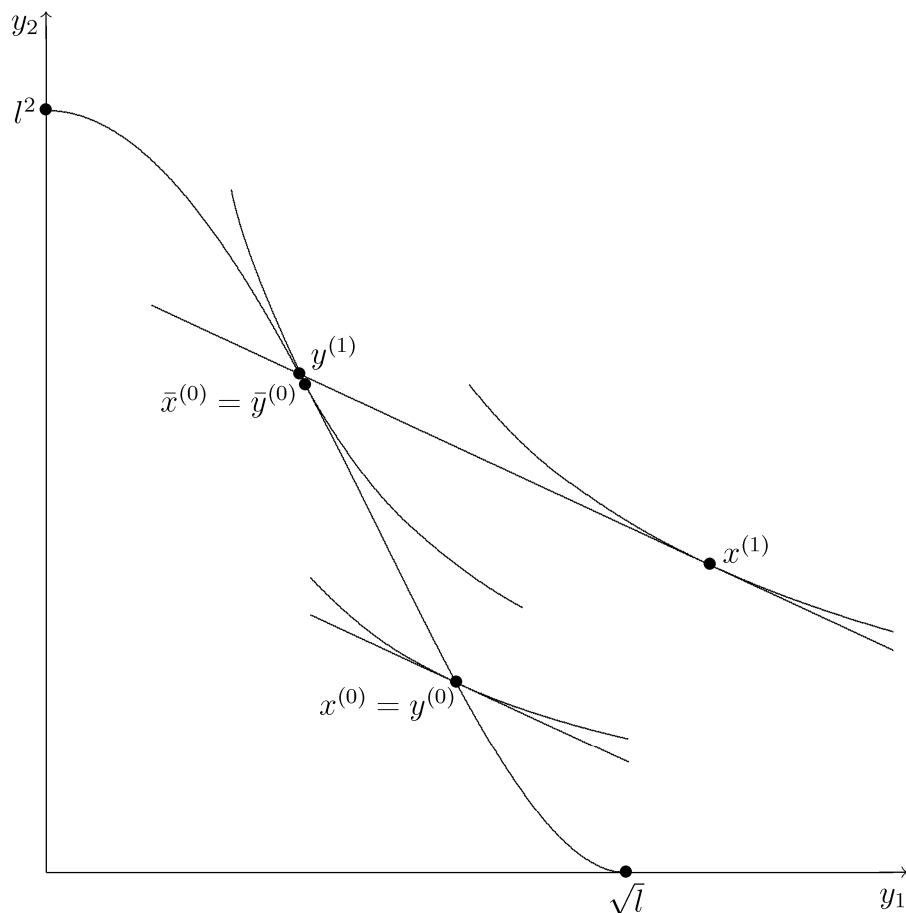


Figure 4

$$\rho_1 = \frac{1}{2}, \rho_2 = 2, l = 1.2, p_1/p_2 = .6^{3/2} = 0.464758002.$$

According to Jones's (1956) interpretation of Ohlin's (1933) definition of comparative advantage, in the first case—where  $0.4 < 0.465$ —our country has a comparative advantage in commodity 2, and in the second case—where  $0.5 > 0.465$ —it has a comparative advantage in commodity 1. In the first case (with the world price ratio  $p_1/p_2 = 0.4$ ), we found that our country will export commodity 2 in case (a) (and gain from trade), but export commodity 1 in case (b) (and lose from trade). Case (a) therefore agrees with Ohlin's definition. In the second case (with world price-ratio  $p_1/p_2 = 0.5$ ), our country exports commodity 2 in both cases, and gains from trade in both cases; yet according to Ohlin's definition, it has a comparative advantage in commodity 1.

What happens if the world price ratio is exactly equal to the autarky price ratio?

In conventional constant-returns-to-scale theory, there will be no trade. In the present case this is just one of two possible equilibrium solutions, and not the optimal one.

Solving (5.5) for  $l = 1.2$  and  $p_1/p_2 = 0.6^{3/2}$  we obtain

$$(5.16) \quad \begin{array}{ll} y_1^{(0)} = 0.774596669241 & \text{and} \quad y_2^{(0)} = 0.36, \\ y_1^{(1)} = 0.478727069164 & \text{and} \quad y_2^{(1)} = 0.942492235950, \end{array}$$

The first pair of values are of course the autarky outputs (3.2), and the second pair the outputs under trade. The corresponding price lines are (see Figure 4)

$$(5.17) \quad \begin{array}{l} 0.464758001545x_1 + x_2 = 0.72 \\ 0.464758001545x_1 + x_2 = 1.1649844719 \end{array}$$

The first of these is precisely the autarky price line (3.4), and the second is the trade price line. Utility is maximized subject to these respective budget constraints when

$$(5.18) \quad \begin{array}{ll} x_1^{(0)} = 0.774596669242 = 0.6^{1/2} & \text{and} \quad x_2^{(0)} = 0.36 \\ x_1^{(1)} = 1.253323738453 & \text{and} \quad x_2^{(1)} = 0.582492235948. \end{array}$$

Again, the first pair correspond to the autarky consumptions (3.2), and only the second pair to consumptions under trade. The utility levels corresponding to these two situations are

$$(5.19) \quad \begin{array}{l} U(x_1^{(0)}, x_2^{(0)}) = 0.671930018225 \\ U(x_1^{(1)}, x_2^{(1)}) = 0.730051346747. \end{array}$$

The country is clearly better off with trade (see Figure 4).

#### 5.4 The borderline case $l = 1.2, p_1/p_2 = 0.506$ (Figure 5)

Now let us consider a second borderline case in which  $l = 1.2$  and  $p_1/p_2 = 2(l/3)^{3/2} = 0.505964425626$ . This is the case in which (5.5) has two repeated positive roots, giving as its unique solution the point of inflection (5.1):

$$(5.1) \quad \begin{array}{l} y_1^{(1)} = y_1^* = \sqrt{l/3} = 0.632455532034 \quad \text{and} \\ y_2^{(1)} = y_2^* = (2l/3)^2 = 0.64. \end{array}$$

The free-trade budget line is

$$(5.2) \quad 0.5059644256265x_1 + x_2 = 0.96,$$

and the consumption levels which maximize utility are

$$(5.3) \quad x_1^{(1)} = 0.948683298052 \quad \text{and} \quad x_2^{(1)} = 0.48.$$

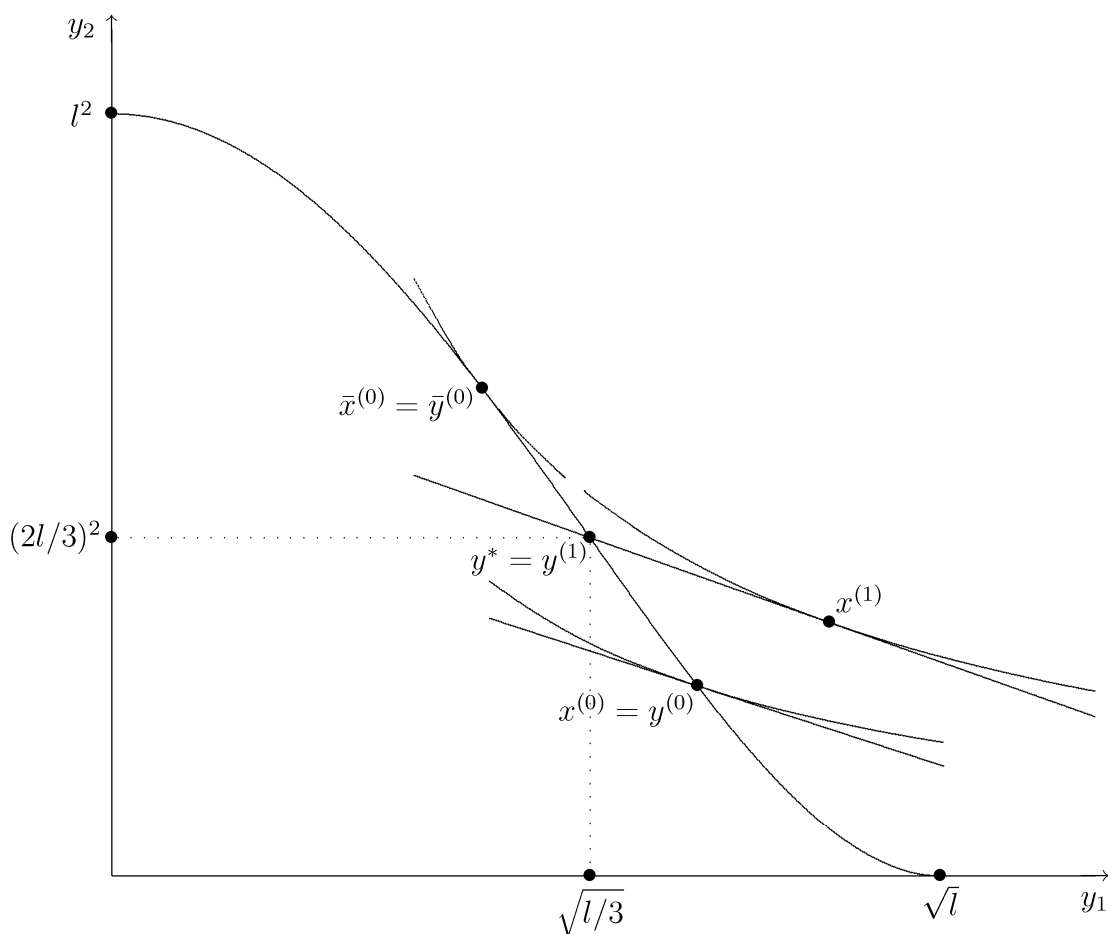


Figure 5

$$\rho_1 = \frac{1}{2}, \rho_2 = 2, l = 1.2, p_1/p_2 = 2(l/3)^{3/2} = .5059644256.$$

The trades are therefore

$$(5.4) \quad \begin{aligned} z_1^{(1)} &= x_1^{(1)} - y_1^{(1)} = +0.316227766018 \quad \text{and} \\ z_2^{(1)} &= x_2^{(1)} - y_2^{(1)} = -0.16. \end{aligned}$$

The utility levels are, under trade and autarky respectively,

$$(5.5) \quad \begin{aligned} U(x_1^{(1)}, x_2^{(1)}) &= 0.674809590229 \\ U(x_1^{(0)}, x_2^{(0)}) &= 0.528067042076. \end{aligned}$$

Thus, again the country gains from trade.

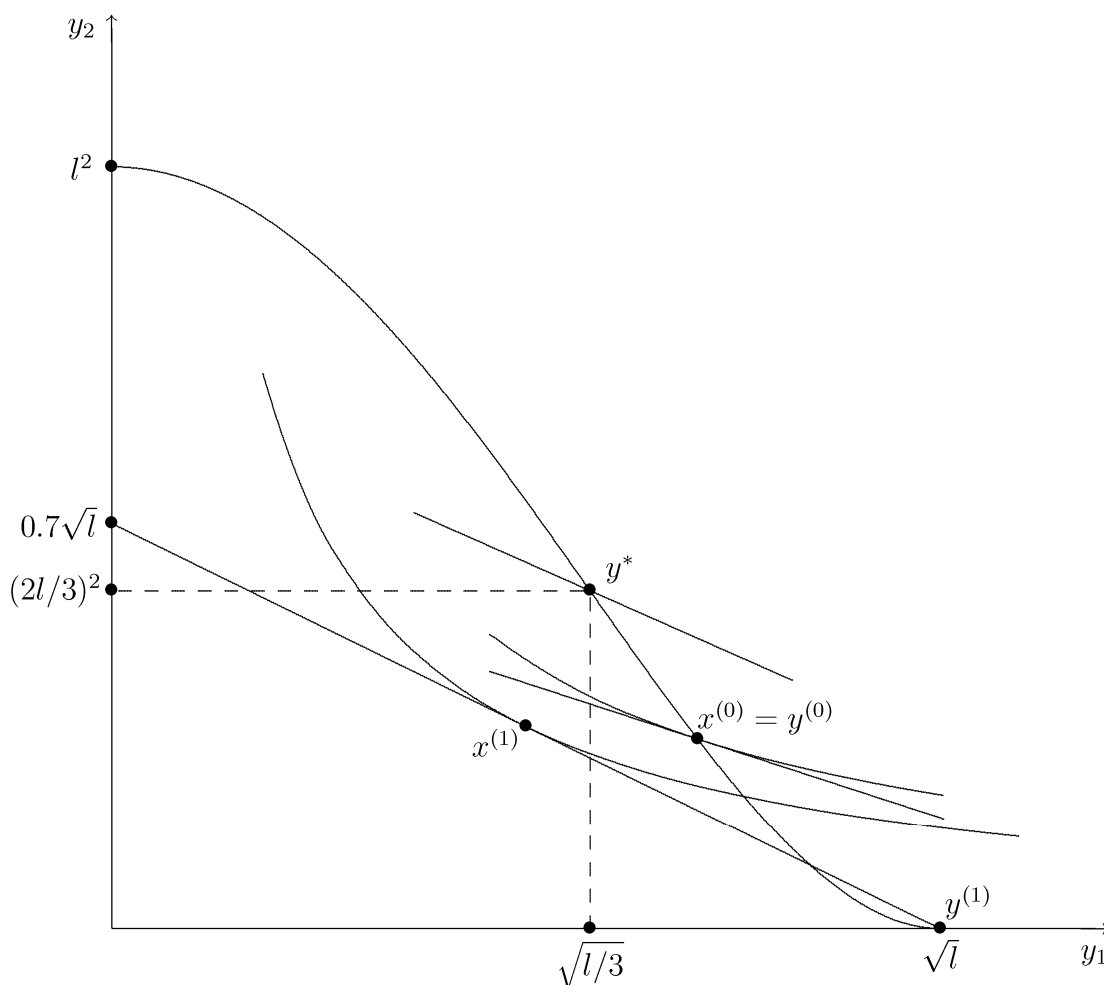


Figure 6

$$\rho_1 = \frac{1}{2}, \rho_2 = 2, l = 1.2; p_1/p_2 = 0.7.$$

### 5.5 The case $l = 1.2, p_1/p_2 = 0.7$ : specialization (Figure 6)

The final case to consider is that in which the inequality in (5.3) is reversed, so that the inequality in (4.2) is strict, i.e., the world price ratio satisfies  $p_1/p_2 > 2(l/3)^{3/2}$ . With  $l = 1.2$  we may take  $p_1/p_2 = 0.7 > 2(0.4)^{3/2} = 0.5059644256265$ . In this case our country necessarily specializes in the decreasing-return good (commodity 1), which is Graham's case (see Graham 1923, 1925). The production vector is  $y^{(1)} = (\sqrt{l}, 0) = (1.09544511501, 0)$  and the price line is  $0.7x_1 + x_2 = 0.7\sqrt{1.2} = 0.766811580507$ , which passes below the autarky point  $x^{(0)} = (0.774596669241, 0.36)$ . Equilibrium consumption is  $x^{(1)} = (0.547722557505, 0.383405790254)$ , yielding a utility of  $U(x^{(1)}) = \sqrt{0.21} = 0.458257569496 < U(x^{(0)}) = 0.6^{5/4} = 0.528067042076$ . Thus the country is worse off under trade than under autarky!

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