Notes on Dynamic Stability of the Marshallian Model of International Trade

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1 Formulation of the model

Let us suppose there are two countries each consuming three commodities—two tradables and one nontradable. The home country (country 1) is initially exporting commodity 1 to and importing commodity 2 from the foreign country (country 2). The consumption, production, and net import of commodity j in country k are denoted x_j^k, y_j^k, and $z_j^k = x_j^k - y_j^k$; thus, $z_k^k < 0, z_j^k > 0$ $(j \neq k, j \neq 3)$, and $z_3^k = 0$ in equilibrium.

Let each country have an aggregate utility function $U^k(x^k) = U^k(x_1^k, x_2^k, x_3^k)$ and a (closed, bounded) production-possibility set $\mathcal{Y}^k$ consisting of all technically feasible bundles $y^k = (y_1^k, y_2^k, y_3^k)$ of output. We may define country k's *net-utility function* as

$$(1) \qquad \tilde{U}^k(z^k) = \max\{U^k(x^k) : x^k \in \mathcal{Y}^k + z^k\}.$$

Geometrically this may be visualized as a three-dimensional version of Meade's (1952) trade-indifference map. The trade-utility function is defined as $\hat{U}^k(z_1^k, z_2^k) = \tilde{U}^k(z_1^k, z_2^k, 0)$. Its contours correspond to Meade's (1952) trade-indifference curves, generalized to take account of nontradables; the z_j^k are trades or net imports (imports if positive, exports if negative). The inverse net-demand functions are defined as the demand (or supply) prices of tradables, equal to the marginal rates of substitution (according to the net-utility function) between the tradables and the nontradable:

$$(2) \qquad \frac{p_j^k}{p_3^k} = \tilde{P}_j^k(z^k) \equiv \frac{\partial \tilde{U}^k / \partial z_j^k}{\partial \tilde{U}^k / \partial z_3^k} \equiv \frac{\tilde{U}_j^k}{\tilde{U}_3^k} \quad (j, k = 1, 2)$$

where p_j^k is the nominal price of commodity j in country k denominated in country k's currency. A special case of interest, prominent in the older literature (Bickerdike 1907, Edgeworth 1908, Robinson 1937—cf. Chipman 1978), is that in which the trade-utility functions are additively separable, i.e.,

$$(3) \qquad \hat{U}^k(z^k) = \hat{u}_j^k(z_j^k) - \hat{u}_k^k(-z_k^k) + \hat{u}_3^k(z_3^k) \quad (j, k) = (1, 2) \text{ or } (2, 1).$$

It is important to note that additive separability of the original utility function $U^k(x^k)$ does not imply additive separability of the trade-utility function $\hat{U}^k(z^k)$.

The *inverse (indirect) trade-demand functions* for country $k = 1, 2$ are defined by

$$(4) \qquad \begin{aligned} \hat{P}_j^k(z_1^k, z_2^k) &= \tilde{P}_j^k(z_1^k, z_2^k, 0) \quad \text{for} \quad j = 1, 2; \\ \hat{P}_3^k(z_1^k, z_2^k) &= z_1^k \hat{P}_1^k(z_1^k, z_2^k) + z_2^k \hat{P}_2^k(z_1^k, z_2^k). \end{aligned}$$

(The third of these equations—following Samuelson's (1950, p. 377) notational convention—defines country k's trade deficit as a function of its trades.) The elasticities of the first two of these inverse functions—or "flexibilities" as they are sometimes called—are defined as

$$(5) \qquad \pi_{ij}^k = \frac{z_j^k}{\hat{P}_i^k} \frac{\partial \hat{P}_i^k}{\partial z_j^k} \quad (i, j, k = 1, 2).$$

Country 1's inverse trade-demand functions must satisfy a budget constraint yielding balanced trade expressed in the prices of its tradables relative to the price of its nontradable:

$$(6) \qquad z_1^1 \hat{P}_1^1(z_1^1, z_2^1) + z_2^1 \hat{P}_2^1(z_1^1, z_2^1) = 0.$$

Likewise for country 2:
$$(7) \qquad z_1^2 \hat{P}_1^2(z_1^2, z_2^2) + z_2^2 \hat{P}_2^2(z_1^2, z_2^2) = 0.$$

These two equations define the Marshallian offer curves for countries 1 and 2 respectively.

Finally, given our choice of notation we must specify the material-balance condition

$$(8) \qquad z_j^1 + z_j^2 = 0 \quad (j = 1, 2).$$

2   Dynamic adjustment and stability

The simplest way to proceed is to postulate the dynamic-adjustment process[1]

$$(9) \qquad \begin{aligned} \dot{z}_1^2 &\propto \hat{D}^1(z_1^2, z_2^1) & \equiv -z_1^2 \hat{P}_1^1(-z_1^2, z_2^1) + z_2^1 \hat{P}_2^1(-z_1^2, z_2^1) \\ \dot{z}_2^1 &\propto \hat{D}^2(z_1^2, z_2^1) & \equiv z_1^2 \hat{P}_1^2(z_1^2, -z_2^1) - z_2^1 \hat{P}_2^2(z_1^2, -z_2^1) \end{aligned}$$

where $\hat{D}^k$ is the deficit in country k's balance of payments on current account, denominated in its external prices measured relatively to the price of its nontradable. For example, the first relation of (9) states that the rate of increase $\dot{z}_1^2$ of country 1's exports to country 2, $-z_1^1 = z_1^2$, is directly proportional to country 1's trade deficit. Assuming that points (z_1^2, z_2^1) (Marshall's "exchange index"[2]) to the left and right of country 1's offer curve, and below and above country 2's offer curve, correspond to deficits and surpluses of their current-account balances respectively, (9) is a possible rendition of Marshall's adjustment process.[3]

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[1] Here, the symbol "$\propto$" means "is proportional to" or "varies as" or "is a positive multiple of."

[2] Marshall 1879, p. 9; 1923, p. 340.

[3] Marshall argued (1879, p. 18; 1923, p. 341) that if exchange took place at a point to the left of country 1's offer curve, say at a point $\left(z_1^2(t), z_2^1(0)\right)$ to the left of the point $\left(z_1^2(0), z_2^1(0)\right)$ on its offer curve (where $z_1^2(t) < z_1^2(0)$), this would mean that country 1 was exporting only $z_1^2(t)$ of commodity 1 in exchange for imports of $z_2^1(0)$ of commodity 2, when it is capable of exporting the larger quantity $z_1^2(0)$ in a competitive equilibrium with zero profits. This would imply, according to Marshall, that when industry 1 was exporting the smaller amount $z_1^2(t)$ in exchange for the same amount of imports, this "must be a trade which affords abnormally high profits"; accordingly, exports of commodity 1 will increase. This argument is not entirely convincing, since if industry 1 is making a profit in the sense that it is earning more than it is spending, and the other industries are breaking even, then the country as a whole must be experiencing a trade surplus. Note that the subsequent description in Marshall (1923, p. 341) is somewhat confused, since Figure 10 on p. 340 should correspond to Figure 7 of the 1879 version, but does not.

The equations $\hat{D}^1(z_1^2, z_2^1) = 0$ and $\hat{D}^2(z_1^2, z_2^1) = 0$ define the Marshallian offer curves for countries 1 and 2 respectively. We verify that, when evaluated at any of these balanced-trade points, the partial derivatives of these functions are given by

$$(10) \qquad \begin{aligned} \frac{\partial \hat{D}^1}{\partial z_1^2} &= -\hat{P}_1^1[1 + \pi_{11}^1 - \pi_{21}^1]; & \frac{\partial \hat{D}^1}{\partial z_2^1} &= \hat{P}_2^1[1 - \pi_{12}^1 + \pi_{22}^1]; \\ \frac{\partial \hat{D}^2}{\partial z_1^2} &= \hat{P}_1^2[1 + \pi_{11}^2 - \pi_{21}^2]; & \frac{\partial \hat{D}^2}{\partial z_2^1} &= -\hat{P}_2^2[1 - \pi_{12}^2 + \pi_{22}^2]; \end{aligned}$$

hence, defining the Jacobian matrix $J = \partial(D^1, D^2)/\partial(z_1^2, z_2^1)$, we see that

$$(11) \qquad \begin{bmatrix} 1/z_1^2 P_1^1 & 0 \\ 0 & 1/z_1^2 P_1^2 \end{bmatrix} J \begin{bmatrix} z_1^2 & 0 \\ 0 & z_2^1 \end{bmatrix} = \begin{bmatrix} -1 - \pi_{11}^1 + \pi_{21}^1 & 1 - \pi_{12}^1 + \pi_{22}^1 \\ 1 + \pi_{11}^2 - \pi_{21}^2 & -1 + \pi_{12}^2 - \pi_{22}^2 \end{bmatrix}.$$

By the implicit-function theorem, provided $\partial \hat{D}^1/\partial z_1^2 \neq 0$ and $\partial \hat{D}^2/\partial z_2^1 \neq 0$, the Marshallian offer functions of the respective countries may be defined explicitly as

$$(12) \qquad \begin{aligned} z_1^2 &= F^1(z_2^1) \quad \text{where} \quad \hat{D}^1(F^1(z_2^1), z_2^1) = 0; \\ z_2^1 &= F^2(z_1^2) \quad \text{where} \quad \hat{D}^2(z_1^2, F^2(z_1^2)) = 0, \end{aligned}$$

expressing the amount of each country's exports as a function of the amount of its imports. The elasticities of these two functions are

$$(13) \qquad \alpha^1 \equiv \frac{z_2^1}{F^1} \frac{\partial F^1}{\partial z_2^1} = \frac{1 - \pi_{12}^1 + \pi_{22}^1}{1 + \pi_{11}^1 - \pi_{21}^1}; \quad \alpha^2 \equiv \frac{z_1^2}{F^2} \frac{\partial F^2}{\partial z_1^2} = \frac{1 - \pi_{21}^2 + \pi_{11}^2}{1 + \pi_{22}^2 - \pi_{12}^2}.$$

These are called the "elasticities of trade" (Alexander, 1951).

From the results of Metzler (1945) and Arrow (1974), for the system (9) to be stable independently of the speeds of adjustment, the principal minors of (11) must be alternately nonpositive and nonnegative, that is, the diagonal elements must be nonpositive and the determinant must be nonnegative. For an equilibrium solution of (9) (a solution of (6), (7), and (8)) to be isolated, the determinant Δ of (11) must be nonzero, hence positive. If the diagonal elements are strictly negative (which we required in order to define the functions F^1 and F^2 in (12)) then in view of the definitions (13) we may write the condition $\Delta > 0$ as

$$(14) \qquad\qquad\qquad\qquad\qquad \alpha^1 \alpha^2 < 1.$$

Writing this inequality as

$$\frac{z_2^1}{F^1} \frac{dF^1}{dz_2^1} \cdot \frac{z_1^2}{F^2} \frac{dF^2}{dz_1^2} = \frac{dF^1}{dz_2^1} \cdot \frac{dF^2}{dz_1^2} < 1$$

and noting that in the (z_1^2, z_2^1) plane the slope of country 2's offer curve is dF^2/dz_1^2 and that of country 1 is the reciprocal of dF^1/dz_2^1, if both slopes are positive then (14) implies $\alpha^2 < 1/\alpha^1$ and therefore country 2's offer curve is less steep than that of country 1 at the intersection point of the two curves. On the other hand, if both slopes are negative then (14) implies $(-\alpha^1)(-\alpha^2) < 1$ or $|\alpha^2| < 1/|\alpha^1|$, so again country 2's offer curve at the

intersection point should be flatter than country 1's for the equilibrium to be stable. If the slopes of the two countries' offer curves have opposite signs then $\alpha^1\alpha^2 < 0$ hence clearly the stability condition (14) is satisfied.

The condition that the diagonal elements of (11) be strictly negative implies in view of (10) that $\partial \hat{D}^1/\partial z_1^2 < 0$ and $\partial \hat{D}^2/\partial z_2^1 < 0$. Supposing $(\bar{z}_1^2, \bar{z}_2^1)$ to be an equilibrium solution, i.e., a pair such that $\hat{D}^1(\bar{z}_1^2, \bar{z}_2^1) = 0$ and $\hat{D}^2(\bar{z}_1^2, \bar{z}_2^1) = 0$, this implies that if $z_1^2 < \bar{z}_1^2$ while $z_2^1 = \bar{z}_2^1$, then $\hat{D}^1(z_1^2, \bar{z}_2^1) > 0$; that is, points to the left of country 1's offer curve are points where country 1 has a trade deficit, and likewise point to the right correspond to surpluses. From the dynamic-adjustment process (9) it follows that points to the left of country 1's offer curve will move rightwards towards the offer curve, and points to the right will move leftwards towards the offer curve. Similarly, points below country 2's offer curve will move upwards towards the offer curve, and point above it will move downwards towards the offer curve. This is the basic Marshallian process.

3   Qualitative analysis of the adjustment process

If we wish to work with the "flexibilities" π_{ij}^k and the elasticities of trade α^k, it is more convenient to deal with the logarithms of the z_1^2, z_2^1. Thus, defining $\zeta_1^2 = \log z_1^2$ and $\zeta_2^1 = \log z_2^1$ and the composite functions $\Phi^1 = \log \circ F^1 \circ \exp$ and $\Phi^2 = \log \circ F^2 \circ \exp$, relations (12) become

(15) $\qquad \zeta_1^2 = \Phi^1(\zeta_2^1) \equiv \log F^1(e^{\zeta_2^1}) \quad \text{and} \quad \zeta_2^1 = \Phi^2(\zeta_1^2) \equiv \log F^2(e^{\zeta_1^2}).$

It then becomes quite straightforward to undertake the qualitative analysis of the stability properties of various equilibria.

We start by rewriting (9) in the form

$$(16) \qquad \begin{aligned} \dot{z}_1^2/z_1^2 &= \dot{\zeta}_1^2 \propto e^{-\zeta_1^2}\hat{D}^1(e^{\zeta_1^2}, e^{\zeta_2^1}) \\ \dot{z}_2^1/z_2^1 &= \dot{\zeta}_2^1 \propto e^{-\zeta_2^1}\hat{D}^2(e^{\zeta_1^2}, e^{\zeta_2^1}) \end{aligned}$$

and then by taking Taylor approximations of the functions $\hat{D}^k(e^{\zeta_1^2}, e^{\zeta_2^1})$ around a point $(\bar{\zeta}_1^2, \bar{\zeta}_2^1) = (\log \bar{z}_1^2, \log \bar{z}_2^1)$:

$$\hat{D}^1(e^{\zeta_1^2}, e^{\zeta_2^1}) = \hat{D}^1(\bar{z}_1^2, \bar{z}_2^1) + \hat{D}_1^1(\bar{z}_1^2, \bar{z}_2^1)\bar{z}_1^2(\zeta_1^2 - \bar{\zeta}_1^2) + \hat{D}_2^1(\bar{z}_1^2, \bar{z}_2^1)\bar{z}_2^1(\zeta_1^2 - \bar{\zeta}_1^2) + R^1(\zeta_1^2, \zeta_2^1)$$
$$\hat{D}^2(e^{\zeta_1^2}, e^{\zeta_2^1}) = \hat{D}^2(\bar{z}_1^2, \bar{z}_2^1) + \hat{D}_1^2(\bar{z}_1^2, \bar{z}_2^1)\bar{z}_2^1(\zeta_2^1 - \bar{\zeta}_1^1) + \hat{D}_2^2(\bar{z}_1^2, \bar{z}_2^1)\bar{z}_2^1(\zeta_2^1 - \bar{\zeta}_1^1) + R^2(\zeta_1^2, \zeta_2^1)$$

(17)
where $\hat{D}_j^k(z_1^2, z_2^1) \equiv \partial \hat{D}^k(z_1^2, z_2^1)/\partial z_j^i$ for $i = 3 - j$ and $R^k(\cdot)$ is the remainder term. If $(\bar{z}_1^2, \bar{z}_2^1) = (e^{\bar{\zeta}_1^2}, e^{\bar{\zeta}_2^1})$ is an *equilibrium* solution then by definition of equilibrium we have $\hat{D}^k(\bar{z}_1^2, \bar{z}_2^1) = 0$ for $k = 1, 2$ hence, using (10) and defining

$$\xi_1 = \zeta_1^2 - \bar{\zeta}_1^2, \quad \xi_2 = \zeta_2^1 - \bar{\zeta}_2^1,$$

(16) becomes, after neglecting the remainder terms,

$$(18) \qquad \begin{aligned} \dot{\xi}_1 &\propto \hat{P}_1^1[-(1 + \pi_{11}^1 - \pi_{21}^1)\xi_1 + (1 - \pi_{12}^1 + \pi_{22}^1)\xi_2] \\ \dot{\xi}_2 &\propto \hat{P}_2^2[\ (1 + \pi_{11}^2 - \pi_{21}^2)\xi_1 - (1 - \pi_{12}^2 + \pi_{22}^2)\xi_2], \end{aligned}$$

4

where use has been made of the fact that, in equilibrium, $z_2^1 P_2^1 = z_1^2 P_1^1$ and $z_1^2 P_1^2 = z_2^1 P_2^2$.

Now let us introduce the speeds of adjustment. The notation "$x \propto y$" means "$x = \sigma y$ for some $\sigma > 0$"; we may therefore write (18) explicitly as

$$
(19) \qquad
\begin{aligned}
\dot{\xi}_1 &= \sigma_1 \hat{P}_1^1 [-(1 + \pi_{11}^1 - \pi_{21}^1)\xi_1 + (1 - \pi_{12}^1 + \pi_{22}^1)\xi_2] \\
\dot{\xi}_2 &= \sigma_2 \hat{P}_2^2 [\ \ (1 + \pi_{11}^2 - \pi_{21}^2)\xi_1 - (1 - \pi_{12}^2 + \pi_{22}^2)\xi_2]
\end{aligned}
$$

where the σ_i are the speeds of adjustment. We shall write this in the form

$$
(20) \qquad
\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix}
=
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}
$$

where

$$
(21) \qquad
A =
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
=
\begin{bmatrix} \sigma_1 \hat{P}_1^1 & 0 \\ 0 & \sigma_2 \hat{P}_2^2 \end{bmatrix}
\begin{bmatrix} -1 - \pi_{11}^1 + \pi_{21}^1 & 1 - \pi_{12}^1 + \pi_{22}^1 \\ 1 + \pi_{11}^2 - \pi_{21}^2 & -1 + \pi_{12}^2 - \pi_{22}^2 \end{bmatrix}.
$$

Here it is understood that the flexibilities $\pi_{ij}^k(z_1^k, z_2^k)$ are evaluated at the given equilibrium point $(\bar{z}_1^k, \bar{z}_2^k)$. Two special cases are of interest: (1) that in which $\sigma_k = 1/\hat{P}_k^k$ for $k = 1, 2$, so that the diagonal matrix in (21) reduces to the identity matrix; and (2) that in which

$$
(22) \qquad \sigma_k = \frac{1}{\hat{P}_k^k[1 + \pi_{kk}^k - \pi_{3-k,k}^k]} \qquad \text{for } k = 1, 2,
$$

in which case the matrix A reduces to

$$
(23) \qquad
A =
\begin{bmatrix} -1 & \alpha^1 \\ \alpha^2 & -1 \end{bmatrix}
$$

as can be seen from (13), and the dynamic-adjustment process is similar to that outlined by Samuelson (1947, pp. 266–7) and further analyzed in Chipman (1987, pp. 935–7).

With the above notation we can write the system of differential equations (19) in the matrix form
$$
(24) \qquad \dot{\xi} = A\xi.
$$

We denote the characteristic roots and vectors of A by λ_i and h^i respectively, so that

$$
Ah^i = \lambda_i, \quad h^i \neq 0.
$$

Since special methods are needed to handle the case of repeated roots, it will be assumed that the characteristic roots of A are distinct. These are the roots of the characteristic equation

$$
\begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = \lambda^2 - (\text{trace}A)\lambda + \det A = 0
$$

which are given by

$$
(25) \qquad \lambda = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{21}a_{12}}}{2}.
$$

Note that for the roots to be distinct it is necessary that the discriminant in (25) be nonvanishing, which implies in particular that $a_{21}a_{12} \neq 0$. Repeated roots could occur in particular if both $a_{11} = a_{22}$ and $a_{21}a_{12} = 0$; this could happen if the countries' offer curves were completely symmetric and they intersected at a point where they both had zero slope. This can certainly not be ruled out. However, special methods would be needed to handle such a case, so we shall assume that equilibrium does not happen to occur at such a point.[4] As noted above, a sufficient condition for the roots to be distinct is that $a_{21}a_{12} \neq 0$ at equilibrium; referring to (13) this implies that $\alpha^1 \alpha^2 \neq 0$ at equilibrium. This will ensure that the characteristic roots of A are distinct, i.e., that $\lambda_1 \neq \lambda_2$.

It is well known that when the characteristic roots of a matrix are distinct, its characteristic vectors are linearly independent (cf., e.g., Hurewicz, 1958, pp. 58–9; Pontryagin, 1962, pp. 278–9). Let $H = [h^1, h^2]$ be the matrix of these chacteristic vectors; then

$$AH = A[h^1, h^2] = [h^1 \lambda_1, h^2 \lambda_2] = H\Lambda,$$

where $\Lambda = \text{diag}\{\lambda_1, \lambda_2\}$ is the diagonal matrix of characteristic roots of A. Since H is nonsingular,

$$H^{-1}AH = \Lambda.$$

Defining the transformation

(26) $$\xi = Hu \quad \text{(i.e., } u = H^{-1}\xi),$$

in terms of the new variables u_1, u_2 we have

$$\dot{u} = H^{-1}\dot{\xi} = H^{-1}A\xi = H^{-1}AHu = \Lambda u$$

hence

$$\dot{u}_i = \lambda_i u_i \quad (i = 1, 2).$$

Thus, $du_i/u_i = \lambda_i dt$, and integrating this equation we obtain $\log u_i = \lambda_i t + c_i$ where c_i is a constant, whence

$$u_i = b_i e^{\lambda_i t} \quad (i = 1, 2)$$

where $b_i = \exp c_i$. Substituting this in (26) we obtain the solution of (24):

(27) $$\xi = h^1 u_1 + h^2 u_2 = h^1 b_1 e^{\lambda_1 t} + h^2 b_2 e^{\lambda_2 t},$$

or

(28) $$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} h_{11} \\ h_{21} \end{bmatrix} b_1 e^{\lambda_1 t} + \begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix} b_2 e^{\lambda_2 t},$$

where $H = [h_{ij}]$.

Now we introduce a further simplification. Since H is nonsingular, h_{11} and h_{12} cannot both be zero. Suppose one of them, say h_{1j}, is zero; then since $h^j \neq 0$ by the definition

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[4]If there is a single repeated root, one must consider separately the cases in which the matrix $I\lambda - A$ has rank 1 or 0. For a discussion see Hurewicz (1958, pp. 80–82), Pontryagin (1962, pp. 121–6).

of a characteristic vector, it follows of course that $h_{2j} \neq 0$. But then the corresponding characteristic root λ_j satisfies

$$\begin{bmatrix} \lambda_j - a_{11} & -a_{12} \\ -a_{21} & \lambda_j - a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ h_{2j} \end{bmatrix} = \begin{bmatrix} -a_{12} \\ \lambda_j - a_{22} \end{bmatrix} h_{2j} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

But this implies $a_{12} = 0$, contradicting the above assumption that $a_{12}a_{21} \neq 0$. From this assumption it therefore follows that both $h_{11} \neq 0$ and $h_{12} \neq 0$. Normalizing, we may then set $h_{11} = h_{12} = 1$ and write $h_{21} = \kappa_1$ and $h_{22} = \kappa_2$. These κ_j are known as *distribution coefficients* (cf. Andronov *et al.*, 1966, pp. 257–8). Dropping subscripts, we see that the distribution coefficients must satisfy

$$\begin{bmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} = \begin{bmatrix} \lambda - a_{11} - a_{12}\kappa \\ -a_{21} + (\lambda - a_{22})\kappa \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Eliminating λ from the pair of equations on the right we obtain

$$a_{12}\kappa^2 + (a_{11} - a_{22})\kappa - a_{21} = 0.$$

Thus, the distribution coefficients are given by the solution of this quadratic equation, namely

(29) $$\kappa = \frac{a_{22} - a_{11} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{21}a_{12}}}{2}.$$

Comparing (29) with (25) we see that

$$\kappa_j = \frac{\lambda_j - a_{11}}{a_{12}} = \frac{a_{21}}{\lambda_j - a_{22}}.$$

Since we may now rewrite (28) as

(30) $$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \kappa_1 \end{bmatrix} b_1 e^{\lambda_1 t} + \begin{bmatrix} 1 \\ \kappa_2 \end{bmatrix} b_2 e^{\lambda_2 t},$$

we see that the transformation (26) has enabled us to express the solution-path in terms of a new coordinate system, the new coordinate axes being the lines through the origin (which is a displaced origin corresponding to the equilibrium state of the system) and the vectors $(1, \kappa_i)'$, with slopes κ_i.

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