Notes on the Heckscher-Ohlin Theorem

John S. Chipman
University of Minnesota

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We assume that there are two countries each producing two commodities with two factors. We denote, for $i, j, k = 1, 2$:

x_j^k = country k's consumption of commodity j;

y_j^k = country k's production of commodity j;

l_i^k = country k's endowment of factor i;

p_j^k = the price of commodity j on country k's markets;

w_i^k = the rental of factor i in country k

We assume that

$$l_1^1/l_2^1 \neq l_1^2/l_2^2$$

(i.e., the countries differ in their relative factor endowments), and choose the suffixes i, k such that

$$l_1^1/l_2^1 > l_1^2/l_2^2. \tag{1}$$

We assume:

1. *Material balance.* This states that the world consumption of each commodity is equal to the world production of this commodity, or equivalently, that one country's export is the other country's import:

$$x_j^1 + x_j^2 = y_j^1 + y_j^2, \quad \text{or} \quad x_j^1 - y_j^1 = y_j^2 - x_j^2. \tag{2}$$

2. *Free trade.* There are no tariffs, transport costs, or other impediments to trade. Thus, $p_j^1 = p_j^2 = p_j$ for $j = 1, 2$, i.e., the prices of the two commodities are equal in the two countries.

3. *Positive production of both commodities in each country.*[1] It follows from this that with competitive markets, prices of the two commodities are equal to their minimum unit costs.

4. *Identical technologies* as between countries, characterized by identical concave, strictly quasi-concave, differentiable, and homogeneous-of-degree-1 production functions

$$y_j^k = f_j(v_{1j}, v_{2j}) \quad (j, k = 1, 2) \tag{3}$$

and thus identical concave, strictly quasi-concave, differentiable, and homogeneous-of-degree-1 minimum-unit-cost functions

$$p_j = g_j(w_1^k, w_2^k) \quad (j, k = 1, 2), \tag{4}$$

where the prices are equal to the commodities' minimum units costs, from assumption 3.

5. *Nonreversal of factor intensities.*[2] Denoting by the homogeneous-of-degree-zero functions

$$b_{ij}(w_1^k, w_2^k) = \partial g_j(w_1^k, w_2^k)/\partial w_i^k \quad (i, j, k = 1, 2) \tag{5}$$

the amount of factor i used to produce one unit of commodity j in country k, as a function of the factor rentals, then for a suitable labelling of the commodities we have

$$|B(w^k)| \equiv \begin{vmatrix} b_{11}(w^k) & b_{21}(w^k) \\ b_{12}(w^k) & b_{22}(w^k) \end{vmatrix} = b_{11}(w^k)b_{12}(w^k)\left[\frac{b_{22}(w^k)}{b_{12}(w^k)} - \frac{b_{21}(w^k)}{b_{11}(w^k)}\right] > 0 \tag{6}$$

for all $w^k = (w_1^k, w_2^k)$. Since the matrix $B(w^k)$ of (6) is (by (5)) the Jacobian of the transformation

$$\begin{aligned} g_1(w_1^k, w_2^k) &= p_1 \\ g_2(w_1^k, w_2^k) &= p_2, \end{aligned} \tag{7}$$

it follows that the solution of (7) is unique, i.e., $w_i^1 = w_i^2 = w_i$ (factor rentals are equalized between the countries). The Rybczynski functions

$$y_j^k = \frac{\partial \Pi(p_1, p_2, l_1^k, l_2^k)}{\partial p_j} = \hat{y}_j(p_1, p_2, l_1^k, l_2^k) \quad (j, k = 1, 2) \tag{8}$$

[1]This assumptions is stronger than needed to establish the Heckscher-Ohlin theorem. If one or both countries specialize in the production of one commodity, the result can be proved by a separate argument.

[2]This assumption is also stronger than needed to establish the Heckscher-Ohlin theorem. It is enough to assume that both countries have their factor endowments in the same diversification cone. This automatically assures assumption 3 as well.

(where Π is the domestic-product function—the same for both countries) are then the solutions of the resource-allocation equations

$$\begin{aligned} b_{11}(\hat{w}(p))y_1^k + b_{12}(\hat{w}(p))y_2^k &= l_1^k \\ b_{21}(\hat{w}(p))y_1^k + b_{22}(\hat{w}(p))y_2^k &= l_2^k, \end{aligned} \tag{9}$$

i.e., the linear functions

$$\begin{aligned} y_1^k &= \hat{y}_1(p_1, p_2, l_1^k, l_2^k) = b^{11}(p)l_1^k + b^{12}(p)l_2^k \\ y_2^k &= \hat{y}_2(p_1, p_2, l_1^k, l_2^k) = b^{21}(p)l_1^k + b^{22}(p)l_2^k, \end{aligned} \tag{10}$$

where the $b^{ij}(p)$ are the elements of the inverse matrix $[B(\hat{w}(p)]^{-1}$ and $\hat{w}(p)$ denoted the inverse of the cost mapping $g(w) = p$. Thus the countries' Rybczynski functions are linear and single-valued, having the same form in the two countries, their values differing only according to the countries' different factor endowments.

6. *Identical homothetic preferences.* These are characterized by demand functions with the property

$$x_j^k = h_j(p_1, p_2, Y^k) = Y^k h_j(p_1, p_2, 1) \tag{11}$$

(where Y^k is country k's national income) so that

$$x_j^1 + x_j^2 = (Y^1 + Y^2)h_j(p_1, p_2, 1) = h_j(p_1, p_2, Y^1 + Y^2). \tag{12}$$

7. *Balanced trade.* For country k this may be stated as either

$$p_1 x_1^k + p_2 x_2^k = p_1 y_1^k + p_2 y_2^k \tag{13}$$

(the value of expenditure equals the value of output, i.e., there is no borrowing or lending by country k), or

$$p_1(x_1^k - y_1^k) + p_2(x_2^k - y_2^k) = 0 \tag{14}$$

(the value of country k's imports (resp. exports) is equal to the value of its exports (resp. imports). Note that if (14) holds for $k = 1$, then by assumption 1 (material balance) it automatically holds for $k = 2$.

Heckscher-Ohlin Theorem. Under the above assumptions, in competitive world equilibrium country 1 will export commodity 1 to and import commodity 2 from country 2.

Proof. The logic of the proof goes as follows. First we will show that from assumptions 2 and 6 we must have

$$\frac{x_1^1}{x_2^1} = \frac{x_1^2}{x_2^2}. \tag{15}$$

Next we will show that from (1) and assumptions 3, 4, and 5, we must have

$$\frac{y_1^1}{y_2^1} > \frac{y_1^2}{y_2^2}. \tag{16}$$

Finally we shall show that from (1), (15), (16), and assumptions 1 and 7, we must have

$$x_1^1 < y_1^1, \quad \text{hence} \quad x_2^2 < y_2^2. \tag{17}$$

To establish (15) we need simply observe from (11) that, owing to assumption 2,

$$\frac{x_1^k}{x_2^k} = \frac{h_1(p_1,p_2,1)}{h_2(p_1,p_2,1)}, \quad \text{independently of } k.$$

To establish (16) we note first that since the domestic-product function $\Pi(p_1,p_2,l_1^k,l_2^k)$ is homogeneous of degree 1 in (l_1^k,l_2^k), so are the Rybczynski functions $\hat{y}_j = \partial\Pi/\partial p_j$, hence (again using assumption 2)

$$\frac{y_1^k}{y_2^k} = \frac{\hat{y}_1(p_1,p_2,l_1^k/l_2^k,1)}{\hat{y}_2(p_1,p_2,l_1^k/l_2^k,1)}.$$

Thus, differentiating with respect to the third argument l_1^k/l_2^k we obtain

$$\frac{\partial(\hat{y}_1/\hat{y}_2)}{\partial(l_1^k/l_2^k)} = \frac{\hat{y}_2\partial\hat{y}_1/\partial l_1^k - \hat{y}_1\partial\hat{y}_2/\partial l_1^k}{(\hat{y}_2)^2}. \tag{18}$$

We now prove that this expression is positive, as follows. As observed in assumption 5, formula (9), the Rybczynski functions are the solution of the linear system

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} l_1^k \\ l_2^k \end{bmatrix}, \tag{19}$$

where the $b_{ij}(\hat{w}(p_1,p_2,l_1^k,l_2^k))$ are (within the diversification cone) independent of (l_1^k,l_2^k). (Here, $\hat{w}_i$ denotes $\partial\Pi/\partial l_i$, which coincides with $g^{-1}(p)$ within the diversification cone.) Denoting this solution as above by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b^{11} & b^{12} \\ b^{21} & b^{22} \end{bmatrix} \begin{bmatrix} l_1^k \\ l_2^k \end{bmatrix}, \tag{20}$$

where

$$\begin{bmatrix} b^{11} & b^{12} \\ b^{21} & b^{22} \end{bmatrix} = |B|^{-1} \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix}, \tag{21}$$

and $|B| > 0$ from (6), we have from (20) and (21)

$$\frac{\partial\hat{y}_1}{\partial l_1^k} = \frac{b_{22}}{|B|} > 0 \quad \text{and} \quad \frac{\partial\hat{y}_2}{\partial l_1^k} = -\frac{b_{21}}{|B|} < 0, \tag{22}$$

so that (18) yields

$$\frac{\partial(\hat{y}_1/\hat{y}_2)}{\partial(l_1^k/l_2^k)} > 0.$$

Thus, given (1), (16) holds.

Finally, by way of contradiction suppose that (17) does not hold, i.e., suppose that

$$x_1^1 \geq y_1^1. \tag{23}$$

Then from (14) we have also

$$x_2^1 \leq y_2^1. \tag{24}$$

From (23) and (24) it follows that

$$\frac{x_1^1}{x_2^1} \geq \frac{y_1^1}{x_2^1} \geq \frac{y_1^1}{y_2^1}. \tag{25}$$

Now from the material-balance condition (2) it follows that the inequalities (23) and (24) are respectively equivalent to the inequalities

$$x_1^2 \leq y_1^2 \quad \text{and} \quad x_2^2 \geq y_2^2. \tag{26}$$

From (26) it then follows that

$$\frac{y_1^2}{y_2^2} \geq \frac{x_1^2}{y_2^2} \geq \frac{x_1^2}{x_2^2}. \tag{27}$$

Putting together (25), (16), and (27) we obtain

$$\frac{x_1^1}{x_2^1} \geq \frac{y_1^1}{y_2^1} > \frac{y_1^2}{y_2^2} \geq \frac{x_1^2}{x_2^2}. \tag{28}$$

But this contradicts assumption (15); thus the supposition (23) has led to a contradiction, and therefore (17) holds. □