

Notes on Graham's Paradox

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1 Introduction

Since it is difficult to transform Graham's model into suitable symbolism (since the names of his two commodities—wheat and watches—both start with the same letter, and he confuses the reader by using A for Britain and B for America), I shall adopt what I hope is a more intuitive notation. Let A be America and B Britain, and let commodity 1 be the commodity (“wheat”) produced under nonincreasing returns, and commodity 2 the commodity (“watches”) produced under increasing returns. (Note that that Graham seems not to clearly distinguish between returns to scale and returns to variable proportions.)

I shall assume that in both countries there is only one factor of production (labor), and that wheat is produced under constant and watches under increasing returns to scale. Symbolically,

$$(1) \quad y_{1k} = c_{1k}v_{1k}, \quad \text{and} \quad y_{2k} = c_{2k}v_{2k}^{\rho_k},$$

where k represents the country ($k = A, B$), y_{jk} is the output of commodity j in country k , v_{jk} is the input of labor into industry j in country k . It is assumed that there are constant returns to scale in industry 1 and increasing returns in industry 2, $\rho_k > 1$ being the degree of returns to scale in industry 2 in country k .

Each country must have a resource-allocation constraint. We write this as

$$(2) \quad v_{1k} + v_{2k} = l_k \quad (k = A, B),$$

where l_k is country k 's labor endowment.

For simplicity, let us assume that the degree of returns to scale is identical between the two countries, so that $\rho_A = \rho_B = \rho > 1$.

Using this simplification and combining (1) and (2), we obtain country k 's production-possibility frontier

$$(3) \quad \frac{y_{1k}}{c_{1k}} + \left(\frac{y_{2k}}{c_{2k}} \right)^{1/\rho} = l_k, \quad \text{or } y_{2k} = c_{2k} \left(l_k - \frac{y_{1k}}{c_{1k}} \right)^\rho \quad (k = A, B).$$

This expresses y_{2k} as a decreasing convex function of y_{1k} , touching the horizontal axis with tangency at $(c_{1k}l_k, 0)$, and the vertical axis at $(0, c_{2k}l_k^\rho)$ with slope $-(c_{2k}/c_{1k})\rho l_k^{\rho-1}$.

2 The world production-possibility frontier

We will first want to characterize the world production-possibility frontier. This is defined, for each world output level $y_1 = y_{1A} + y_{1B}$, as the maximum world output level $y_2 = y_{2A} + y_{2B}$ subject to $y_{2k} = c_{2k}(l_k - y_{1k}/c_{1k})^\rho$ for $k = A, B$ as well as to $y_{jk} \geq 0$ for $j = 1, 2, k = A, B$.

To derive the world production-possibility frontier $y_2 = \hat{y}_2(y_1)$, let us set

$$(4) \quad \begin{aligned} y_2 = y_{2A} + y_{2B} &= c_{2A}(l_A - y_{1A}/c_{1A})^\rho + c_{2B}(l_B - y_{1B}/c_{1B})^\rho \\ &= c_{2A}(l_A + y_{1B}/c_{1A} - y_1/c_{1A})^\rho + c_{2B}(l_B - y_{1B}/c_{1B})^\rho \\ &= \frac{c_{2A}}{c_{1A}^\rho}(c_{1A}l_A + y_{1B} - y_1)^\rho + \frac{c_{2B}}{c_{1B}^\rho}(c_{1B}l_B - y_{1B})^\rho \\ &\equiv f_B(y_{1B}, y_1). \end{aligned}$$

At this point let us introduce a further simplification and postpone consideration of the general case. Let us assume identical technologies in the two countries, i.e., $c_{jk} = 1$ for $j = 1, 2, k = A, B$. We then have

$$\frac{\partial f_B}{\partial y_{1B}} = \rho [(l_A + y_{1B} - y_1)^{\rho-1} - (l_B - y_{1B})^{\rho-1}],$$

which vanishes at

$$(5) \quad \begin{aligned} y_{1B} &= \frac{1}{2}y_1 - \frac{1}{2}(l_A - l_B) \equiv y_{1B}^*, \quad \text{hence} \\ y_{1A} &= \frac{1}{2}y_1 + \frac{1}{2}(l_A - l_B) \equiv y_{1A}^*. \end{aligned}$$

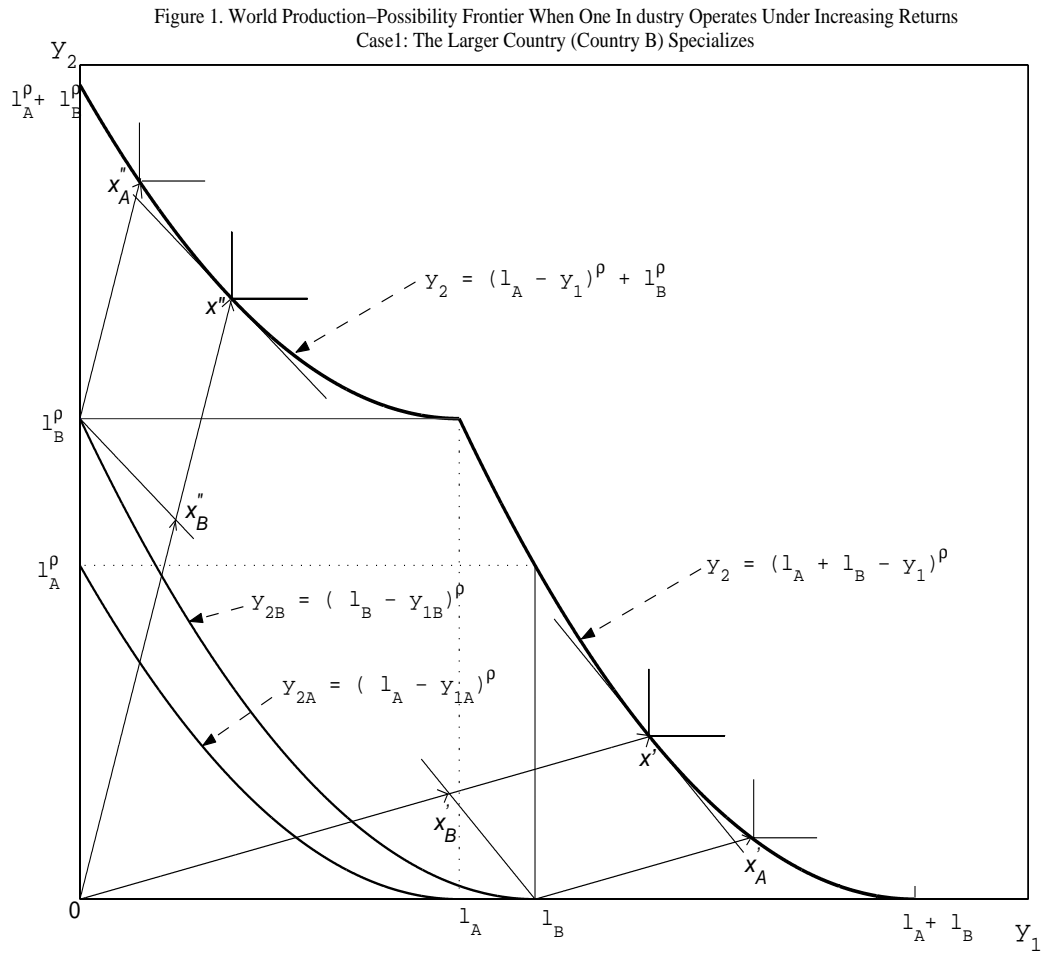
The function $f_{1B}(\cdot, y_1)$ is minimized at this value of y_{1B} , since

$$\begin{aligned} \frac{\partial^2 f_B}{\partial y_{1B}^2} &= \rho(\rho - 1) [(l_A + y_{1B} - y_1)^{\rho-2} + (l_B - y_{1B})^{\rho-2}] \\ &= 2\left(\frac{1}{2}\right)^{\rho-2}(l_A + l_B - y_1)^{\rho-2} > 0 \end{aligned}$$

at the value (5). It follows that f_B is decreasing for $y_{1B} < y_{1B}^*$ and increasing for $y_{1B} > y_{1B}^*$. Therefore $f_B(y_{1B}, y_1)$ has local maxima at $y_{1B} = 0$ and $y_{1B} = l_B$. We thus have

$$(6) \quad \hat{y}_2(y_1) = \begin{cases} f_B(0, y_1) = (l_A - y_1)^\rho + l_B^\rho & \text{for } 0 \leq y_1 \leq l_A; \\ f_B(l_B, y_1) = (l_A + l_B - y_1)^\rho & \text{for } l_A \leq y_1 \leq l_A + l_B. \end{cases}$$

This is the formula for the world production-possibility frontier in the case of identical technologies between the two countries. It is graphed in Figure 1 for the case $l_A = 1, l_B = 1.2$, and $\rho = 2$.



3 World equilibrium

The world production-possibility set (the set of all nonnegative bundles of world output which are less than or equal to some bundle on the world production-possibility frontier), is mathematically the sum of the two countries' production-possibility sets. Country A 's production-possibility set is the triangular region between l_A on the horizontal axis, l_A^ρ on the vertical axis, and the origin; country B 's is the larger region bounded by l_B , l_B^ρ , and the origin. We may visualize the world production-possibility set as the locus obtained by sliding country A 's production-possibility set down country B 's. It is not hard to see that for $0 \leq y_1 \leq l_A$, country B specializes in commodity 2 (watches), producing l_B^ρ units, while country 1 moves from specializing in commodity 2 all the way to specializing in commodity 1 (wheat), producing l_A units, so that at the point (l_A, l_B^ρ) , country B specializes in commodity 2 and country A in commodity 1. After country A 's production-possibility set has "slid" all the way down to the horizontal axis, country B specializes in commodity 1 (producing l_B units), and country A moves from specializing in commodity 2 (at point l_B) to diversifying and finally specializing in commodity 1 (at point $l_A + l_B$). Thus, in the regions $(0, l_A)$ and $(l_B, l_A + l_B)$, country B specializes; only in the region (l_A, l_B) does it diversify, while country A specializes in commodity 1.

We assume that preferences are aggregable within and between countries, so that a world utility function may be defined representing the collective preferences of the individuals in the two countries. If these preferences exhibit a fairly large elasticity of substitution, then it is probable that maximization of world utility subject to the world production-possibility set will yield an equilibrium at the kink, $(y_1, y_2) = (l_A, l_B^\rho)$. Each country will specialize, country A producing l_A units of commodity 1 (the constant-return good) and country B producing l_B^ρ units of commodity 2 (the increasing-return good). World prices will depend upon the slope of the world indifference curve at (l_A, l_B^ρ) , which we must assume will lie between the slope of country A 's production-possibility frontier at $(y_{1A}, y_{2A}) = (l_A, 0)$ (which is 0), and that of country B 's production-possibility frontier at $(y_{B1}, y_{B2}) = (0, l_B^\rho)$, which is $-\rho l_B^{\rho-1} < 0$. Since country k 's budget set at world prices dominates its production-possibility set, for each $k = A, B$, clearly it is better off under free trade than in the competitive equilibrium under autarky.¹ Thus, each country will be better off under free trade.

However, Graham believed that consumer demand was unresponsive to

¹That such a competitive equilibrium will exist under parametric external economies of scale was shown in Chipman (1970). Note, however, that it is not Pareto-optimal; in conformity with Marshall's rule, Pareto optimality would require taxing the constant-return industry and subsidizing the increasing-return industry. We do not consider this question here, however.

price changes; he also believed that it was erratic. Thus Graham's assumptions are better represented by a world utility function of the form

$$(7) \quad U(x_1, x_2) = \min \left(\frac{x_1}{a_1}, \frac{x_2}{a_2} \right),$$

where the a_i s are subject to random disturbances. The above result would then only be achieved if $a_2/a_1 = l_B^\rho/l_A$. This may be ruled out as infinitely improbable.

Let us then first consider the case $a_2/a_1 < l_A^\rho/l_B$, i.e., there is (worldwide) a relatively strong demand for wheat relative to watches. Then one of the countries (namely country B) specializes in wheat (commodity 1), producing l_B units. World output is obtained at the intersection of the ray $(x_1, x_2) = \lambda(a_1, a_2)$ ($\lambda \geq 0$) with the locus $(y_1, (l_A + l_B - y_1)^\rho)$, i.e., by solving the equation

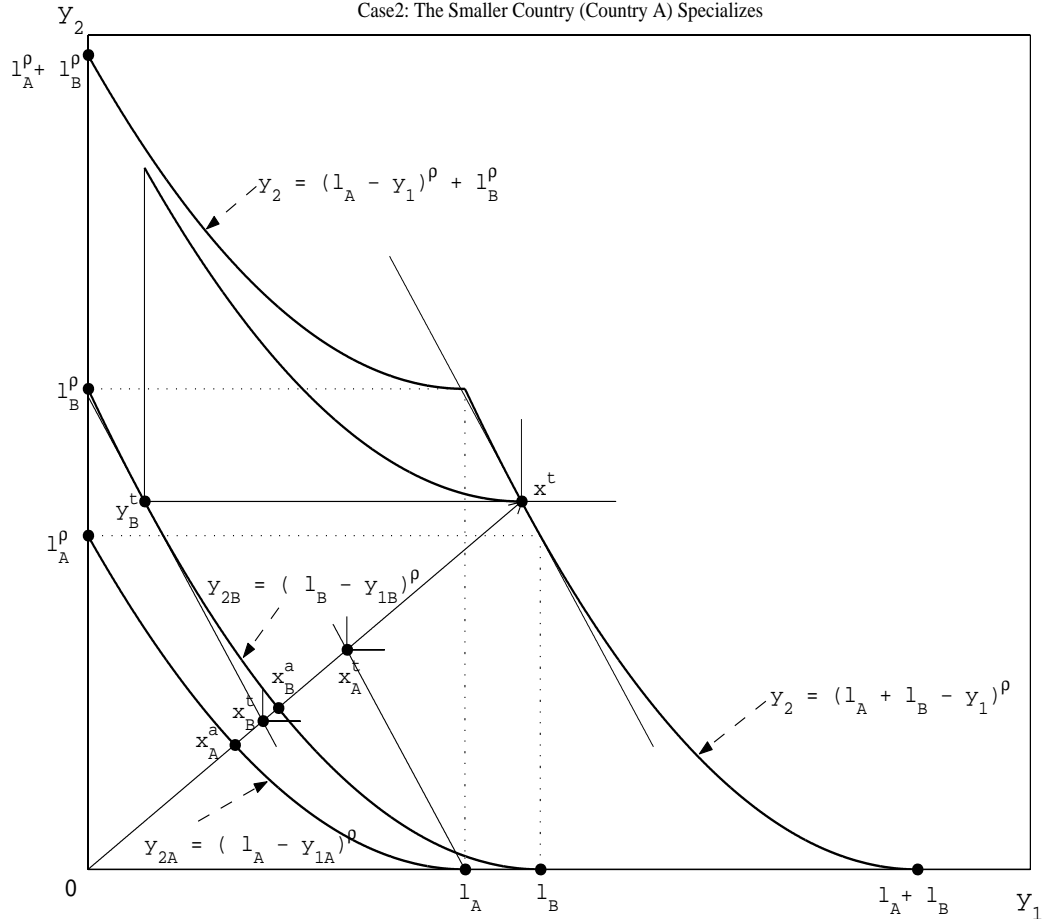
$$\frac{l_A + l_B - x_1^\rho}{x_1} = \frac{a_2}{a_1}$$

for x_1 and setting $x_2 = (a_2/a_1)x_1$. It will not be necessary to solve this equation, however. Let us simply denote this intersection point by $x' = (x'_1, x'_2)$ (see Figure 1). Draw the tangent to the curve $y_2 = (l_A + l_B - y_1)^\rho$ at $x' = y' = (y'_1, y'_2)$. Its slope is the world equilibrium price ratio p_1/p_2 (that this is a competitive equilibrium despite the fact the the tangent lies below the curve is shown in Chipman 1970). Draw a line parallel to this tangent going through the point $(l_B, 0)$ and intersecting the ray going through x' . This is a portion of country B 's budget set, so the intersection point, x'_B , is country B 's equilibrium consumption bundle (which is clearly bigger than the bundle it consumes under autarky, where the ray through x'_B intersects country B 's production-possibility frontier joining l_B and l_B^ρ). Now draw a ray through $(l_B, 0)$ (the origin of country A 's displaced production-possibility set) and parallel to the ray through x' . The point x'_A on the boundary of country A 's displaced production-possibility set corresponds to what country A will consume under autarky. Now since world consumption under free trade is x' , and country B consumes x'_B of this, the length of the side of the parallelogram—the distance between x'_B and x' —represents the length of country A 's consumption vector under free trade. But from the opposite side of the parallelogram it is clear that this length is less than that of the distance between l_B and x'_A . Thus, *country A is worse off under free trade than under autarky!* It is nevertheless true, as is obvious from the diagram, that if country B were to grant country A a sufficient subsidy, both countries could be better off under free trade than under autarky; but there would be no incentive for country B to do this, unless country A were to threaten it with a prohibitive tariff.

Exactly the same argument applies to the part of the world production-possibility set contained in the interval $(0, l_A)$. This time it is supposed that

$a_2/a_1 > l_B^\rho/l_A$. The ray $\lambda a = \lambda(a_1, a_2)$ ($\lambda \geq 1$) is drawn which hits the segment $y_2 = (l_A - y_1)^\rho + l_B^\rho$ at x'' ; this is the free-trade equilibrium. We draw the tangent at x'' , and a line parallel to it through l_B^ρ . x'' represents world consumption, and x_B'' country B's consumption, under free trade. We draw a line through $(0, l_B^\rho)$, parallel to the ray from 0 to x'' , and hitting country A's displaced production-possibility frontier at x_A'' . The difference between x_A'' and $(0, l_B^\rho)$ is country A's consumption under autarky; but this is clearly greater than the difference between x'' and x_B'' . The shorter segment of the ray from $(0, l_B^\rho)$ to x_A'' cut off by the parallelogram represents what country A consumes under free trade. Again, therefore, *country A is worse off under free trade than under autarky!*

Figure 2. World Production-Possibility Frontier When One Industry Operates Under Increasing Returns
Case2: The Smaller Country (Country A) Specializes



It remains to consider the case $l_A^\rho/l_B \leq a_2/a_1 \leq l_B^\rho/l_A$, in which the consumption ray passes through the segment of the world production-possibility frontier corresponding to the interval $[l_A, l_B]$ (not shown in Figure 1). We have already seen that both countries are better off under free trade at the kink point (l_A, l_B^ρ) . What about the rest of that interval?

To follow the argument we may look at Figure 2. A consumption ray conforming to the above assumption is shown to hit the world production-possibility frontier at the point x^t (the t superscript stands for free-trade equilibrium; an a superscript will stand for an autarky equilibrium). The tangent to the world production-possibility frontier at x^t is shown, as well as country A 's displaced production-possibility set whose origin is the point y_B^t , which lies on country B 's production-possibility frontier, which at that point and indeed over the whole interval $l_A^\rho < y_2 < l_B^\rho$ has the same slope as the world production-possibility frontier at the same value of y_2 ; thus the tangent at y_B^t has necessarily the same slope as the tangent at x^t . The world equilibrium point $x^t = y^t$ is the sum of country B 's output vector y_B^t and country A 's output vector $(l_A, 0)$. The tangent line at y_B^t when extended to the two axes is country B 's budget line at the equilibrium world prices; consequently, the point x_B^t is country B 's equilibrium consumption vector under free trade. But its equilibrium consumption vector under autarky is the point x_B^a , and clearly $x_B^a > x_B^t$, i.e., *in this case country B is worse off under free trade than under autarky!*

As for country A , drawing from country A 's output vector $(l_A, 0)$ a budget line parallel to the tangents through x^t and y_B^t , country A 's equilibrium consumption vector under free trade is shown as x_A^t ; but its consumption vector under autarky is $x_A^a < x_A^t$; thus, in this case, *country A is better off under free trade than under autarky, and enough so that if it transferred a sufficient amount of the two goods to country B, both countries would be better off under free trade than under autarky.*

Thus we may conclude: At the extreme points $(l_A + l_B, 0)$ and $(0, l_A^\rho + l_B^\rho)$, where consumers in the two countries desire only one or the other of the two commodities, there is neither gain nor loss to either country from moving from autarky to free trade. In all other cases, except that of the singular point (l_A, l_B^ρ) , where both countries gain, one country always diversifies while the other specializes. The specializing country is always better off, and the diversifying country always worse off, under free trade than under autarky. The larger of the two countries (as measured by its labor endowment) specializes in the constant-return good (commodity 1) if there is a sufficiently large relative world demand for this good, and specializes in the increasing-return good (commodity 2) if there is a sufficiently large relative world demand for that good. Unless there is no world demand for the increasing-return good, the smaller country (country A) will never specialize in this good; and when-

ever the larger country diversifies, the smaller country will specialize in the constant-return good. If world preferences are intermediate between those exhibiting a high relative preference for one or the other of the two goods, and if there is a sufficiently great difference in the sizes of the two countries (as measured by $l_B - l_A$), then the smaller country may be expected to gain, and the larger country to lose, from trade. In the limiting case of equal sizes ($l_A = l_B$), then in all cases other than the three exceptional cases of world equilibrium $((l_A + l_B, 0), (l_A, l_B^l), (0, l_B^l))$, both countries are necessarily worse off under free trade than under autarky.

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