Notes on trade-demand functions with two produced tradables, one nontradable, and two factors

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With three products and two factors, the production-possibility frontier is a ruled surface, hence the domestic-product function is not differentiable. For the same reasons, the price of the nontradable is determined uniquely from the prices of the tradables via the cost equations:

$$\begin{aligned} g_1(w_1, w_2) &= p_1; \\ g_2(w_1, w_2) &= p_2. \end{aligned} \tag{1}$$

Given the factor endowments, these equations have a unique solution

$$\begin{aligned} w_1 &= \hat{w}_1(p_1, p_2); \\ w_2 &= \hat{w}_2(p_1, p_2). \end{aligned} \tag{2}$$

The price of the nontradable is then determined from

$$p_3 = \hat{p}_3(p_1, p_2) \equiv g_3(\hat{w}_1(p_1, p_2), \hat{w}_2(p_1, p_2)). \tag{3}$$

The Jacobian matrix of (2) is the inverse of the Jacobian matrix of (1):

$$\begin{bmatrix} \partial\hat{w}_1/\partial p_1 & \partial\hat{w}_1/\partial p_2 \\ \partial\hat{w}_2/\partial p_1 & \partial\hat{w}_2/\partial p_2 \end{bmatrix} = \begin{bmatrix} b^{11} & b^{21} \\ b^{12} & b^{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}^{-1} = (B^{1\prime})^{-1}. \tag{4}$$

Thus, from (3) we have

$$\begin{aligned} \left(\frac{\partial\hat{p}_3}{\partial p_1}, \; \frac{\partial\hat{p}_3}{\partial p_2} \right) &= \left(\frac{\partial g_3}{\partial w_1}\frac{\partial\hat{w}_1}{\partial p_1} + \frac{\partial g_3}{\partial w_2}\frac{\partial\hat{w}_2}{\partial p_1}, \; \frac{\partial g_3}{\partial w_1}\frac{\partial\hat{w}_1}{\partial p_2} + \frac{\partial g_3}{\partial w_2}\frac{\partial\hat{w}_2}{\partial p_2} \right) \\ &= \left(b_{13}b^{11} + b_{23}b^{12}, \; b_{13}b^{21} + b_{23}b^{22} \right) \\ &= \begin{bmatrix} b_{13} & b_{23} \end{bmatrix} \begin{bmatrix} b^{11} & b^{21} \\ b^{12} & b^{22} \end{bmatrix} = B^{3\prime}(B^{1\prime})^{-1}. \end{aligned} \tag{5}$$

Let us define the *domestic-cost function* by

$$\Phi(w_1, w_2; y_1, y_2, y_3) = \sum_{k=1}^{3} g_k(w_1, w_2) y_k. \tag{6}$$

It expresses the domestic product as a function of factor rentals and commodity outputs. Its partial derivative with respect to the ith factor rental is the ith *factor-demand function*:

$$\phi_i(w_1, w_2; y_1, y_2, y_3) \equiv \frac{\partial \Phi(w_1, w_2; y_1, y_2, y_3)}{\partial w_i} = \sum_{k=1}^{3} b_{ik}(w_1, w_2) y_k. \tag{7}$$

We note for future use that the Hessian matrix of $\Phi(w; y)$ with respect to w,[1]

$$\begin{aligned} H(w; y) &= \begin{bmatrix} \dfrac{\partial \phi_1}{\partial w_1} & \dfrac{\partial \phi_1}{\partial w_2} \\ \dfrac{\partial \phi_2}{\partial w_1} & \dfrac{\partial \phi_2}{\partial w_2} \end{bmatrix} \\ &= \left[\frac{\partial^2 \Phi(w; y)}{\partial w_i \partial w_j}\right]_{i,j=1,2} = \sum_{k=1}^{3} \left[\frac{\partial^2 g_k(w)}{\partial w_i \partial w_j}\right]_{i,j=1,2} y_k, \end{aligned} \tag{8}$$

is symmetric and negative semi-definite, being a nonnegative linear combination of the Hessians of the minimum-unit-cost functions g_k, which are symmetric and negative semi-definite since the cost functions g_k are differentiable and concave. Further, since the g_k are homogeneous of degree 1, the ϕ_i are homogeneous of degree 0 and thus (from Euler's theorem),

$$H(w; y) w = \begin{bmatrix} \dfrac{\partial^2 \Phi}{\partial w_1^2} & \dfrac{\partial^2 \Phi}{\partial w_1 \partial w_2} \\ \dfrac{\partial^2 \Phi}{\partial w_2 \partial w_1} & \dfrac{\partial^2 \Phi}{\partial w_2^2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \phi_1}{\partial w_1} & \dfrac{\partial \phi_1}{\partial w_2} \\ \dfrac{\partial \phi_2}{\partial w_1} & \dfrac{\partial \phi_2}{\partial w_2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{9}$$

The three outputs are then obtained from the following three equations:

$$\begin{aligned} \phi_1\Big(\hat{w}_1(p_1, p_2), \hat{w}_2(p_2, p_2), y_1, y_2, y_3\Big) &= l_1; \\ \phi_2\Big(\hat{w}_1(p_1, p_2), \hat{w}_2(p_2, p_2), y_1, y_2, y_3\Big) &= l_2; \\ h_3\Big(p_1, p_2, \hat{p}_3(p_1, p_2), p_1 y_1 + p_2 y_2 + \hat{p}_3(p_1, p_2) y_3 + D\Big) &= y_3. \end{aligned} \tag{10}$$

[1]This notation departs from that of Chipman (1981) where Φ stood for the Hessian of the domestic-cost function and no further use was made of this domestic-cost function.

We now take differentials of (10):

$$
(11)\qquad
\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ -c_3p_1 & -c_3p_2 & 1-c_3p_3 \end{bmatrix}
\begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \end{bmatrix}
$$

$$
= \begin{bmatrix}
-\frac{\partial\phi_1}{\partial w_1}\frac{\partial\hat{w}_1}{\partial p_1} - \frac{\partial\phi_1}{\partial w_2}\frac{\partial\hat{w}_2}{\partial p_1} & -\frac{\partial\phi_1}{\partial w_1}\frac{\partial\hat{w}_1}{\partial p_2} - \frac{\partial\phi_1}{\partial w_2}\frac{\partial\hat{w}_2}{\partial p_2} \\
-\frac{\partial\phi_2}{\partial w_1}\frac{\partial\hat{w}_1}{\partial p_1} - \frac{\partial\phi_2}{\partial w_2}\frac{\partial\hat{w}_2}{\partial p_1} & -\frac{\partial\phi_2}{\partial w_1}\frac{\partial\hat{w}_1}{\partial p_2} - \frac{\partial\phi_2}{\partial w_2}\frac{\partial\hat{w}_2}{\partial p_2} \\
s_{31} - c_3z_1 + s_{33}\frac{\partial\hat{p}_3}{\partial p_1} & s_{32} - c_3z_2 + s_{33}\frac{\partial\hat{p}_3}{\partial p_2}
\end{bmatrix}
\begin{bmatrix} dp_1 \\ dp_2 \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix} dD + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dl_1 \\ dl_2 \end{bmatrix}.
$$

From (8) we may write the first two rows of the 3×2 coefficient matrix in (11) of the differentials of the world prices as

$$
(12)\qquad
-\begin{bmatrix} \frac{\partial\phi_1}{\partial w_1} & \frac{\partial\phi_1}{\partial w_2} \\ \frac{\partial\phi_2}{\partial w_1} & \frac{\partial\phi_2}{\partial w_2} \end{bmatrix}
\begin{bmatrix} \frac{\partial\hat{w}_1}{\partial p_1} & \frac{\partial\hat{w}_1}{\partial p_2} \\ \frac{\partial\hat{w}_2}{\partial p_1} & \frac{\partial\hat{w}_2}{\partial p_2} \end{bmatrix}
= -H(w;y)(B^{1\prime}(w))^{-1}
$$

where $(B^{1\prime})^{-1}$ is defined by (4). From (5), the third row of this matrix may be written

$$
(13)\qquad (s_{31}, s_{32}) - c_3(z_1, z_2) + s_{33}B^{3\prime}(B^{1\prime})^{-1}.
$$

The coefficient matrix in (11) of the differentials of the three outputs has as its inverse, as may be verified,

$$
(14)\qquad
\begin{bmatrix} B^1 & B^3 \\ -c_3(p_1,p_2) & 1-c_3p_3 \end{bmatrix}^{-1} =
$$

$$
\begin{bmatrix} (B^1)^{-1} - (B^1)^{-1}B^3c_3(p_1,p_2)(B^1)^{-1} & -(B^1)^{-1}B^3 \\ c_3(p_1,p_2)(B^1)^{-1} & 1 \end{bmatrix}.
$$

Now we note that, in view of (4), the system of equations (1) may be written $(w_1, w_2)B^1 = (p_1, p_2)$, and likewise the inverse system (2) may be written $(p_1, p_2)(B^1)^{-1} = (w_1, w_2)$. Further, from (3) we have $(w_1, w_2)B^3 = p_3$ where B^3 is defined by (5). We then verify from (14) that

$$
(p_1, p_2, p_3)\begin{bmatrix} B^1 & B^3 \\ -c_3(p_1,p_2) & 1-c_3p_3 \end{bmatrix}^{-1} = (w_1, w_2, 0),
$$

hence, using (9), the following envelope condition holds:

$$(p_1, p_2, p_3)\begin{bmatrix} \dfrac{\partial \tilde{y}_1}{\partial p_1} & \dfrac{\partial \tilde{y}_1}{\partial p_2} \\ \dfrac{\partial \tilde{y}_2}{\partial p_1} & \dfrac{\partial \tilde{y}_2}{\partial p_2} \\ \dfrac{\partial \tilde{y}_3}{\partial p_1} & \dfrac{\partial \tilde{y}_3}{\partial p_2} \end{bmatrix}$$

$$= (w_1, w_2, 0)\begin{bmatrix} -H(w;y)(B^{1\prime}(w))^{-1} \\ (s_{31}, s_{32}) - c_3(z_1, z_2) + s_{33}B^{3\prime}(B^{1\prime})^{-1} \end{bmatrix}$$

$$= (0, 0). \tag{15}$$

Thus we have, denoting the solutions of (10) as $\tilde{y}_i(p_1, p_2, D, l)$ for $i = 1, 2, 3$, and using (14), (12), (13), and (9),

$$\begin{bmatrix} \dfrac{\partial \tilde{y}_1}{\partial p_1} & \dfrac{\partial \tilde{y}_1}{\partial p_2} \\ \dfrac{\partial \tilde{y}_2}{\partial p_1} & \dfrac{\partial \tilde{y}_2}{\partial p_2} \\ \dfrac{\partial \tilde{y}_3}{\partial p_1} & \dfrac{\partial \tilde{y}_3}{\partial p_2} \end{bmatrix} = \begin{bmatrix} (B^1)^{-1} - (B^1)^{-1}B^3c_3(w_1, w_2) & -(B^1)^{-1}B^3 \\ c_3(w_1, w_2) & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} -H(B^{1\prime})^{-1} \\ (s_{31}, s_{32}) - c_3(z_1, z_2) + s_{33}B^{3\prime}(B^{1\prime})^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -(B^1)^{-1}H(B^{1\prime})^{-1} - (B^1)^{-1}B^3\partial\tilde{y}_3/\partial(p_1, p_2) \\ \partial\tilde{y}_3/\partial(p_1, p_2) \end{bmatrix} \tag{16}$$

where

$$\frac{\partial \tilde{y}_3}{\partial(p_1, p_2)} = (s_{31}, s_{32}) - c_3(z_1, z_2) + s_{33}B^{3\prime}(B^{1\prime})^{-1}.$$

From (11) and (14) it follows that

$$\begin{bmatrix} \dfrac{\partial \tilde{y}_1}{\partial D} \\ \dfrac{\partial \tilde{y}_2}{\partial D} \end{bmatrix} = -(B^1)^{-1}B^3c_3, \tag{17}$$

hence from (16):

$$\begin{bmatrix} \dfrac{\partial \tilde{y}_1}{\partial p_1} + \dfrac{\partial \tilde{y}_1}{\partial D}z_1 & \dfrac{\partial \tilde{y}_1}{\partial p_2} + \dfrac{\partial \tilde{y}_1}{\partial D}z_2 \\ \dfrac{\partial \tilde{y}_2}{\partial p_1} + \dfrac{\partial \tilde{y}_2}{\partial D}z_1 & \dfrac{\partial \tilde{y}_2}{\partial p_2} + \dfrac{\partial \tilde{y}_2}{\partial D}z_2 \end{bmatrix} =$$

$$-(B^1)^{-1}H(B^{1\prime})^{-1} - (B^1)^{-1}B^3[(s_{31}, s_{32}) + s_{33}B^{3\prime}(B^{1\prime})^{-1}]. \tag{18}$$

In order to compute the trade-Slutsky matrix it remains to compute the partial derivatives of the functions

$$\tilde{x}_i(p_1, p_2, D, l) \equiv$$

$$h_i\Big(p_1, p_2, \hat{p}_3(p_1, p_2), p_1\tilde{y}_1(\cdot) + p_2\tilde{y}_2(\cdot) + \hat{p}_3(p_1, p_2)\tilde{y}_3(\cdot) + D\Big) \quad (i = 1, 2). \tag{19}$$

Making use of (15) we have

$$\begin{bmatrix} \dfrac{\partial \tilde{x}_1}{\partial p_1} & \dfrac{\partial \tilde{x}_1}{\partial p_2} \\ \dfrac{\partial \tilde{x}_2}{\partial p_1} & \dfrac{\partial \tilde{x}_2}{\partial p_2} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \begin{bmatrix} z_1 & z_2 \end{bmatrix} + \begin{bmatrix} s_{13} \\ s_{23} \end{bmatrix} B^{3\prime}(B^{1\prime})^{-1}. \tag{20}$$

Since

$$\frac{\partial \tilde{x}_i}{\partial D} = \frac{\partial h_i}{\partial Y} = c_i \quad (i = 1, 2), \tag{21}$$

it follows that

$$\begin{bmatrix} \dfrac{\partial \tilde{x}_1}{\partial p_1} + \dfrac{\partial \tilde{x}_1}{\partial D} z_1 & \dfrac{\partial \tilde{x}_1}{\partial p_2} + \dfrac{\partial \tilde{x}_1}{\partial D} z_2 \\ \dfrac{\partial \tilde{x}_2}{\partial p_1} + \dfrac{\partial \tilde{x}_2}{\partial D} z_1 & \dfrac{\partial \tilde{x}_2}{\partial p_2} + \dfrac{\partial \tilde{x}_2}{\partial D} z_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} + \begin{bmatrix} s_{13} \\ s_{23} \end{bmatrix} B^{3\prime}(B^{1\prime})^{-1}. \tag{22}$$

The trade-demand function is defined by

$$\hat{h}_i(p_1, p_2, D, l) = \tilde{x}_i(p_1, p_2, D, l) - \tilde{y}_i(p_1, p_2, D, l) \quad (i = 1, 2),$$

hence, subtracting (18) from (22) we obtain the trade-Slutsky matrix:

$$\begin{aligned} \hat{S} &= \begin{bmatrix} \hat{s}_{11} & \hat{s}_{12} \\ \hat{s}_{21} & \hat{s}_{22} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\partial \hat{h}_1}{\partial p_1} + \dfrac{\partial \hat{h}_1}{\partial D}\hat{h}_1 & \dfrac{\partial \hat{h}_1}{\partial p_2} + \dfrac{\partial \hat{h}_1}{\partial D}\hat{h}_2 \\ \dfrac{\partial \hat{h}_2}{\partial p_1} + \dfrac{\partial \hat{h}_2}{\partial D}\hat{h}_1 & \dfrac{\partial \hat{h}_2}{\partial p_2} + \dfrac{\partial \hat{h}_2}{\partial D}\hat{h}_2 \end{bmatrix} \\ &= \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} + \begin{bmatrix} s_{13} \\ s_{23} \end{bmatrix} B^{3\prime}(B^{1\prime})^{-1} + (B^1)^{-1}B^3 \begin{bmatrix} s_{31} & s_{32} \end{bmatrix} \\ &\quad + (B^1)^{-1}B^3 s_{33} B^{3\prime}(B^{1\prime})^{-1} + (B^1)^{-1}H(B^{1\prime})^{-1} \\ &= \begin{bmatrix} I_2 & (B^1)^{-1}B^3 \end{bmatrix} \begin{bmatrix} S_{11} & S_{13} \\ S_{31} & s_{33} \end{bmatrix} \begin{bmatrix} I_2 \\ B^{3\prime}(B^{1\prime})^{-1} \end{bmatrix} + (B^1)^{-1}H(B^{1\prime})^{-1}, \\ &= (B^1)^{-1}\{BSB' + H\}(B^{1\prime})^{-1}, \end{aligned} \tag{23}$$

where

$$S_{11} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, \quad S_{31} = \begin{bmatrix} s_{31} & s_{32} \end{bmatrix}, \quad S_{13} = \begin{bmatrix} s_{13} \\ s_{12} \end{bmatrix},$$

and thus

$$S = \begin{bmatrix} S_{11} & S_{13} \\ S_{31} & s_{33} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}, \quad \text{and where} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}.$$

Since S and H are symmetric and negative semi-definite, so is $\hat{S}$.

References

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