Notes on trade-demand functions with two produced tradables, one nontradable, and three factors

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Here we take up the case of three factors and three commodities, and assume that world prices are such that all three commodities are produced in strictly positive amounts. Then the domestic-product function Π is twice differentiable and yields single-valued Rybczynski functions for the three commodities, the third of which is assumed to be nontradable. The function $\tilde{p}_3(p_1, p_2, D, l)$ relating the price of the nontradable to the prices of the two tradables, the trade deficit, and the factor-endowment vector, is defined implicitly by

$$(1) \qquad h_3\Big(p_1, p_2, \tilde{p}_3(\cdot), \Pi(p_1, p_2, \tilde{p}_3(\cdot), l) + D\Big) - \hat{y}_3(p_1, p_2, \tilde{p}_3(\cdot), l) = 0.$$

The trade-demand functions for the tradables are then defined for $i = 1, 2$ by

$$(2) \; \hat{h}_i(p_1, p_2, D; l) = h_i(p_1, p_2, \tilde{p}_3(\cdot), \Pi(p_1, p_2, \tilde{p}_3(\cdot), l^1) + D) - \hat{y}_i(p_1, p_2, \tilde{p}_3(\cdot), l).$$

From (2) we have for $i, j = 1, 2$ (and using $\partial\Pi/\partial p_j = y_j$)

$$(3) \qquad \begin{aligned} \frac{\partial \hat{h}_i}{\partial p_j} &= \frac{\partial h_i}{\partial p_j} + \frac{\partial h_i}{\partial p_3}\frac{\partial \tilde{p}_3}{\partial p_j} + \frac{\partial h_i}{\partial Y}\left[y_j + y_3\frac{\partial \tilde{p}_3}{\partial p_j}\right] - \frac{\partial \hat{y}_i}{\partial p_j} - \frac{\partial \hat{y}_i}{\partial p_3}\frac{\partial \tilde{p}_3}{\partial p_j} \\ &= s_{ij} - t_{ij} - c_i z_j + (s_{i3} - t_{i3})\frac{\partial \tilde{p}_3}{\partial p_j}, \end{aligned}$$

where $t_{ij} = \partial\hat{y}_i/\partial p_j$, and (using $\partial\Pi/\partial p_3 = y_3 = x_3$)

$$(4) \qquad \begin{aligned} \frac{\partial \hat{h}_i}{\partial D} &= \frac{\partial h_i}{\partial p_3}\frac{\partial \tilde{p}_3}{\partial D} + \frac{\partial h_i}{\partial Y}\left[x_3\frac{\partial \tilde{p}_3}{\partial D} + 1\right] - \frac{\partial \hat{y}_i}{\partial p_3}\frac{\partial \tilde{p}_3}{\partial D} \\ &= (s_{i3} - t_{i3})\frac{\partial \tilde{p}_3}{\partial D} + c_i, \end{aligned}$$

where

$$s_{ij} = \frac{\partial h_i}{\partial p_j} + \frac{\partial h_i}{\partial Y}h_j \quad \text{and} \quad c_i = \frac{\partial h_i}{\partial Y}$$

and where we have used the fact that $x_i - y_i = z_i$ and $z_3 = 0$.

Differentiating the identity (1) with respect to p_j and D we obtain the partial derivatives of $\tilde{p}_3$ with respect to p_j and D:

$$\text{(5)} \qquad \frac{\partial \tilde{p}_3}{\partial p_j} = -\frac{s_{3j} - t_{3j} - c_3 z_j}{s_{33} - t_{33}} \quad \text{and} \quad \frac{\partial \tilde{p}_3}{\partial D} = -\frac{c_3}{s_{33} - t_{33}}.$$

Substituting (5) into (3) and (4) we obtain

$$\text{(6)} \qquad \frac{\partial \hat{h}_i}{\partial p_j} = s_{ij} - t_{ij} - c_i z_j - \frac{s_{i3} - t_{i3}}{s_{33} - t_{33}}(s_{3j} - t_{3j} - c_3 z_j)$$

and

$$\text{(7)} \qquad \frac{\partial \hat{h}_i}{\partial D} = -\frac{s_{i3} - t_{i3}}{s_{33} - t_{33}} c_3 + c_i,$$

whence

$$\text{(8)} \qquad \frac{\partial \hat{h}_i}{\partial D} \hat{h}_j = -\frac{s_{i3} - t_{i3}}{s_{33} - t_{33}} c_3 z_j + c_i z_j.$$

Summing (6) and (8) we finally obtain

$$\text{(9)} \qquad \hat{s}_{ij} \equiv \frac{\partial \hat{h}_i}{\partial p_j} + \frac{\partial \hat{h}_i}{\partial D} \hat{h}_j = s_{ij} - t_{ij} - (s_{i3} - t_{i3})(s_{33} - t_{33})^{-1}(s_{3j} - t_{3j}).$$

The trade-Slutsky matrix may therefore be written

$$\begin{aligned} \hat{S} = \begin{bmatrix} \hat{s}_{11} & \hat{s}_{12} \\ \hat{s}_{21} & \hat{s}_{22} \end{bmatrix} &= \begin{bmatrix} s_{11} - t_{11} & s_{12} - t_{12} \\ s_{21} - t_{21} & s_{22} - t_{22} \end{bmatrix} \\ \text{(10)} \qquad & - \begin{bmatrix} s_{13} - t_{13} \\ s_{23} - t_{23} \end{bmatrix} (s_{33} - t_{33})^{-1} \begin{bmatrix} s_{31} - t_{31} & s_{32} - t_{32} \end{bmatrix}. \end{aligned}$$

This matrix is clearly symmetric. To show that it is negative semi-definite, let

$$S - T = \begin{bmatrix} s_{11} - t_{11} & s_{12} - t_{12} & s_{13} - t_{13} \\ s_{21} - t_{21} & s_{22} - t_{22} & s_{23} - t_{23} \\ s_{31} - t_{31} & s_{32} - t_{32} & s_{33} - t_{33} \end{bmatrix}$$

denote the net-Slutsky matrix, which is clearly negative semi-definite, say of rank ρ. Then there exists a $\rho \times (n_1 + n_2 + n_3) = \rho \times 3$ matrix $R = [R_1, R_2, R_3]$ such that

$$R'R = -(S - T).$$

Written out, this is (assuming $\rho = 2$),

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix}$$

hence

$$R'R = \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \\ r_{13} & r_{23} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix} = - \begin{bmatrix} s_{11}-t_{11} & s_{12}-t_{12} & s_{13}-t_{13} \\ s_{21}-t_{21} & s_{22}-t_{22} & s_{23}-t_{23} \\ s_{31}-t_{31} & s_{32}-t_{32} & s_{33}-t_{33} \end{bmatrix}.$$

We have clearly

$$[R_1, R_2]'[R_1, R_2] = \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = - \begin{bmatrix} s_{11}-t_{11} & s_{12}-t_{12} \\ s_{21}-t_{21} & s_{22}-t_{22} \end{bmatrix},$$

which is the first matrix on the right in (10). Likewise,

$$R_3'R_3 = [r_{13} \quad r_{23}] \begin{bmatrix} r_{13} \\ r_{23} \end{bmatrix} = -(s_{33}-t_{33}).$$

Finally,

$$R_3'[R_1, R_2] = [r_{13} \quad r_{23}] \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = -[s_{31}-t_{31} \quad s_{32}-t_{32}],$$

which is the last matrix on the right in (10). Thus, (10) may be written

$$-\hat{S} = [R_1, R_2]'[I_r - R_3(R_3'R_3)^{-1}R_3'][R_1, R_2].$$

The matrix $R_3(R_3'R_3)^{-1}R_3'$ is idempotent of rank $n_3 = 1$, hence the matrix $I_r - R_3(R_3'R_3)^{-1}R_3'$ is idempotent of rank $\rho - 1$, and therefore positive semi-definite. Therefore $\hat{S}$ is negative semi-definite.

It follows from the theorem of Hurwicz and Uzawa (1971) that the two trade-demand functions $z_i = \hat{h}_i(p_1, p_2, D, l)$ $(i = 1, 2)$ are generated by maximizing a trade-utility function $\hat{U}(z_1, z_2)$ (where $z_i = x_i - y_i$ for $i = 1, 2$) subject to a balance-of-payments constraint $p_1 z_1 + p_2 z_2 \leq D$.

References

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