The Envelope Theorem, Euler and Bellman Equations, without Differentiability*

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Abstract

We extend the standard Bellman's theory of dynamic programming and the theory of recursive contracts with *forward-looking* constraints of Marcet and Marimon (2019) to encompass non-differentiability of the value function associated with non-unique solutions or multipliers. The envelope theorem provides the link between the Bellman equation and the Euler equations, but it may fail to do so if the value function is non-differentiable. We introduce an *envelope selection condition* which restores this link. In standard single-agent dynamic programming, ignoring the envelope selection condition may result in *inconsistent multipliers*, but not in non-optimal outcomes. In recursive contracts it can result in inconsistent promises and non-optimal outcomes. Planner problems with recursive preferences are a special case of recursive contracts and, therefore, solutions can be *dynamically inconsistent* if they are not unique. A recursive method of solving dynamic optimization problems with non-differentiable value function involves expanding the co-state and imposing the envelope selection condition.

Keywords: Dynamic programming, Euler equations, Envelope Theorem, Planner's problems, Recursive contracts, Recursive preferences, Value Function Differentiability.

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1 Introduction

The Euler equation and the Bellman equation are the two basic tools used to analyse dynamic optimization problems. Euler equations are the first-order inter-temporal *necessary conditions* for optimal solutions and, under standard concavity-convexity assumptions, they are also *sufficient conditions*, provided that a transversality condition holds. Euler equations are usually second-order difference equations. The Bellman equation allows the transformation of an infinite-horizon optimization problem into a recursive problem, resulting in time-independent policy functions determining the actions as functions of the states. The envelope theorem provides the bridge between the Bellman equation and the Euler equations, establishing the necessity of the latter for the former. This bridge allows us to reduce the second-order difference equation system of Euler equations to a first-order system, fully determined by the policy function of the Bellman equation with corresponding initial conditions, provided that the value function is differentiable.

Differentiability makes the bridge between the Bellman equation and the Euler equation tight. If the value function is differentiable, the state provides univocal information about the derivative and, therefore, the inter-temporal change of values across states (Bellman) is uniquely associated with the change of marginal values (Euler) via the envelope theorem. However, the value function may not be differentiable when solutions are non-unique or constraints are binding. In fact, as we show, the envelope theorem implies that a convex value function is non-differentiable if the solutions are non-unique, and concave value function is non-differentiable if the multipliers of the binding constraints are non-unique. In these cases, knowing the state and its value does not provide univocal information about the derivative, which is needed to recover the Euler equations from the Bellman equation.

Recursive methods of dynamic programming have been widely applied in macroeconomics over the last 30 years since the publication of Stokey *et al.* (1989). Usual assumptions such as interiority of optimal paths imply differentiability of the concave value function and enable the application of the standard envelope theorem. However, the issue of non-differentiability cannot be ignored in a wide range of currently used models. Models where households, firms, or countries, may face – possibly, *forward-looking* – binding constraints at optimal choices are, nowadays, more the norm than the exception. As we show, the differentiability problem caused by binding constraints or non-unique solutions is particularly pervasive in dynamic models with *forward-looking* constraints.[1] Forward-looking constraints are constraints

[1] Koeppl (2006) provided an early example of non-differentiabilty of the value function in a dynamic model of risk-sharing with enforcement constraints.

where feasibility of current actions depends on future actions. Incentive constraints stemming from limited enforcement or moral hazard usually take the form of forward-looking constraints. Dynamic planner's problems play an important role in macroeconomics because of the equivalence between the solution to planner's problems and competitive equilibria, provided by the welfare theorems. Planner's problems usually have value functions which are convex, with respect to the vector of Pareto weights. Solutions may not be unique and, therefore, the value function non-differentiable with respect to the Pareto weights. Non-uniqueness arises when the Pareto frontier is not strictly concave, for example, due to linear preferences (e.g. of consumers, firms, or banks), non-convex feasible sets, or the use of lotteries to convexify them; as it is done in general equilibrium models with incentive problems or externalities, see Prescott and Townsend (1984) and Kilenthong and Townsend (2021).

This paper extends the standard Bellman's theory of dynamic programming and the theory of recursive contracts of Marcet and Marimon (2019) – (MM19) from now on – to encompass non-differentiability of the value function resulting from the presence of binding constraints or non-unique solutions. In particular, this involves three major contributions: First, we generalize the envelope theorem using methods of subdifferential calculus and saddle-point analysis (see Rockafellar (1970))). The new envelope theorem plays a critical role in the applications to dynamic programming. Second – this is the main contribution of the paper – we identify an *envelope selection* condition in dynamic programming, and show that it is necessary and sufficient to recover the Euler equation from the Bellman equation when the value function is not differentiable. Third, we extend the theory of recursive contracts to recursive preferences of Koopmans (1960), showing that dynamic planner's problems with recursive preferences can be modelled as recursive contracts.

We derive the envelope theorem for static constrained optimization problems in Section 2 without assuming differentiability of the value function or interiority of the solutions. Our result extends the envelope theorem for directional derivatives of Milgrom and Segal (2002, Corollary 5) to maximization problems with non-compact choice sets. It applies to value functions arising in recursive dynamic programming. Further, we provide characterizations of the superdifferential of a concave value function and the subdifferential of a convex value function. We derive several sufficient conditions for differentiability of the value function from the envelope theorem. For example, if there is a unique saddle-point, the value function is differentiable and the standard form of the envelope theorem holds. If the value function is concave, a sufficient condition for differentiability is that the saddle-point multiplier be unique. If it is convex, a sufficient condition is that the solution be unique. The well-known result of Benveniste and Scheinkman (1979) stating that the concave value function is differentiable at an interior solution follows from our results because the saddle-point multiplier is zero and hence unique at an interior solution.[2]

In the main part of the paper, Section 3, we analyse dynamic models where the problem of differentiability of the value function may arise, and lead to time-inconsistent solutions or multipliers. We introduce the *envelope selection condition* which guarantees that solutions and multipliers generated from the Bellman equation satisfy the Euler equations. The envelope selection is a consistency condition on solutions and multipliers, and it involves subgradients of the value function. The recursive method of solving dynamic programming problems can be extended to provide consistent solutions and multipliers by expanding the co-state to include a subgradient of the value function, and taking a policy function from the saddle-point Bellman equation that satisfies the envelope selection condition. We show these results first in the context of deterministic recursive contracts: partnership problems with (forward-looking) limited enforcement constraints. It is in this framework that Messner and Pavoni (2016) constructed an example with linear preferences, resulting in non-unique solutions and non-differentiable convex value function, in which the saddle-point Bellman equation generates outcomes which are non-optimal or do not satisfy enforcement constraints. Our recursive method of solving recursive contracts involves expanding the co-state to include a subgradient of the value function and imposing the envelope selection condition (i.e. a subroutine within a recursive contracts algorithm). The envelope selection condition is equivalent to the *intertemporal consistency condition* in MM19, which, in turn, is the 'promise keeping condition'. If the value function is differentiable, the envelope selection condition is redundant and, accordingly, there is no need to expand the co-state.

Second, we analyze the deterministic dynamic Pareto planner's problem with *recursive preferences* of Lucas and Stokey (1984). Recursive utilities play an important role in macroeconomics, in particular, in stochastic models which can accommodate the important Epstein and Zin (1989) utility. We extend the theory of recursive contracts to recursive preferences. In fact, we show that the planner's problem is a special case of recursive contracts and, therefore, may have the differentiability problem if there are multiple solutions, either because the period-utility functions are linear or lotteries are used by the planner to convexify the set of feasible values (Examples 3 and 4). We also show that when lotteries over future values are used, even if preferences are time-separable, their representation as recursive preferences helps

[2]The result of Rincón-Zapatero and Santos (2009) that the value function in concave dynamic programming problems is differentiable if the multiplier is unique follows from our results as well.

to define and characterise the saddle-point Bellman equation of the planner's problem. In the absence of the envelope selection condition, the recursive method of solving the planner's problem may deliver time-inconsistent paths of consumptions and continuation utilities to the agents even without limited enforcement constraints. The envelope selection condition recovers the Euler equations and the consistency of solutions. Our saddle-point approach to the planner's problem is an alternative to the approaches of Lucas and Stokey (1984) and, more recently, Pavoni *et al.* (2018). Those approaches are subject to the problem of differentiability of the value function and potential inconsistency of solutions as well.

Third, we consider the standard dynamic programming problem with discounted time-separable utilities and no forward looking constraints. We show that in problems with binding *backward looking* constraints and multiple multipliers (resulting in non-differentiability of the concave value function), there may be inconsistency of multipliers, but not solutions. The envelope selection condition makes the multipliers to follow a time-consistent path of the Euler equations. We present an example in Section 3 and a general discussion of the standard dynamic programming in Appendix F.

Dynamic macroeconomics abounds in optimization problems with forward-looking constraints (see Ljungqvist and Sargent (2018)). Recursive contracts (MM19) are nowadays part of the graduate toolkit to formulate and solve dynamic incentive and limited-commitment problems, see Messner, Pavoni and Sleet (2012) and Mele (2014), with a broad range of applications in economics. Some examples are: optimal taxation (Aiyagari *et al.* (2002)), international business cycles (Kehoe and Perri (2003), debt contracts and risk-sharing mechanisms (Ábrahám *et al.*, (2020)), dynamic political economy (Acemoglu *et al.* (2011)), dynamic household behavior (Voena (2015)). Other work that closely relates to ours is Cole and Kubler (2012). They provide an example of a dynamic principal-agent problem, where lotteries convexify a non-convex set of feasible contracts and, therefore, the value function of the planner's problem is not differentiable. As in Messner and Pavoni (2016) they discuss how recursive contracts with multiple solutions can fail to select the optimal ones, which is precisely the problem we analyse, solve and provide an algorithmic solution for the general class of recursive contract in this paper.[3] Pavoni *et al.* (2018) develop a slightly different approach to dynamic contracts with forward-looking constraints and recursive preferences based on duality theory, but they do not take advantage of the saddle-point structure of recursive contracts. They identify the problem of multiple solutions to the recursive method but do not offer any remedy other than requiring unique solutions. Instead, we show that recursive utilities, and the cor-

[3]Cole and Kubler (2012) provide a solution tailored to the two-agent partnership problem they consider.

responding planner's problems, have a saddle-point structure and, therefore, can be analysed as recursive contracts and, for example, the result on existence of the saddle-point value function in (MM19) applies with minor modifications. Our result is for the deterministic case and provides the guideline for the more general stochastic case. The extension of the co-state and the *envelope selection condition*, proposed here, can be easily incorporated into existing programs that compute recursive contracts, which in turn use relatively standard computational methods in dynamic optimisation, thereby expanding the class of problems that the graduate toolkit can handle.

The envelope theorem has long history in mathematics and economics, as well. Earlier versions of it for non-parametric constraints have been known as Danskin's Theorem (see Oyama and Takenawa (2018) and references therein). Our approach provides representation of directional derivatives in optimization problems with parametric constraints using saddle-points of the Lagrangian, and extends the results of Milgrom and Segal (2002). A slightly different approach using multipliers of the Kuhn-Tucker first-order conditions instead of saddle-point multipliers in problems with differentiable objective and constraint functions has been developed in Dubeau and Govin (1982) and Morand *et al.* (2015, 2018)).[4] Recent papers by Oyama and Takenawa (2018) and Clausen and Strub (2016) provide further explorations of the conditions for differentiability of the value function.

The paper is organized as follows: Sections 2 covers the static envelope theorem and provides an example of how our results apply. Section 3 contains the analysis of recursive contracts for the partnership problem with forward-looking participation constraints (subsection 3.1), the Pareto planner's problem with recursive utilities – with an example, showing how (subsection 3.2), and the standard dynamic programming (subsection 3.3). All subsections include examples where the value function is non-differentiable and the saddle-point Bellman equation generates non-optimal outcomes or inconsistent multipliers. The appendix contains an extension of the envelope theorem to constrained saddle-point problems, proofs of the main results, and a discussion of the standard dynamic programming.

2 The Envelope Theorem

This section gives a synopsis of the envelope theorem for parametric constrained maximization problems and the differentiability of the value function. An extension to saddle-point problems is presented in

[4]Extensions of the envelope theorem to non-smooth optimization problems using generalized Kuhn-Tucker multipliers can be found in Morand *et al.* (2015, 2018).

Appendix C for the use in recursive contracts in Section 3.

We consider the following problem:

$$\max_{y \in Y} \; f(x,y) \tag{1}$$

$$\text{s.t.} \;\; h_1(x,y) \geq 0, \; \ldots, \; h_k(x,y) \geq 0. \tag{2}$$

Parameter x lies in the set $X \subset \Re^m$. Choice variable y lies in $Y \subset \Re^n$. Objective function f and each constraint function h_i are real-valued functions on $Y \times X$.[5] The value function of the problem (1–2) is denoted by $V(x)$.

The Lagrangian function associated with (1) is $\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda h(x,y)$, where $\lambda \in \Re^k_+$ is a vector of (positive) multipliers.[6] The saddle-point of $\mathcal{L}$ is a pair (y^*, λ^*), where $y^* \in Y$ is called a saddle-point solution and $\lambda^* \in \Re^k_+$ a saddle-point multiplier, such that

$$\mathcal{L}(x,y,\lambda^*) \leq \mathcal{L}(x,y^*,\lambda^*) \leq \mathcal{L}(x,y^*,\lambda), \tag{3}$$

for every $y \in Y$ and $\lambda \in \Re^k_+$. The *saddle-point* operator defined by (3) is denoted[7] by SP so that

$$\mathcal{L}(x,y^*,\lambda^*) = \text{SP} \min_{\lambda \geq 0} \max_{y \in Y} \mathcal{L}(x,y,\lambda). \tag{4}$$

The set of saddle-points of $\mathcal{L}$ is a product of two sets $Y^*(x) \times \Lambda^*(x)$, where $Y^*(x) \subset Y$ and $\Lambda^*(x) \subset \Re^k_+$, see Lemma 1, Appendix A. The slackness condition $\lambda_i^* h_i(x,y^*) = 0$ for every i, for every $(y^*, \lambda^*) \in Y^*(x) \times \Lambda^*(x)$, implies that the saddle-point value of (4) is equal to $V(x)$.

We impose the following conditions, for every i,

A1. Y is convex; f and h_i are continuous functions of (x,y),

A2. The constraint set $\Gamma(x) = \{y \in Y : h(x,y) \geq 0\}$ is compact for every $x \in X$.

A3. The correspondence $\Gamma : X \to Y$ is continuous.

A4. For every $x \in X$, there exists $\hat{y}_i \in Y$ such that $h_i(x,\hat{y}_i) > 0$ and $h_j(x,\hat{y}_i) \geq 0$ for $j \neq i$.

A5. $Y^*(x) \times \Lambda^*(x) \neq \emptyset$ for every $x \in X$.

[5] We abstract away from equality constraints. Typical budgetary, resource, or incentive constraints can be stated as inequalities.

[6] We use the product notation: $\lambda h(x,y) = \sum_{i=1}^k \lambda_i h_i(x,y)$.

[7] This SP notation was introduced in Marcet and Marimon (2019). The $\min$ and $\max$ operators merely indicate which variables are being minimised and maximised, respectively, in the saddle-point.

Assumption A1 is standard. A2 guarantees the existence of a solution to (1–2). A3 guarantees continuity of the value function while A4 is a weak form of the Slater's condition. Assumption A5 says that the set of saddle-points is nonempty. It holds if functions f and h_i are concave in y and A4 holds.[8] If A5 holds, then the saddle-point solutions $Y^*(x)$ are precisely the solutions to (1–2), see the proof of Lemma 1, Appendix A. We refer to them simply as solutions.

The envelope theorem is best stated in terms of directional derivatives of the value function V. We first consider one-dimensional parameter set X – a convex subset of the real line. Directional derivatives are then the left- and right-hand derivatives defined as $V'(x+) = \lim_{t\to 0+}[V(x+t) - V(x)]/t$ and $V'(x-) = \lim_{t\to 0-}[V(x+t) - V(x)]/t$ if the limits exist.

We have the following result:

Theorem 1. *Suppose that $X \subset \Re$, conditions A1-A5 hold, and partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial h_i}{\partial x}$ are continuous functions of (x, y). Then the value function V is right- and left-hand differentiable at every $x \in intX$ and the directional derivatives are*

$$V'(x+) = \max_{y^*\in Y^*(x)} \min_{\lambda^*\in\Lambda^*(x)} \big[\frac{\partial f}{\partial x}(x, y^*) + \lambda^* \frac{\partial h}{\partial x}(x, y^*)\big] \tag{5}$$

and

$$V'(x-) = \min_{y^*\in Y^*(x)} \max_{\lambda^*\in\Lambda^*(x)} \big[\frac{\partial f}{\partial x}(x, y^*) + \lambda^* \frac{\partial h}{\partial x}(x, y^*)\big], \tag{6}$$

where the order of maximum and minimum does not matter.

Theorem 1 generalizes Corollary 5 in Milgrom and Segal (2002) by discarding assumptions of concavity of functions f and h_i and compactness of the choice set Y, and weakening the Slater's condition. In applications to dynamic programming in Section 3 and Appendix F, the choice set is not compact.

The value function V on $X \subset \Re$ is differentiable at x if the one-sided derivatives are equal to each other. Sufficient conditions for differentiability follow from Theorem 1.

Corollary 1. *Under the assumptions of Theorem 1, each of the following conditions is sufficient for differentiability of value function V at $x \in intX$:*

(i) there is a unique saddle-point,

[8] Lemma 36.2 in Rockafellar (1970) provides necessary and sufficient conditions for existence of a saddle-point without concavity of f or h_i.

(ii) there is a unique solution and h_i does not depend on x for every i.

(iii) there is a unique saddle-point multiplier and $\frac{\partial f}{\partial x}$ and $\frac{\partial h_i}{\partial x}$ do not depend on y, for every i.

A result related to (iii) can be found in Kim (1993). A sufficient condition for uniqueness of saddle-point solution is that f be strictly concave and h_i be concave in y. A sufficient condition for uniqueness of the multiplier is a Constrained Qualification condition, see Appendix B.

For a multidimensional parameter set X in $\Re^m$, the directional derivative of the value function V at $x \in X$ in the direction $\hat{x} \in \Re^m$ such that $x+\hat{x} \in X$ is defined as $V'(x;\hat{x}) = \lim_{t\to 0+}[V(x+t\hat{x})-V(x)]/t$. Applying Theorem 1 to the value function $V(x+t\hat{x})$ of single variable t, it follows that, if $D_x f(x,y)$ and $D_x h_i(x,y)$ are continuous functions of (x,y), then

$$V'(x;\hat{x}) = \max_{y^*\in Y^*(x)} \min_{\lambda^*\in\Lambda^*(x)} \left[D_x f(x,y^*) + \lambda^* D_x h(x,y^*)\right] \hat{x}, \tag{7}$$

and the order of maximum and minimum does not matter. The value function V is (Gateaux) differentiable at $x \in X$ if the gradient vector $DV(x)$ exists and $V'(x;\hat{x}) = DV(x)\hat{x}$ for every direction $\hat{x} \in \Re^m$. Any condition of Corollary 1 for multivariate functions f and h_i is sufficient for differentiability of value function V. In that case, it holds

$$DV(x) = D_x f(x,y^*) + \lambda^* D_x h(x,y^*) \tag{8}$$

for every $(y^*,\lambda^*) \in Y^*(x) \times \Lambda^*(x)$.

If the value function is concave or convex, the envelope theorem can be stated using the superdifferential or the subdifferential. The superdifferential $\partial V(x)$ of concave V is the set of all vectors $\phi \in \Re^m$ such that $V(x') + \phi(x-x') \leq V(x)$ for every $x' \in X$. The value function is concave if the objective function f and all constraint functions h_i are concave functions of (x,y). We have the following:

Theorem 2. *Suppose that conditions A1-A5 hold, derivatives $D_x f$ and $D_x h_i$ are continuous functions of (x,y) for every i, and V is concave. Then*

$$\partial V(x) = \bigcap_{y^*\in Y^*(x)} \bigcup_{\lambda^*\in\Lambda^*(x)} \{D_x f(x,y^*) + \lambda^* D_x h(x,y^*)\} \tag{9}$$

for every $x \in intX$.

It follows that a sufficient condition for differentiability of concave value function V at $x \in intX$ is that the saddle-point multiplier be unique, or alternatively that h_i does not depend on x for every i (for the latter, see Corollary 3 in Milgrom and Segal (2002)).

The subdifferential $\partial V(x)$ of convex V is defined by reversing the inequality in the definition of the superdifferential.[9] The value function is convex, if the objective function $f(y, \cdot)$ is convex in x for every $y \in Y$ and all constraint functions h_i are independent of x. We have the following:

Theorem 3. *Suppose that conditions A1-A5 hold, derivatives $D_x f$ and $D_x h_i$ are continuous functions of (x, y) for every i, and V is convex. Then*

$$\partial V(x) = \bigcap_{\lambda^* \in \Lambda^*(x)} co\Big(\bigcup_{y^* \in Y^*(x)} \{D_x f(x, y^*) + \lambda^* D_x h(x, y^*)\}\Big), \tag{10}$$

for every $x \in intX$, where co( ) denotes the convex hull.

A sufficient condition for differentiability of convex value function is that the saddle-point solution be unique. We conclude with an example.

Example 1. *(A planner's problem)*

Consider the resource allocation problem in an economy with k agents. The planner's problem is

$$\max_{c \geq 0} \sum_{i=1}^{k} \mu_i u_i(c_i) \tag{11}$$

$$\text{s.t.} \sum_{i=1}^{n} c_i \leq x, \tag{12}$$

where $\mu = (\mu_1, \ldots, \mu_k) \in \Re^k_{++}$ is a vector ofF welfare weights and $x \in \Re^L_{++}$ represents total resources. Utility functions u_i are continuous so that conditions A1-A4 hold. Let $V(x, \mu)$ be the value of (11). It follows from Corollary 1 (iii) that V is differentiable in x if the saddle-point multiplier λ^* of constraint (12) exists and is unique. If utility functions u_i are differentiable, then the CQ condition (see Appendix B) holds, implying that the multiplier is unique. The derivative is $D_x V = \lambda^*$. V is a convex function of μ. The subdifferential $\partial_\mu V$ is (by Theorem 3) the convex hull of the set of vectors $(u_1(c_1^*), \ldots, u_k(c_k^*))$ over all saddle-point solutions c^*. V is differentiable in μ if the saddle-point solution exists and is unique.

[9] We use the same notation for the superdifferential and the subdifferential as is customary in the literature.

Consider two agents with single good and linear utilities $u_i(c) = c$. Let the welfare weights be $\mu_1 = \mu$ and $\mu_2 = 1 - \mu$ for $\mu \in [0, 1]$. The value function is $V(x, \mu) = \max\{\mu, 1 - \mu\}x$. For every $\mu \neq \frac{1}{2}$, V is differentiable with respect to μ and the solution c^* is unique. For $\mu = \frac{1}{2}$, V is not differentiable, and there are multiple solutions. V is differentiable with respect to x.

3 The envelope selection condition

Euler equations characterize the solutions of the infinite-horizon problems. However, when these problems are analyzed as dynamic programming problems, the Bellman equation and the envelope theorem may not be equivalent to the Euler equations if the value function is not differentiable. We identify this inconsistency and introduce an *envelope selection* condition which is necessary and sufficient to recover the Euler equations, and to guarantee correct solutions. We apply the envelope theorem and the results on the differentiability of the value function and on the characterisation of the subdifferentials of Section 2 and Appendix C.

This section consists of three subsections. First, we show the role of the *envelope selection* condition in *recursive contracts* (MM19). Second, we show that the inconsistency can emerge in the planner problem with *recursive preferences* of Lucas and Stokey (1984). Thereby we extend *recursive contracts* to *recursive preferences*. Third, we show that the inconsistency may also arise in the standard dynamic programming with binding *backward looking* constraints. In this case, it is inconsistency of multipliers but not suboptimality of solutions.

3.1 The partnership problem with limited enforcement

We consider a canonical deterministic partnership problem[10] with *limited enforcement* and discounted time-separable utilities. Each agent receives an endowment of $y_{i,t}$, and has an outside option with utility $v_i(y_{i,t})$ if she leaves the partnership at date t. The planner's problem is

[10](MM19) considers recursive contracts in a more general dynamic stochastic formulation. The approach presented in this section can be easily extended to their general setup with *infinite-horizon forward-looking* constraints.

$$
\begin{aligned}
V_\mu(y_0) \quad = \quad & \max_{\{c_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{m} \mu_i \, u(c_{i,t}) & (13) \\
& \text{s.t. } \sum_{i=1}^{m} y_{i,t} - \sum_{i=1}^{m} c_{i,t} \geq 0, & (14) \\
& \sum_{n=0}^{\infty} \beta^n \, u(c_{i,t+n}) - v_i(y_{i,t}) \geq 0, & (15) \\
& c_{i,t} \geq 0, \quad \text{for all } i, t \geq 0,
\end{aligned}
$$

where the sequence $\{c_t\}$ is bounded, i.e. $\{c_t\} \in \ell_\infty^m$. The sequence of incomes $\{y_t\}$ is bounded and follows a law of motion $y_{t+1} = g(y_t)$ for some $g : \Re_{++}^m \to \Re_{++}^m$ and given initial income vector y_0. We impose the following conditions:

P1. The function $u : \Re_+ \to \Re_+$ is bounded, increasing, concave, and differentiable. Furthermore $\mu_i > 0$ for every i, and $\beta \in (0, 1)$.

P2. Given $y_0 \in \Re_{++}^m$, there exists $\epsilon > 0$ and $\{\hat{c}_t\}_{t=0}^\infty \in \ell_\infty^m$, with $\hat{c}_{i,t} > 0$ for every i and t, such that constraints (14) and (15), hold with ϵ instead of 0 on the right-hand side.

A convenient way of analysing problem (13) is to consider the following constrained saddle-point problem resulting from writing the Lagrangean with all the forward-looking constraints (15), rearranging terms, and collecting the partial sums of multipliers into weights $\mu_{i,t}$ – see (MM19) for details:

$$
\begin{aligned}
\text{SP} \max_{\{c_t\}_{t=0}^\infty} \min_{\{\mu_t\}_{t=1}^\infty} \quad & \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{m} \left[\mu_{i,t+1} \left(u(c_{i,t}) - v_i(y_{i,t}) \right) + \mu_{i,t} v_i(y_{i,t}) \right] & (16) \\
& \text{s.t. } \sum_{i=1}^{m} c_{i,t} \leq \sum_{i=1}^{m} y_{i,t} \\
& \mu_{i,t+1} \geq \mu_{i,t} \\
& c_{i,t} \geq 0, \quad \text{for all } i, t,
\end{aligned}
$$

where $\mu_0 = \mu$, and $\{c_t\} \in \ell_\infty^m$ and $\{\mu_t\} \in \ell_\infty^m$. The unconstrained saddle-point problem for the Lagrangian of (16) is

$$\text{SP} \max_{\{c_t,\lambda_{t+1}\}_{t=0}^{\infty}} \min_{\{\gamma_t,\mu_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t \sum_{i=1}^{m}\big\{\mu_{i,t+1}\left(u(c_{i,t})-v_i(y_{i,t})\right)+\mu_{i,t}v_i(y_{i,t}) \qquad (17)$$
$$-\lambda_{i,t+1}(\mu_{i,t+1}-\mu_{i,t})+\gamma_t\left(y_{i,t}-c_{i,t}\right)\big\}.$$

If $\{c_t^*\}\in\ell_\infty^m$ is a solution to (13), then under assumptions P1-P2 there exists a bounded sequence of weights $\{\mu_t^*\}\in\ell_\infty^m$ and summable sequences of multipliers $\{\lambda_t^*\}\in\ell_1^m$ and $\{\gamma_t^*\}\in\ell_1$ (see Dechert (1992, Theorem 2)) such that $\{c_t^*,\lambda_{t+1}^*,\mu_{t+1}^*,\gamma_t^*\}$ is a saddle-point of (17).[11] Conversely, if $\{c_t^*,\lambda_{t+1}^*,\mu_{t+1}^*,\gamma_t^*\}$ is a saddle-point of (17), then constraints (14) and (15) are satisfied and $\{c_t^*\}$ is a solution to (13).

The first-order necessary conditions with respect to $\mu_{i,t+1}$ for saddle-point $\{c_t^*,\lambda_{t+1}^*,\mu_{t+1}^*,\gamma_t^*\}$ of (17) are

$$u(c_{i,t}^*)-\left(v_i(y_{i,t})+\lambda_{i,t+1}^*\right)+\beta\left(v_i(y_{i,t+1})+\lambda_{i,t+2}^*\right)=0 \qquad (18)$$

for every i and $t\geq 0$. The respective condition with respect to $c_{i,t}$ is $\mu_{i,t+1}^*u'(c_{i,t}^*)=\gamma_t^*$. Equation (18) is the *intertemporal Euler equation* for the partnership problem. The Euler equation together with first-order conditions for $c_{i,t}$, the constraints, and complementary slackness conditions for γ_t^* and $\lambda_{i,t+1}^*$, are a system of second-order difference equations.

Euler equations and a transversality condition are sufficient conditions for a solution to (17).

Proposition 1. *Suppose that condition P1 holds. Let $\{c_t^*,\lambda_{t+1}^*,\mu_{t+1}^*,\gamma_t^*\}$, with $\{c_t^*\}_{t=0}^{\infty}\in\ell_\infty^m$, $\{\mu_t^*\}_{t=1}^{\infty}\in\ell_\infty^m$, $\mu_0^*=\mu$, $\lambda_{t+1}^*\geq 0$ and $\gamma_t^*\geq 0$ for every t, satisfy the Euler equations (18), the constraints and complementary slackness conditions*[12]*, and the first-order conditions with respect to $c_{i,t}$. If the transversality condition*

$$\lim_{t\to\infty}\beta^t[v_i(y_{i,t})+\lambda_{i,t+1}^*]=0 \qquad (19)$$

holds for every i, then $\{c_t^*,\lambda_{t+1}^*,\mu_{t+1}^*,\gamma_t^*\}$ is a saddle-point of (17). In particular, $\{c_t^*\}$ is a solution to the partnership problem (13).

Proof: see Appendix D.

The constrained saddle-point problem (16) has recursive structure that can be expressed by the follow-

[11]In the derivation of (16) weights μ_t^* are obtained as partial sums of a summable sequence and therefore are a convergent, hence bounded, sequence.

[12]These are: $\lambda_{i,t+1}^*(\mu_{i,t+1}^*-\mu_{i,t}^*)=0,\ \mu_{i,t+1}^*-\mu_{i,t}^*\geq 0$ for all i, and $\gamma_t^*\sum_{i=1}^m\left(y_{i,t}-c_{i,t}^*\right)=0,\ \sum_{i=1}^m\left(y_{i,t}-c_{i,t}^*\right)\geq 0.$

ing saddle-point Bellman equation:

$$W(y,\mu) = \text{SP}\min_{\mu'}\max_{c}\sum_{i=1}^{m}\left[\mu_i'(u(c_i)-v_i(y_i))+\mu_i v_i(y_i)\right]+\beta W(y',\mu') \tag{20}$$

$$\text{s.t.} \sum_{i=1}^{m} c_i \leq \sum_{i=1}^{m} y_i \tag{21}$$

$$\mu_i' \geq \mu_i \tag{22}$$

$$c_i \geq 0, \text{ for all } i,$$

where $y' = g(y)$. The existence of function W satisfying equation (20) has been established in (MM19; Theorem 3) under assumptions P1 and P2. The value function W is convex and homogeneous of degree one with respect to μ. (MM19; Theorem 1) implies that the value function V_μ satisfies equation (20), that is, $W(y_0,\mu) = V_\mu(y_0)$ for every $\mu \in \Re^m_+$.

A saddle-point of the Bellman equation (20) is denoted by $(c^*, \lambda^*, \mu^{*\prime}, \gamma^*)$ where γ^* is a multiplier of constraint (21) and λ^* is a multiplier of (22). The set of all saddle-points is a product of two sets $M^*(y,\mu)$ and $N^*(y,\mu)$ so that $(c^*, \lambda^*) \in M^*(y,\mu)$ and $(\mu^{*\prime}, \gamma^*) \in N^*(y,\mu)$ for every saddle-point $(c^*, \lambda^*, \mu^{*\prime}, \gamma^*)$. Under assumptions P1-P2, the sets $M^*(y,\mu)$ and $N^*(y,\mu)$ are non-empty. The envelope Theorem 4 for saddle-point problems, see Appendix C, can be applied to W.[13] Eq. (98) implies that the subdifferential of W is

$$\partial_\mu W(y,\mu) = v(y) + \{\lambda^* \mid (c^*, \lambda^*) \in M^*(y,\mu) \text{ for some } c^*\}. \tag{23}$$

Function W is differentiable with respect to μ at (y,μ) if and only if there is unique multiplier λ^* that is common to all solutions c^*.

For every sequence $\{c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*\}_{t=0}^\infty$ which is a saddle-point of (17), $(c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*)$ is a saddle-point of the saddle-point Bellman equation (20) at (y_t, μ_t^*) for every t. It follows that for every solution $\{c_t^*\}$ to partnership problem (13), there exist weights $\{\mu_t^*\}$ such that (c_t^*, μ_{t+1}^*) is a solution to the saddle-point Bellman equation for every t. However, the converse result requires an envelope selection condition that involves subgradients from the first-order conditions of (20). The first-order conditions with respect to μ' for saddle-point $(c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*)$ at (y_t, μ_t^*) state that there exists subgradient vector

[13]The constraint set given by (22) can be compactified under P2, see (MM19; Section 4.)

$\phi^*_{t+1} \in \partial_\mu W(y_{t+1}, \mu^*_{t+1})$ such that

$$u(c^*_{i,t}) - \big(v_i(y_{i,t}) + \lambda^*_{i,t+1}\big) + \beta\phi^*_{i,t+1} = 0 \tag{24}$$

for every i.

Proposition 2. *Suppose that conditions P1-P2 hold. Let $\{c^*_t, \lambda^*_{t+1}, \mu^*_{t+1}, \gamma^*_t\}$ be a sequence of saddle-points generated by saddle-point Bellman equation (20) starting at (y_0, μ), with $\{c^*_t\}_{t=0}^\infty \in \ell^m_\infty$ and $\{\mu^*_t\}_{t=1}^\infty \in \ell^m_\infty$, and let $\{\phi^*_t\}_{t=1}^\infty$ be the corresponding sequence of subgradients with $\phi^*_{t+1} \in \partial_\mu W(y_{t+1}, \mu^*_{t+1})$ satisfying (24) for $t \geq 0$. If the following envelope selection condition*

$$\phi^*_{i,t} = v_i(y_{i,t}) + \lambda^*_{i,t+1} \tag{25}$$

holds for every i and every $t \geq 1$, then $\{c^*_t, \lambda^*_{t+1}, \mu^*_{t+1}, \gamma^*_t\}$ is a saddle-point of (17) and $\{c^*_t\}$ is a solution to (13).

Proof: The first-order conditions (24) together with the envelope selection condition (25) imply the Euler equation (18) for every $t \geq 0$. The complementary slackness conditions and the first-order condition with respect to $c_{i,t}$ follow from the respective conditions for (20). Since W is bounded and convex, the sequence of subgradients $\{\phi^*_t\}$ is bounded. The transversality condition (19), which by (25) can be written as $\lim_{t\to\infty} \beta^t \phi^*_t = 0$, holds. The conclusion follows from Proposition 1. □

Proposition 2 corresponds to – and provides an alternative proof of – (MM19, Theorem 2 and its Corollary) where the envelope selection condition (25) is replaced by the *intertemporal consistency condition*

$$\phi^*_{i,t} = u(c^*_{i,t}) + \beta\phi^*_{i,t+1}. \tag{26}$$

Because of the first-order condition (24), those two conditions are equivalent in this context.

The *envelope selection condition* (25) guarantees *consistency* of multipliers and solutions generated by the saddle-point Bellman equation. It can be dispensed with if the saddle-point multiplier is unique, which is a sufficient condition for value function W to be differentiable in μ. Inconsistency of multipliers may occur if, given $(c^*_t, \lambda^*_{t+1}, \mu^*_{t+1}, \gamma^*_t)$ with the corresponding subgradient $\phi^*_{t+1} \in \partial_\mu W(y_{t+1}, \mu^*_{t+1})$ from the first-order condition (24) at t, at $t+1$ multiplier λ^*_{t+2} is chosen without satisfying envelope selection condition (25). This is likely to happen if the saddle-point Bellman equation is solved only knowing (y_{t+1}, μ^*_{t+1}). Then the Euler equation (18) is not satisfied for λ^*_{t+1} and λ^*_{t+2}, and the sequence $\{c^*_t, \lambda^*_{t+1}, \mu^*_{t+1}, \gamma^*_t\}$ need not be a saddle-point of (17). Since the multiplier λ^*_{t+1} has to be chosen together with consumption c^*_t in the set $M^*(y_t, \mu^*_t)$, an inconsistent choice of the multiplier may lead to consumption that is either suboptimal, or violates the participation constraints.

Proposition 2 does not provide a recursive method of generating consistent solutions from the saddle-point Bellman equation, but it clearly suggests what that method should be: extend the co-state to (μ, ϕ) with $\phi \in \partial_\mu W(y, \mu)$, and impose the envelope selection condition. To that end, we define the *selective value function* as the value function of the saddle-point Bellman equation (20) with the additional restriction that the saddle-point satisfies the envelope selection condition. That is:

$$W^s(\,y,\,\mu;\phi) = \text{SP}\min_{\mu'}\max_{c,\lambda}\sum_{i=1}^{m}\left[\mu'_i(u(c_i) - v_i(y_i)) + \mu_i v_i(y_i) - \lambda_i\left(\mu'_i - \mu_i\right)\right] + \beta W(y', \mu') \tag{27}$$

$$\text{s.t. } v(y_i) + \lambda_i = \phi_i, \tag{28}$$

$$\sum_{i=1}^{m} c_i \leq \sum_{i=1}^{m} y_i,$$

$$c_i \geq 0, \lambda_i \geq 0, \text{ for all } i,$$

where $y' = g(y)$.[14] It holds that $W^s(\,y,\,\mu;\phi) = W(\,y,\,\mu)$ for $\phi \in \partial_\mu W(y, \mu)$, but the (saddle-point) solutions to (27) may be a proper subset of solutions of (20). The first-order condition for saddle-point $(c^*, \lambda^*, \mu^{*\prime}, \gamma^*)$ of (27) with respect to μ is

$$u(c^*_i) - (v_i(y_i) + \lambda^*_i) + \beta\phi^{*\prime}_i = 0, \tag{29}$$

where $\phi^{*\prime} \in \partial_\mu W(y', \mu^{*\prime})$. It holds that

$$W^s(\,y,\,\mu;\phi) = \sum_{i=1}^{m}\left[\mu^{*\prime}_i(u(c^*_i) - v_i(y_i)) + \mu_i v_i(y_i)\right] + \beta W^s(y', \mu^{*\prime}; \phi^{*\prime}).$$

The policy functions $\varphi : \Re^{3m}_+ \to \Re^{2m}_+$ and $\ell : \Re^{3m}_+ \to \Re^{m+1}_+$ are defined by $\varphi(y,\,\mu,\phi) = (c^*, \lambda^*, \mu^{*\prime}, \gamma^*)$ such that $(c^*, \lambda^*, \mu^{*\prime}, \gamma^*)$ is a saddle-point of (27) and $\ell(y,\,\mu,\phi) = \phi^{*\prime}$ where $\phi^{*\prime} \in \partial_\mu W(y', \mu^{*\prime})$ satisfies the first-order condition (29).

[14]Note that the multiplier λ is uniquely determined by constraint (53).

Policy functions (φ, ℓ) can be used to generate sequences of saddle-points $\{c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*\}$ and subgradients $\{\phi_{t+1}^*\}$ such that $(c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*) = \varphi(y_t, \mu_t^*, \phi_t^*)$ and $\phi_{t+1}^* = \ell(y_t, \mu_t^*, \phi_t^*)$ for every $t \geq 0$, with initial state y_0 and co-state (μ_0^*, ϕ_0^*) where $\mu_0^* = \mu$ and $\phi_0^* \in \partial_\mu W(y_0, \mu)$. It follows from Proposition 1 that $\{c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*\}_{t=0}^\infty$ is a saddle-point of (17).[15] The sequence $\{c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*, \phi_{t+1}^*\}$ can be found by solving the system of equations (25) and (24) together with first-order conditions w.r. $c_{i,t}$ and complementary slackness conditions. All these equations are first-order difference equations.

The next Corollary 2 summarizes our results for recursive contracts. It extends the sufficiency result in (MM19; Theorem 2 and its Corollary) by providing a recursive algorithm for solving recursive contracts with forward-looking constraints.

Corollary 2. *Suppose that conditions P1-P2 hold. If $\{c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*\}$ and $\{\phi_{t+1}^*\}$ are sequences of saddle-points and subgradients generated by policy functions (φ, ℓ) of (27), starting from $\mu_0^* = \mu$ and $\phi_0^* \in \partial_\mu W(y_0, \mu)$, with $\{c_t^*\}_{t=0}^\infty \in \ell_\infty^m$ and $\{\mu_t^*\}_{t=0}^\infty \in \ell_\infty^m$, then $\{c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*\}$ is a saddle-point of (17) at μ and $\{c_t^*\}$ is a solution to (13). Furthermore, if at date $\tau + 1$ a new sequence $\{\widehat{c}_t^*, \widehat{\lambda}_{t+1}^*, \widehat{\mu}_{t+1}^*, \widehat{\gamma}_t^*\}_{t=\tau+1}^\infty$ is generated using possibly different policy functions $(\widehat{\varphi}, \widehat{\ell})$ starting from initial state y_τ and co-state $(\mu_\tau^*, \phi_\tau^*)$, then $(\{c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_t^*\}_{t=1}^\tau, \{\widehat{c}_t^*, \widehat{\lambda}_{t+1}^*, \widehat{\mu}_{t+1}^*, \widehat{\gamma}_t^*\}_{t=\tau+1}^\infty)$ is also a saddle-point of (17) at μ and $\{c_t^*\}$ is a solution to (13).*

In sum, the intertemporal Euler equation (18) along with other first-order conditions are necessary and sufficient for a solution to the partnership problem (13). Saddle-point Bellman equation can be used to generate the solution recursively, but the envelope selection condition is required to guarantee that the Euler equation are satisfied. Such sequence can be generated by a policy function of the selective value function. Example 2 illustrates these results.

Example 2. *(Messner and Pavoni (2004) revisited)*

[15]Note that co-state ϕ_t^* can be replaced by λ_{t+1}^*. This is so because the envelope selection condition (23) is a one-to-one relationship between the subgradient ϕ_t^* and the multiplier λ_{t+1}^*.

Consider a partnership problem (13) with two agents and linear utilities,

$$
\begin{aligned}
W(\mu) \quad = \quad & \max_{\{c_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{2} \mu_i c_{i,t} && (30)\\
& \text{s.t. } \; c_{1,t} + c_{2,t} \leq y, \\
& \quad \sum_{j=0}^{\infty} \beta^j c_{1,t+j} \geq 0, \;\; \sum_{j=0}^{\infty} \beta^j c_{2,t+j} \geq b(1-\beta)^{-1}, && (31)\\
& \quad c_{i,t} \geq 0, \; i = 1,2, \; \text{for all } t \geq 0,
\end{aligned}
$$

where $0 < b < y$, $\mu_i > 0$ for $i = 1, 2$, and $0 < \beta < 1$. Note that agent 1's participation constraint is not binding. The value function is

$$
W(\mu) = \begin{cases} (1-\beta)^{-1}[\mu_1(y-b) + \mu_2 b] & \text{if } \; \mu_1 \geq \mu_2 \\ (1-\beta)^{-1}\mu_2 y & \text{if } \; \mu_1 \leq \mu_2. \end{cases} \tag{32}
$$

If $\mu_1 > \mu_2$, the constant sequence $c^*_{1,t} = y - b$ and $c^*_{2,t} = b$ is a solution to (30), but we shall see that there are many other solutions. The value function W is convex and differentiable if $\mu_1 \neq \mu_2$, but it is not differentiable at $\mu_1 = \mu_2$ where the sub-differential is

$$
\partial W(\mu) = (1-\beta)^{-1} co\{(y-b, b), (0, y)\}. \tag{33}
$$

The saddle-point Bellman equation (20) is

$$
W(\mu) = \text{SP } \min_{\mu'} \max_{c} \left\{ \sum_{i=1}^{2} \mu'_i c_i - (\mu'_2 - \mu_2) b (1-\beta)^{-1} + \beta W(\mu') \right\} \tag{34}
$$

$$
\text{s.t. } c_1 + c_2 \leq y, \; \mu'_i \geq \mu_i, \; c_i \geq 0, \; i = 1, 2.
$$

A sequence of saddle-points $\{c^*_t, \lambda^*_{t+1}, \mu^*_{t+1}, \gamma^*_{t+1}\}$ of (34) generated by a policy function can be found by recursively solving a system of equations consisting of the first-order conditions

$$
\begin{aligned}
\mu^*_{i,t+1} - \gamma^*_{t+1} \quad &= \quad 0, && (35)\\
c^*_{1,t} - \lambda^*_{1,t+1} + \beta \phi^*_{1,t+1} \quad &= \quad 0 && (36)\\
c^*_{2,t} - (b(1-\beta)^{-1} + \lambda^*_{2,t+1}) + \beta \phi^*_{2,t+1} \quad &= \quad 0, && (37)
\end{aligned}
$$

with $\phi^*_{t+1} \in \partial W(\mu^*_{t+1})$, the complementary slackness conditions for λ^*_{t+1}, and the envelope selection conditions

$$\phi^*_{1,t} = \lambda^*_{1,t+1}, \text{ and } \quad \phi^*_{2,t} = b(1-\beta)^{-1} + \lambda^*_{2,t+1}. \tag{38}$$

Suppose that the initial state is $\mu^*_0 = \mu$ such that $\mu_1 > \mu_2$. Since W is differentiable at μ^*_0, the initial co-state is $\phi^*_0 = DW(\mu^*_0)$, that is, $\phi^*_{1,0} = (y-b)(1-\beta)^{-1}$ and $\phi^*_{2,0} = b(1-\beta)^{-1}$. From equations (38) we obtain $\lambda^*_{1,1} = (y-b)(1-\beta)^{-1}$ and $\lambda^*_{2,1} = 0$, and using complementary slackness $\mu^*_{1,1} = \mu_1$ and $\mu^*_{2,1} = \mu^*_{1,1}$. Because W is not differentiable at μ^*_1, $\phi^*_{1,1}$ and $\phi^*_{2,1}$ can be arbitrary, as long as they satisfy: $\phi^*_{1,1} + \phi^*_{2,1} = y(1-\beta)^{-1}$, $\phi^*_{1,1} \geq 0$ and $\phi^*_{2,1} \geq b(1-\beta)^{-1}$ (see (33)). The consumption plan c^*_0 can be any selection satisfying $c^*_{1,0} = (y-b)(1-\beta)^{-1} - \beta\phi^*_{1,1}$ and $c^*_{2,0} = b(1-\beta)^{-1} - \beta\phi^*_{2,1}$. Selection of c^*_0 determines ϕ^*_1. We have thus derived $\phi^*_1 = \ell(\mu^*_0, \phi^*_0)$, and $(c^*_0, \mu^*_1, \lambda^*_1) = \varphi(\mu^*_0, \phi^*_0)$ for a policy function (φ, ℓ).[16]

Next, iteration of equations (35) - (38) with given state μ^*_1 and co-state ϕ^*_1 gives $\mu^*_2 = \mu^*_1$, $c^*_{1,1} = \phi^*_{1,1} - \beta\phi^*_{1,2}$, $c^*_{2,1} = \phi^*_{2,1} - \beta\phi^*_{2,2}$ where $\phi^*_{1,2} + \phi^*_{2,2} = y(1-\beta)^{-1}$, $\phi^*_{1,2} \geq 0$ and $\phi^*_{2,2} \geq b(1-\beta)^{-1}$. Again, a selection of c^*_1 determines ϕ^*_2. With this step, we have derived $\phi^*_2 = \ell(\mu^*_1, \phi^*_1)$, and $(c^*_1, \mu^*_2, \lambda^*_2) = \varphi(\mu^*_1, \phi^*_1)$. All subsequent iterations follow the same pattern with $\mu^*_t = \mu^*_1$ for all $t > 1$, and μ^*_t being the point of non-differentiability of value function W. Corollary 5 implies that every sequence generated in this way is an optimal solution to (30). For example, consumption sequence $c^*_{1,t} = y - b$ and $c^*_{2,t} = b$ is optimal.

There are sequences of saddle-points of the Bellman equation (34) that are not optimal solutions to (30) because the envelope selection conditions are not satisfied. For example, if $\beta y \geq b$, then the sequences $c^*_{1,t} = y$, $c^*_{2,t} = 0$, for all t, with μ^*_t as before and the corresponding subgradients satisfy equations (35 - 36), but the envelope selection conditions (38) are violated for every $t \geq 1$. This sequence violates participation constraints (31) and hence is not a solution to (30).

3.2 Pareto optimal allocations with recursive utilities

In this section we consider the planner's problem of constructing Pareto optimal allocations with *recursive preferences*. As in (13), the planner's objective is a Benthamite welfare function with positive Pareto weights. We abstract away from limited enforcement constraints. The optimization problem is

[16] To simplify notation, we eliminated multipliers γ^*_t (given by (35)) from consideration.

$$
\begin{aligned}
V(y_0, \mu_0) \quad &\equiv \quad \max_{\{c_t\}_0^\infty} \sum_{i=1}^{m} \mu_{i,0} U_{i,0}(c_i) &(39)\\
\text{s.t.} \quad & \quad \sum_{i=1}^{m} c_{i,t} \le y_t, &(40)\\
& \quad c_{i,t} \ge 0, \text{ for all } i, t \ge 0,
\end{aligned}
$$

where the sequences $\{c_{i,t}\}$ are bounded, i.e. $\{c_{i,t}\} \in \ell_\infty^+$ for every i. The sequence $\{y_t\}$ of aggregate endowment follows the law of motion $y_{t+1} = g(y_t)$ for some $g : \Re_{++}^m \to \Re_{++}^m$, with initial value y_0. We impose the following assumption that parallels P2 of Section (13):

R1. The sequence $\{y_t\}$ is bounded, and there exists $\epsilon > 0$ such that $y_t > \epsilon$ for every $t \ge 0$.

Recursive preferences have been introduced by Koopmans (1960) to relax the restrictions of discounted time-separable utilities while maintaining recursivity. They are specified by utility functions $U_t(c)$ for every date $t \ge 0$ which are *history independent*, that is, utility $U_t(c)$ depends only on consumption sequence $(c_t, c_{t+1}, \dots)$ from date-t on.[17] Date-t utility $U_t(c)$ is related to date-$(t+1)$ continuation utility $U_{t+1}(c)$ via stationary aggregator $F : \Re_+ \times \Re_+ \to \Re$, so that

$$
U_t(c) = F(c_t, U_{t+1}(c)), \quad t = 0, \dots, \tag{41}
$$

for every $\{c_t\} \in \ell_\infty^+$.[18] Recursive preferences are *dynamically consistent*. Examples of recursive preferences are the standard discounted time-separable utility with $F(c, v) = u(c) + \beta v$, and the Epstein-Uzawa utility with $F(c, v) = u(c) + \beta(c)v$, where $\beta(c) = e^{-u(c)}$.

Aggregator function F_i of every agent i is assumed to satisfy the following conditions,

R2. $F_i : \Re_+ \times \Re_+ \to \Re$ is bounded, strictly increasing, concave, differentiable, and $F_i(0, 0) = 0$.

R3. There exists $0 < \beta < 1$ such that $|F_i(c, v) - F_i(c, v')| \le \beta |v - v'|$ for every $c \in R_+$ and $v, v' \in R_+$.

Assumptions R2 and R3 guarantee that a unique sequence of utilities $\{U_{i,t}(c)\} \in \ell_\infty^+$ defined recur-

[17] For simplicity, we write c is the notation $U_t(c)$ even though U_t depends only on $(c_t, c_{t+1}, \dots)$. Note that - unlike in Koopmans (1960) and Lucas and Stokey (1984) - utility functions U_t are time-dependent.

[18] We do not restrict recursive preferences to satisfy Koopmans' condition of *future-independence*, i.e. preferences over c_t are independent of future consumption $(c_{t+1}, \dots)$, although this condition is not restrictive when there is a single consumption good at every date. See Backus *et al.* (2004) for a discussion.

sively by the aggregator F_i exist for every $c \in \ell_\infty^+$, see Lucas and Stokey (1984, Theorem 1). Assumption R3 is known as the Blackwell condition.

Using the recursive structure of utilities (41), the Pareto problem (39) can be restated as *recursive contract* problem, akin to (13). That is,

$$V(y_0, \mu_0) = \max_{\{c_t, v_{t+1}\}_{t=0}^{\infty}} \sum_{i=1}^{m} \mu_{i,0} F_i(c_{i,0}, v_{i,1}) \tag{42}$$

$$\begin{aligned} \text{s.t.} \quad & F_i(c_{i,t}, v_{i,t+1}) - v_{i,t} \geq 0, \text{ for all } i, t \geq 1, \\ & y_t - \sum_{i=1}^{m} c_{i,t} \geq 0, \\ & c_{i,t} \geq 0, \; v_{i,t+1} \geq 0, \text{ for all } i, t \geq 0, \end{aligned} \tag{43}$$

where $\{c_{i,t}\}_{t=0}^{\infty} \in \ell_\infty^+$ and $\{v_{i,t}\}_{t=1}^{\infty} \in \ell_\infty^+$ for every i. Constraints (43) have been written as inequality constraint, because aggregator functions are strictly increasing. It is easy to see that, under R2 and R3, $\{c_t^*\}$ is a solution to (39) if an only if $\{c_t^*, v_{t+1}^*\}$ solves (42), where $v_{i,t}^* = F_i(c_{i,t}^*, v_{i,t+1}^*)$ for every $t \geq 1$ and every i.

Problem (42) is a special case of recursive contracts of Marcet and Marimon (2019) with *one-period forward-looking constraints*, see Appendix E. It can be analyzed as a saddle-point problem of the Lagrangian which has the same value function. It is convenient to use multipliers scaled by β^t, with the discount factor β of assumption R3, so that $\beta^t \mu_t^i$ is the multiplier of date-t constraint (43) for $t \geq 1$. Rearranging terms, we can write the saddle-point problem as

$$V(y_0, \mu_0) = \text{SP} \min_{\{\mu_t\}_{t=1}^{\infty}} \max_{\{c_t, v_{t+1}\}} \sum_{t=0}^{\infty} \sum_{i=1}^{m} \beta^t [\mu_{i,t} F_i(c_{i,t}, v_{i,t+1}) - \beta \mu_{i,t+1} v_{i,t+1}] \tag{44}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_i c_{i,t} \leq y_t, \\ & c_{i,t} \geq 0, \; v_{i,t+1} \geq 0, \; \mu_{i,t+1} \geq 0, \text{ for all } i, t \geq 0, \end{aligned}$$

If $\{c_t^*, v_{t+1}^*\}$ is a solution to (42) with $\{c_t^*\} \in \ell_\infty^m$ and $\{v_t^*\} \in \ell_\infty^m$, then under assumptions R1 - R3 there exists a sequence of multipliers $\{\mu_t^*\}$ and $\{\gamma_t^*\}$ (of the feasibility constraint) such that $\{\beta^t \mu_t^*\} \in \ell_1^m$ and $\{\gamma_t^*\} \in \ell_1^m$ (see Dechert (1992, Theorem 2)) such that $\{c_t^*, \mu_{t+1}^*, v_{t+1}^*, \gamma_t^*\}$ is a saddle-point of (44).

The necessary first-order conditions with respect to $c_{i,t}$ and $v_{i,t+1}$ for a saddle-point $\{c_t^*, \mu_{t+1}^*, v_{t+1}^*, \gamma_t^*\}$

of (44), where γ_t^* is a multiplier of the feasibility constraint, are $\beta^t \mu_{i,t}^* \partial_1 F_i(c_{i,t}^*, v_{i,t+1}^*) = \gamma_t^*$ and $\mu_{i,t}^* \partial_2 F_i(c_{i,t}^*, v_{i,t+1}^*) = \beta \mu_{i,t+1}^*$. The first-order condition with respect to $\mu_{i,t}$ is

$$F_i(c_{i,t}^*, v_{i,t+1}^*) = v_{i,t}^* \tag{45}$$

for every i and $t \geq 0$. Equations (45) are the *intertemporal Euler equations.* They guarantee consistency of continuation utilities with consumption plans at optimal solutions.

Euler equations and a transversality condition are sufficient conditions for a solution to (44).

Proposition 3. *Suppose that conditions R2 and R3 hold. Let* $\{c_t^*, v_t^*, \mu_t^*, \gamma_t^*\}$*, with* $\{c_t^*\}_{t=0}^\infty \in \ell_\infty^m$*,* $\{v_t^*\}_{t=1}^\infty \in \ell_\infty^m$*,* $\mu_0^* = \mu$*, satisfy the Euler equations (45), the constraints and complementary slackness conditions and the first-order conditions with respect to* $c_{i,t}$ *and* $v_{i,t+1}$*. If the transversality condition*

$$\lim_{t \to \infty} \beta^t \mu_{i,t}^* v_{i,t}^* = 0 \tag{46}$$

holds for every i*, then* $\{c_t^*, v_t^*, \mu_t^*, \gamma_t^*\}$ *is a saddle-point of (44). In particular,* $\{c_t^*\}$ *is a solution to the Pareto problem (39).*

Proof: The proof is analogous to that of Proposition 1.

The *saddle-point Bellman equation* of the problem (44) is

$$V(y, \mu) = \text{SP} \min_{\mu'} \max_{c,v} \sum_{i=1}^m [\mu_i F_i(c_i, v_i) - \beta \mu_i' v_i] + \beta V(y', \mu') \tag{47}$$

$$\text{s.t.} \sum_i c_i \leq y,$$

$$c_i \geq 0,\ v_i \geq 0,\ \mu_i' \geq 0, \text{ for all } i,$$

where $y' = g(y)$.

If assumptions R2 - R3 are satisfied, it follows from Marcet and Marimon (2019, Theorem 3) that there exists a continuous and bounded value function satisfying (47).[19] The value function V is convex and homogeneous of degree one in μ.

The set of all saddle-points of (47) is a product of two sets $M^*(y, \mu)$ and $N^*(y, \mu)$ so that $(c^*, v^*) \in$

[19] See Appendix E for details on how Marcet and Marimon (2019, Theorem 3) applies to recursive preferences.

$M^*(y,\mu)$ and $(\mu'^*, \gamma^*) \in N^*(y,\mu)$ for every saddle-point $(c^*, v^*, \mu'^*, \gamma^*)$. The envelope Theorem 4 of Appendix C and, in particular, equation (98) imply that

$$\partial_{\mu_i} V(y,\mu) = \{F_i(c_i^*, v_i^*) : (c^*, v^*) \in M^*(y,\mu)\}, \tag{48}$$

Clearly, V is differentiable in μ if the saddle-point maximizer is unique. A sufficient condition for this is strict concavity of the aggregators F_i.

Let $(\mu_{t+1}^*, c_t^*, v_{t+1}^*, \gamma_t^*)$ be a saddle-point Bellman equation (47) at date t with state (μ_t^*, y_t). Let ϕ_t^* be the subdifferential satisfying FOC for the optimality of μ_t^* at $t-1$, that is

$$-v_{i,t}^* + \phi_{i,t}^* = 0, \tag{49}$$

and $\phi_{i,t}^* \in \partial_{\mu_i} V(y_t, \mu_t^*)$. The *envelope selection condition* for a solution (c_t^*, v_{t+1}^*) at date t is

$$\phi_{i,t}^* = F_i(c_{i,t}^*, v_{i,t+1}^*). \tag{50}$$

If imposed, this condition implies - together with (49) - that the Euler equation

$$v_{i,t}^* = F_i(c_{i,t}^*, v_{i,t+1}^*) \tag{51}$$

holds (see (45)). The envelope selection condition guarantees that continuation utilities are consistent with consumption plans. If the value function is differentiable, the envelope selection condition is redundant.

Proposition 4. *Suppose that conditions R1-R3 hold. Let $\{c_t^*, v_t^*, \mu_t^*, \gamma_t^*\}$ be a sequence of saddle-points generated by saddle-point Bellman equation (47) starting at (y_0, μ_0), with $\{c_t^*\}_{t=0}^\infty \in \ell_\infty^m$, $\{v_t^*\}_{t=1}^\infty \in \ell_\infty^m$, and $\{\beta^t \mu_t^*\}_{t=1}^\infty \in \ell_1^m$, and let $\{\phi_t^*\}_{t=1}^\infty$ be the corresponding sequence of subgradients with $\phi_{t+1}^* \in \partial_\mu V(y_{t+1}, \mu_{t+1}^*)$ satisfying (49) for $t \geq 0$. If the envelope selection condition* (50) *holds for every i and every $t \geq 1$, then $\{c_t^*, v_t^*, \mu_t^*, \gamma_t^*\}$ is a saddle-point of (44) and $\{c_t^*\}$ is a solution to (39).*

Proof: The proof is analogous to that of Proposition 2. The transversality condition (46) holds because $\{\beta^t \mu_t^*\} \in \ell_1^m$, and $\{v_t^*\}$ is bounded. $\square$

Imposing the envelope selection condition (50) requires adding ϕ_t^* as co-state to (y_t, μ_t^*) at every $t \geq 0$. As in Section 3.3, we define the *selective value function* of state (y,μ) and co-state ϕ, where

$\phi \in \partial_\mu V(y, \mu)$, that is:

$$V^s(y, \mu; \phi) = \text{SP} \ \min_{\mu'} \max_{c,\, v} \sum_{i=1}^{m} [\mu_i F_i(c_i, v_i) - \beta \mu_i' v_i] + \beta V(y', \mu') \tag{52}$$

$$\begin{aligned} \text{s.t.} \ \ & \phi_i = F_i(c_i, v_i), && (53) \\ & \sum_{i=1}^{m} c_i \leq y, \\ & c_i \geq 0, \ v_i \geq 0, \ \mu_i' \geq 0, \text{ for all } \ i, \end{aligned}$$

where $y' = g(y)$. It holds that $V^s(y, \mu; \phi) = V(y, \mu)$ for $\phi \in \partial_\mu V(y, \mu)$, but the (saddle-point) solutions to (52) may be a proper subset of solutions of (47).

The policy functions $\varphi : \Re_+^{3m} \to \Re_+^{2m}$ and $\ell : \Re_+^{3m} \to \Re_+^{m+1}$ are defined by $\varphi(y, \mu, \phi) = (c^*, v^*, \mu^{*\prime}, \gamma^*)$ such that $(c^*, v^*, \mu^{*\prime}, \gamma^*)$ is a saddle-point of (52) and $\ell(y, \mu, \phi) = \phi^{*\prime}$ where $\phi^{*\prime} \in \partial_\mu V(y', \mu^{*\prime})$ satisfies the first-order condition $-v_i^* + \phi_i^{*\prime} = 0$. Policy functions (φ, ℓ) can be used to generate sequences of saddle-points of (52) that are saddle-point solution to (44) therefore a solution to (39) as well.

The following example shows that, if value function V is non-differentiable and there are multiple solutions to (47), then the envelope selection condition is indispensable.

Example 3. *(Planner's recursive choices may be time-inconsistent)*

Let there be two agents with the same aggregator that is linear in current consumption and strictly concave in continuation utility. We take $F_i = F$ for $i = 1, 2$, where

$$F(c, v) = c + \beta \ln(v + 1), \tag{54}$$

where $\beta < 1$, see Koopmans et al (1964).[20] Further, let $g(y) = y$ so that the aggregate endowment is constant over time, with $y > 0$. Aggregator (54) satisfies conditions R2 and R3.[21] It follows that there exists stationary utility function $U : \ell_+^\infty \to R$ such that date-t utility $U(c_t, \dots)$ satisfies the recursive

[20] Koopmans et al (1964, pg 97) consider the aggregator $F(c, v) = \ln(1 + c + \beta v)$. It is well known that the transformed aggregator $\hat{F}(c, v) = h \circ F(c, h^{-1}(v))$ is ordinally equivalent for any strictly increasing function h, as it gives rise to recursive utility function $h \circ U$. Taking $h(x) = e^x - 1$, we obtain (54).

[21] In fact F is not bounded in c, but this does not affect the analysis.

relation (41). It holds

$$U(c) = c_0 + \beta \lim_{T\to\infty} \ln\{c_1 + \beta \ln\{c_2 + \dots .\beta \ln(c_T + 1)\}\},$$

see Lucas and Stokey (1984). Utility function U is linear in date-0 consumption and strictly concave in date-t consumption for $t \geq 1$.

Pareto optimal allocations that maximize the welfare function with equal weights $\bar{\mu} = (\mu, \mu)$ are of the form $c_i = (c_{i,0}, \frac{y}{2}, \frac{y}{2}, \dots)$ for $i = 1, 2$, with arbitrary $c_{i,0} \geq 0$ such that $c_{1,0} + c_{2,0} = y$. Date-0 utility of consumption plan c_i obtain as follows: For the constant consumption plan with $c_{i,0} = \frac{y}{2}$, continuation utility is time-invariant $\bar{v}$, and it obtains from $\frac{y}{2} + \beta \ln(\bar{v} + 1) = \bar{v}$. It is easy to see that there is unique $\bar{v} > 0$ solving this equation. Date-0 utility of any other c_i is $c_{i,0} + \bar{v} - \frac{y}{2}$.

The value function V at equal weights $\bar{\mu}$ is $V(\bar{\mu}) = 2\mu\bar{v}$. Because of multiple solutions to (39) at $\bar{\mu}$, the convex value function V is non-differentiable at $\bar{\mu}$. The subdifferential $\partial_\mu V(y, \bar{\mu})$ obtains from (48) as

$$\partial_\mu V(y, \bar{\mu}) = \{(\alpha y + \bar{v} - \frac{y}{2}, (1-\alpha)y + \bar{v} - \frac{y}{2}) : \alpha \in [0, 1]\}. \tag{55}$$

The first-order conditions for saddle-point (c^*, v^*, μ^*) of Bellman equation (47) are $-v_i^* + \phi_i^* = 0$ for some $\phi_i^* \in \partial_{\mu_i} V(y, \mu^*)$, and $\frac{\mu_i}{v_i^*+1} = \mu_i^*$. At $\bar{\mu}$, equal weights $\mu_1^* = \mu_2^*$ are a solution to (47) together with equal continuation utilities $v_1^* = v_2^*$, and with $\phi_1^* = \phi_2^*$. Eq. (55) implies that $\phi_i^* = \bar{v}$, and therefore $v_i^* = \bar{v}$. The first-order conditions put no restriction on solutions (c_1^*, c_2^*) beyond feasibility.

When solving the saddle-point Bellman equation with a policy function of the selective value function, consumption plans are determined by the envelope selection condition (see (50))

$$\phi_{i,t}^* = c_{i,t}^* + \beta \ln(v_{i,t+1}^* + 1). \tag{56}$$

It guarantees that the consumption sequence is consistent with continuation utilities. Let the initial state be $\bar{\mu}_0 = (\mu_0, \mu_0)$ and the co-state be $\phi_0^* \in \partial_\mu V(y, \bar{\mu}_0)$. The sequence $\{v_t^*, \mu_t^*\}$, where $\mu_{t+1}^* = \frac{\mu_t^*}{\bar{v}+1}$ and $v_{i,t}^* = \bar{v}$ solves recursively the first-order conditions with corresponding subgradient co-states $\phi_{i,t}^* = \bar{v}$ for $t \geq 1$. The consumption sequence implied by the envelope selection condition (56) is $c_{i,t}^* = \frac{y}{2}$, for $t \geq 1$ and date-0 consumption $c_{i,0}^*$ is determined by condition (56) with the initial co-state ϕ_0^*; that is, the initial co-state provides a parametrization of date-0 solutions. By Proposition 4, the sequence $\{c_t^*\}$ is a solution to

(39). Ignoring the envelope selection condition (56) in solving the saddle-point Bellman equation at equal-weights $\bar{\mu}_0$, would lead to *dynamically inconsistent* solutions, letting one to conclude that an arbitrary sequence of feasible consumptions can be the planner's choice. In this example, *dynamically consistent* solutions are semi-stationary – i.e. consumption is constant after the first period – this is not the case in the next example.

Pareto optimal allocations with lotteries

Following the pioneering work of Prescott and Townsend (1984), convexification through lotteries has been a standard practice in macroeconomics since incentive problems, as well as externalities (Kilenthong and Townsend (2021), often result in non-convex feasible sets. The following example shows that in dynamic economic problems, where lotteries are over agents' values – in particular, continuation values – the envelope selection condition can play a key role. It also shows how our saddle-point representation of recursive preferences can help to formulate these problems recursively.

Example 4. *(Convexification with lotteries)*

Consider an economy with two infinitely-lived agents and two consumption goods which can be produced with two technologies using labour as the unique input. Agents have time-separable preferences with a discount rate $\beta \in (0, 1)$ and on current consumption represented by:

$$u_1(c) = c_1^{\gamma} c_2^{1-\gamma} \text{ and } u_2(c) = c_1^{1-\gamma} c_2^{\gamma}, \text{ where } \gamma \in (1/2, 1).$$

Agents jointly provide one unit of labour without disutility, which can be allocated either to technology 1 or to technology 2, with $n \in [0, 1]$ being the fraction of labour used in the production of good 1 and $(1 - n)$ the fraction fraction used in the production of good 2. There is only one plant and, therefore, only one technology can be used at any given period, but technologies can be changed between periods without adjustment costs. The two technology frontiers are:

$$\begin{aligned} T_1 &= \{(c_1, c_2) = (n, \alpha(1 - n)) : n \in [0, 1]\} \\ T_2 &= \{(c_1, c_2) = (\alpha n, 1 - n) : n \in [0, 1]\}, \end{aligned}$$

where $\alpha \in (0, 1)$. The planner chooses at the beginning of any period the technology – say, $s \in \{s^1, s^2\}$, where s^1 denotes the choice of technology T_1, – and, conditional on the technological choice, how to allocate the unit the labour and consumption goods among the two agents.

We provide first a characterization of Pareto frontiers corresponding to the planner using only one technology. Let $(n^*(s^j,\mu), c^*(s^j,\mu))$ be the choice of $(n,c) \in [0,1] \times \mathrm{R}^4_+$ that solves the one-period Pareto problem with weights $\mu = (\mu_1, \mu_2)$ for technology T_j. The Pareto frontier corresponding to T_j is described by

$$VT_j(\mu) = (1-\beta)^{-1}\left[\mu_1 u_1(c_1^*(s^j,\mu)) + \mu_2 u_2(c_2^*(s^j,\mu))\right], \quad j = 1, 2. \tag{57}$$

Figure 1 shows how the graphs of $VT_1(\mu)$ and $VT_2(\mu)$ intersect, which happens since $\gamma > 1/2$ and

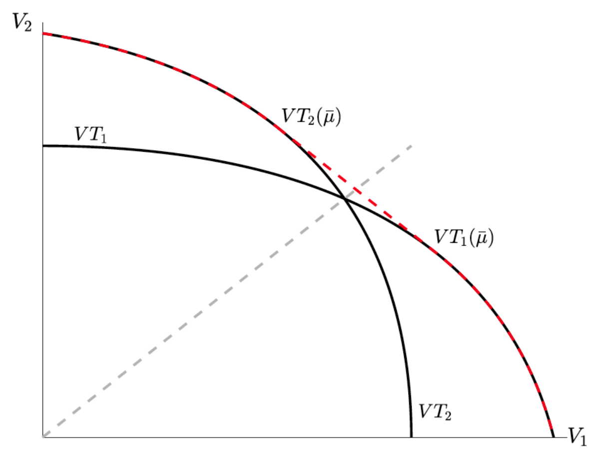

Figure 1: The Pareto frontier with (dashed), and without lotteries

$\alpha \in (0,1)$. Without lotteries, the one-period Pareto frontier is the upper contour of the two functions (scaled down by $(1-\beta)$). It generates a non-convex set of feasible values. With lotteries, as we assume, the set of feasible values is convex, including the flat portion of slope -1, both for the one-period and the infinite-horizon problem[22]

The value function W of the planner who can choose the technology, as a function of the Pareto weights, is

$$W(\mu) = (1-\beta)^{-1} \max\{VT_1(\mu), VT_2(\mu)\}. \tag{58}$$

[22]The planner in the infinite-horizon problem can expand the one-period possibility frontier with deterministic programs by changing technologies over periods of different length. However, not all points on the frontier with lotteries can be achieved that way. More importantly, such deterministic programs with changing technologies are subject to the same selection problem as with lotteries.

Function W is convex. Because of the symmetry of technologies, it holds $W(\mu) = (1-\beta)^{-1}VT_1(\mu)$ for $\mu_1 > \mu_2$, and $W(\mu) = (1-\beta)^{-1}VT_2(\mu)$ for $\mu_2 > \mu_1$. For equal weights denoted by $\bar{\mu}$ with $\bar{\mu}_1 = \bar{\mu}_2$, we have $W(\bar{\mu}) = (1-\beta)^{-1}VT_1(\bar{\mu}) = (1-\beta)^{-1}VT_2(\bar{\mu})$. Value function W is non-differentiable at $\bar{\mu}$ where its subdifferential is

$$\partial W(\bar{\mu}) = (1-\beta)^{-1}\text{co}\{\partial VT_1(\bar{\mu}), \partial VT_2(\bar{\mu})\}. \tag{59}$$

Let $\ell = (\pi; \{n(s^j), c(s^j)\}_{j=1,2})$ denote a one-period lottery which takes a T_1-feasible allocation $(n(s^1), c(s^1))$ with probability π and a T_2-feasible allocation $(n(s^2), c(s^2))$ with probability $1-\pi$. Let $\mathcal{L}$ be the convex and compact set of such lotteries. Expected utility of lottery ℓ to agent i is $E_\pi[u_i(c_i(s))]$. The planner's problem with lotteries is

$$\begin{aligned} W(\mu) \quad &= \quad \max_{\{\ell_t\}_0^\infty} \sum_{i=1}^{2} \mu_i \sum_{t=0}^{\infty} \beta^t \, E_{\pi_t}[u_i(c_{i,t}(s))] \\ \text{s.t.} \quad & \qquad \ell_t \in \mathcal{L}, \quad \forall t \geq 0. \end{aligned} \tag{60}$$

For every solution $\{\ell_t^*\}$ to (60) at μ, the support of lottery ℓ_t^* are the solutions to one-period Pareto problems, that is, ℓ_t^* takes $(n^*(s^1,\mu), c^*(s^1,\mu))$ with probability π_t^*, and $(n^*(s^2,\mu), c^*(s^2,\mu))$ with probability $1-\pi_t^*$. Note that, for every $\{\ell_t\}_{t=0}^\infty$ there is a one-period lottery $\bar{\ell}$ such that $\sum_{t=0}^\infty \beta^t \, E_{\pi_t}[u_i(c_{i,t}(s))] = (1-\beta)^{-1}E_{\bar{\pi}}[u_i(c_i(s))$, i.e. the discounted expected utility of $\{\ell_t\}_{t=0}^\infty$ can be achieved with a stationary lottery. In particular, if $\{\ell_t^*\}_{t=0}^\infty$ is a solution to (60), then $\bar{\pi}^* = 1$ for $\mu_1 > \mu_2$, $\bar{\pi}^* = 0$ for $\mu_1 < \mu_2$, and $\bar{\pi}^* \in [0,1]$ for $\mu = \bar{\mu}$.

Let $U_{i,t}(L) = \sum_{\tau=t}^\infty \beta^{\tau-t} E_{\pi_\tau}[u_i(c_{i,\tau}(s))]$ denote date-t discounted expected utility of the sequence of lotteries $L = \{\ell_t\}$ so that date-0 utilities $U_{i,0}(L)$ for $i = 1,2$ are featured in the planner's problem (60). This utility function has recursive representation $U_{i,t}(L) = F_i(\ell_t, U_{i,t+1}(L))$ with the aggregator

$$F_i(\ell, v_i) = E_\pi[u_i(c_i(s)] + \beta v_i \tag{61}$$

for $\ell \in \mathcal{L}$ and $v_i \geq 0$. The planner's problem (60) has exactly the same structure as the problem (39) of Pareto optimal consumption allocation with recursive utilities. The saddle-point Bellman equation is

$$\begin{aligned} W(\mu) \quad &= \quad \min_{\mu'} \max_{\ell, v} \sum_{i=1}^{2} [\mu_i F_i(\ell, v_i) - \beta \mu_i' v_i] + \beta W(\mu') \\ \text{s.t.} \quad & \qquad \ell \in \mathcal{L}, \; v_i \geq 0, \; \mu_i' \geq 0, \text{ for } i = 1,2, \end{aligned} \tag{62}$$

as (47). The envelope Theorem 4 of Appendix C implies that

$$\begin{aligned} \partial_{\mu_i} W(\mu) &= \{F_i(\ell^*, v_i^*) : (\ell^*, v_i^*) \in M^*(\mu)\} \\ &= \{E_{\pi^*}[u_i(c_i^*(s)] + \beta v_i^* : (\ell^*, v_i^*) \in M^*(\mu)\}, \end{aligned} \tag{63}$$

where $M^*(\mu)$ is the set of solutions to (62) at μ. If $\mu_1 \neq \mu_2$, then $\partial_{\mu_i} W(\mu)$ is a singleton, in patricular, if $\mu_1 > \mu_2$, then $\partial_{\mu_1} W(\mu) = F_1(\ell^*, v_i^*) = (1-\beta)^{-1} u_1(c_1^*(s^1, \mu))$. At $\bar{\mu}$, we have

$$\partial_{\mu_i} W(\overline{\mu}) = \left\{(1-\beta)^{-1}\left[\bar{\pi}^* u_i(c^*(s^1, \overline{\mu})) + (1-\bar{\pi}^*) u_i(c^*(s^2, \overline{\mu}))\right] : \bar{\pi}^* \in [0,1]\right\},$$

which follows from the fact that any $\bar{\pi}^* \in [0,1]$ can be part of a one-period optimal lottery at $\bar{\mu}$.

Let $\{\mu_{t+1}^*, \ell_t^*, v_{t+1}^*\}$ be a sequence of recursive solutions to the saddle-point Bellman equation (62) starting at μ_0. The first-order condition with respect to v_i implies that $\mu_{i,t}^* = \mu_{i,0}$ for all $t \geq 1$ and i. Let $\phi_{i,t}^* \in \partial_{\mu_i} W(\mu_t^*)$ be the corresponding sequence of subgradients satisfying the first-order condition with respect to μ_i', that is $v_{i,t}^* = \phi_{i,t}^*$. The envelope selection condition is $\phi_{i,t}^* = F_i(\ell_t^*, v_{i,t+1}^*) = E_{\pi_t^*}[u_i(c_{i,t}^*(s)] + \beta v_{i,t+1}^*$, as in (50). It implies that

$$v_{i,t}^* = \sum_{\tau=t}^{\infty} \beta^{\tau-t} E_{\pi_\tau^*}[u_i(c_{i,\tau}^*(s, \mu_\tau^*))],$$

that is $v_{i,t}^* = U_{i,t}(L^*)$. Thus, it guarantees that $v_{i,t}^*$ is the continuation utility of the sequence of lotteries $L^* = \{\ell_t^*\}$. For $\mu_0 = \bar{\mu}$, where the subdifferential (63) is not a singleton, ignoring the envelope selection condition would lead to dynamically inconsistent sequence of continuation utilities.

In sum, there are two lessons from this example. First, that even if agents' preferences are time separable, when the planner uses lotteries to convexify the set of feasible values – effectively, lotteries over agents' continuation values – the recursive formulation of the infinite-horizon planner's problem is obtained by using its recursive preference representation on current and future values. Second, since lotteries precisely reflect indifferent choices for the planner which, in general, are not indifferent for the agents, the *envelope selection* condition plays a key role. It guarantees that the planner's allocations are intertemporally consistent for the agents, so that "promises are kept": the 'principle of optimality' continues to hold also for the agents.

3.3 Envelope selection in dynamic optimization

In this subsection we show that the inconsistency of the Euler equations on one side, and the Bellman equation and the envelope theorem on the other side may be present in the standard dynamic programming with binding *backward looking* constraints, if the value function is not differentiable. Backward looking constraints satisfy Koopmans' *future independence* i.e., the feasibility of an action at t is independent of actions taken at $t+1, \ldots$. We show that the inconsistency affects saddle-point multipliers but not solutions. The *envelope selection* condition disposes of the inconsistency. We provide an example which illustrates these insights. The general results can be found in Appendix F.

Example 5. *(A dynastic problem)*

Consider a dynasty where each generation leaves a bequest to the next – say a landholding. Preferences of generation t are represented by the utility function $u(x_t) + \alpha u(x_{t+1})$, where x_t is the received bequest and x_{t+1} the bequest given to the next generation. We assume that u is differentiable, increasing and concave, and $0 < \alpha$. Bequests must be non-negative and satisfy two constraints: a resource constraint, $x_t + x_{t+1} \leq 2y$, and a non-expansion constraint (as it may be the case with land) $x_{t+1} \leq x_t$. The allocation problem of the dynasty is to maximise the discounted utility of all future generations as follows

$$V(x_0) = \max_{\{x_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[u(x_t) + \alpha u(x_{t+1})\right] \tag{64}$$

$$\text{s.t. } x_t + x_{t+1} \leq 2y, \;\; x_{t+1} \leq x_t, \;\; x_t \geq 0, \;\; t \geq 0, \tag{65}$$

for given $x_0 \in [0, 2y]$. Note that problem (64) does *not* contain *forward-looking* constraints.

The value function V is

$$V(x) = \begin{cases} (1+\alpha)(1-\beta)^{-1}u(x) & \text{if } \; x \leq y \\ u(x) + (\alpha+\beta)(1-\beta)^{-1}u(2y-x) & \text{if } \; x \geq y, \end{cases} \tag{66}$$

and it is concave. Function V is not differentiable at y where the super-differential is

$$\partial V(y) = [1 - (\alpha + 2\beta), 1 + \alpha](1-\beta)^{-1}u'(y). \tag{67}$$

The Euler equations for a saddle-point of (64) are

$$\alpha u'(x_t) - \lambda^*_{1,t} - \lambda^*_{2,t} + \beta\big[u'(x_{t+1}) - \lambda^*_{1,t+1} + \lambda^*_{2,t+1}\big] = 0 \tag{68}$$

for $t \geq 1$, where $\lambda^*_{1,t}$ and $\lambda^*_{2,t}$ are the multipliers of the first and the second constraints in (65), respectively. The unique solution to (64) for $x_0 = y$ is the constant sequence $x^*_t = y$, where both constraints are binding and the slackness conditions are vacuous. Therefore the saddle-point multipliers are arbitrary positive solutions to difference equations (68) that satisfy the standard transversality condition.

The saddle-point Bellman equation is

$$V(x) = \text{SP} \min_{\lambda \geq 0} \max_{x' \geq 0} \big\{u(x) + \alpha u(x') + \lambda_1(2y - x' - x) + \lambda_2(x - x') + \beta V(x')\big\}. \tag{69}$$

The unique (saddle-point) solution to (69) at $x = y$ is $x'^* = y$. Starting from $x_0 = y$, the saddle-point Bellman equation generates the sequence of solutions $x^*_t = y$. The first-order condition at x^*_t is

$$\alpha u'(x^*_{t+1}) - \lambda^*_{1,t+1} - \lambda^*_{2,t+1} + \beta \phi^*_{t+1} = 0, \tag{70}$$

for some $\phi^*_{t+1} \in \partial V(x^*_{t+1})$. For $x^*_{t+1} = y$, condition (70) can be written more explicitly using (67) as

$$(1-\beta)^{-1} u'(y)[1 - \beta(\alpha + 2\beta)] \leq \lambda^*_{1,t+1} + \lambda^*_{2,t+1} \leq (1-\beta)^{-1} u'(y)[1 + 2\alpha - \alpha\beta], \tag{71}$$

and is the only restriction on multipliers generated by the saddle-point Bellman equation starting from $x_0 = 2$. The set of multipliers (71) is a superset of multipliers satisfying the Euler equation (68).

The envelope selection condition, see (109) in Appendix F, is

$$\phi^*_t = u'(x^*_t) - \lambda^*_{1,t+1} + \lambda^*_{2,t+1}. \tag{72}$$

If imposed, it restricts - together with (70) - the set of saddle-point multipliers to those satisfying the Euler equation (68). Thus the set of multipliers generated by the saddle-point Bellman equation and satisfying (72) is the same as the set of saddle-point multipliers of (64).

4 Conclusions

The main problem that we have addressed in this paper is *h*ow to make dynamically consistent selections in dynamic programming. In standard dynamic programming – say, a single-agent problem with a concave value function and *backward-looking* constraints – there may be multiple solutions even if the value function is not differentiable, but in this case the selection of an action from a policy correspondence is not a problem: constraints (not including future actions) are not affected by the selection and all the possible selections have the same optimal value. However, when multiplicity of solutions involve future values, or promises, the selection is, in fact, a commitment to follow the path where such values, or promises, should realise – i.e. the commitment must be recorded, which is what our 'envelope selection' does. Dynamic programming problems with forward-looking constraints, or promises, have a saddle-point structure and recursive solutions to the SP-Bellman equation that satisfy the envelope selection condition, and only those solutions, are *dynamically consistent*.

In sum, this paper makes three contributions to constrained optimization problems when the value function may be non-differentiable. First, it extends the envelope theorem to non-concave problems and provides novel characterizations of super- and sub-differentials of concave and convex value functions. Second, it uncovers time-inconsistency arising when the Bellman equation restarted at a later state results in solutions and/or multipliers for which the Euler equations fail to hold, and the continuation solution or multipliers are not part of the infinite-horizon optimization. In the presence of forward-looking constraints or recursive utilities, this time-inconsistency can turn into inconsistency of solutions, that is, non-optimal or infeasible outcomes. In the standard dynamic programming problems, the solutions to the saddle-point Bellman equation are time-consistent, but the multipliers may not be. The paper introduces an envelope selection condition, which is shown to restore the Euler equations for solutions and multipliers of the saddle-point Bellman equations. It also provides a recursive algorithm by extending the co-state of the value function to account for the envelope selection condition when the function may not be differentiable. The third contribution is the formulation of recursive utilities as recursive contracts which, in turn, broadens the scope of recursive contract theory.

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Appendix

A Proofs in Section 2

Lemma 1. Under assumptions A1-A5, the following hold:

(i) The set of saddle-points of (3) is a product of two sets $Y^*(x) \times \Lambda^*(x)$
where $Y^*(x) \subset Y$ and $\Lambda^*(x) \subset \Re^k_+$.

(ii) The sets $Y^*(x)$ and $\Lambda^*(x)$ are compact for every $x \in X$.

(ii) The correspondences Y^* and Λ^* are upper hemi-continuous on X.

Proof: It follows from Lemma 36.2 in Rockafellar (1970) that if the set of saddle-points is non-empty (assumption A5), then the set of saddle-points solutions is

$$Y^*(x) = \text{argmax}_{y \in Y} \min_{\lambda \in \Re^k_+} [f(x,y) + \lambda h(x,y)], \tag{73}$$

while the set of saddle-point multipliers is

$$\Lambda^*(x) = \text{argmin}_{\lambda \in \Re^k_+} \max_{y \in Y} [f(x,y) + \lambda h(x,y)], \tag{74}$$

Consequently, the set of saddle-points is the product $Y^*(x) \times \Lambda^*(x)$. This proves (i).

For parts (ii) and (iii), we first note that the function $\min_{\lambda \in \Re^k_+} [f(x,y) + \lambda h(x,y)]$ which is maximized over y in (73) equals $f(x,y)$ whenever $h(x,y) \geq 0$ and $-\infty$ if $h(x,y) \ngeq 0$. This implies that $Y^*(x)$ is the set of solutions to the constrained maximization problem (1–2). Since the constraint correspondence Γ is compact-valued and continuous, the Maximum Theorem implies that Y^* is compact-valued and upper hemi-continuous on X. Furthermore, the value function V is continuous on X.

To show the desired properties of $\Lambda^*(x)$, we proceed as follows: Let $\hat{y}_i$ be of the weak Slater's condition A4. The saddle-point property (3) implies that

$$f(x, \hat{y}_i) + \lambda^* h(x, \hat{y}_i) \leq V(x) \tag{75}$$

for every saddle-point multiplier λ^* at x. Using A4, we obtain

$$\lambda_i^* \leq \frac{V(x) - f(x, \hat{y}_i)}{h_i(x, \hat{y}_i)}. \tag{76}$$

We denote the RHS of (76) by $\bar{\lambda}_i(x)$ and note that it is a continuous function of x. Consider a sequence $\{x_n\}$ with $x_n \in X$ converging to $x \in X$. Let $\{\lambda_n^*\}$ be a sequence of saddle-point multipliers $\lambda_n^* \in \Lambda^*(x_n)$. Since $0 \leq \lambda_n^* \leq \bar{\lambda}(x_n)$ and $\bar{\lambda}(x_n)$ converges to $\bar{\lambda}(x)$, it follows that the sequence $\{\lambda_n^*\}$ has a convergent subsequence (for which we use the same notation) with the limit denoted by λ^*. Let $\{y_n^*\}$ be sequence of solutions with $y_n^* \in Y^*(x_n)$. Since correspondence Y^* is upper hemi-continuous, there exists a subsequence (retaining the same notation) converging to some $y^* \in Y^*(x)$. It is now easy to show taking the limits as x_n converges to x in saddle-point inequalities (3) that λ^* is a saddle-point multiplier at x, that is, $\lambda^* \in \Lambda^*(x)$. This shows that Λ^* is upper hemi-continuous and compact-valued on X. $\Box$

Proof of Theorem 1: We shall prove that equations (5) and (6) hold for arbitrary $x_0 \in intX$. Let $\Delta f(s, y)$ denote the difference quotient of function f with respect to x at x_0, that is

$$\Delta f(s, y) = \frac{f(x_0 + s, y) - f(x_0, y)}{s}$$

for $s \neq 0$. For $s = 0$, we set $\Delta f(0, y) = \frac{\partial f}{\partial x}(x_0, y)$. Assumptions of Theorem 1 imply that function $\Delta f(s, y)$ is continuous in (s, y) on $U_0 \times Y$ for some neighborhood U_0 of 0 in $\Re$.

Similar notation $\Delta h_i(s, y)$ is used for each function h_i, and $\Delta \mathcal{L}(s, y, \lambda)$ for the Lagrangian. Functions $\Delta h_i(s, y)$ are continuous in (s, y). Note that $\Delta \mathcal{L}(s, y, \lambda) = \Delta f(s, y) + \lambda \Delta h(s, y)$, where we use the scalar-product notation $\lambda \Delta h(s, y) = \sum_i \lambda_i \Delta h_i(s, y)$.

Let (y_s^*, λ_s^*) denote a saddle-point of Lagrangian $\mathcal{L}$ at $x_0 + s$, that is, $\lambda_s^* \in \Lambda^*(x_0 + s)$ and $y_s^* \in Y^*(x_0 + s)$, where the two sets are non-empty by A5. Saddle-point property (3) together with (4) imply that

$$V(x_0 + s) \geq \mathcal{L}(x_0 + s, y_0^*, \lambda_s^*) \tag{77}$$

and

$$V(x_0) \leq \mathcal{L}(x_0, y_0^*, \lambda_s^*). \tag{78}$$

Subtracting (78) from (77) and dividing the result on both sides by $s > 0$, we obtain

$$\frac{V(x_0 + s) - V(x_0)}{s} \geq \Delta \mathcal{L}(s, y_0^*, \lambda_s^*) = \Delta f(s, y_0^*) + \lambda_s^* \Delta h(s, y_0^*). \tag{79}$$

Since (79) holds for every $y_0^* \in Y^*(x_0)$, we can take the maximum on the right-hand side and obtain

$$\frac{V(x_0+s)-V(x_0)}{s} \geq \max_{y_0^* \in Y^*(x_0)} [\Delta f(s, y_0^*) + \lambda_s^* \Delta h(s, y_0^*)]. \tag{80}$$

Consider function Ψ defined as

$$\Psi(s, \lambda) = \max_{y_0^* \in Y^*(x_0)} [\Delta f(s, y_0^*) + \lambda \Delta h(s, y_0^*)] \tag{81}$$

so that the expression on the right-hand side of (80) is $\Psi(s, \lambda_s^*)$. Since $Y^*(x_0)$ is compact by Lemma 1 (ii), it follows from the Maximum Theorem that Ψ is a continuous function of (s, λ). Further, since $\lambda_s^* \in \Lambda^*(x_0+s)$ and Λ^* is an upper hemi-continuous correspondence by Lemma 1 (iii), we obtain

$$\liminf_{s \to 0+} \Psi(s, \lambda_s^*) \geq \min_{\lambda_0^* \in \Lambda^*(x_0)} \Psi(0, \lambda_0^*) = \min_{\lambda_0^* \in \Lambda^*(x_0)} \max_{y_0^* \in Y^*(x_0)} [\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)] \tag{82}$$

where we used the scalar-product notation $\lambda_0^* \frac{\partial h}{\partial x} = \sum_i \lambda_{i0}^* \frac{\partial h_i}{\partial x}$. It follows from (82) and (80) that

$$\liminf_{s \to 0+} \frac{V(x_0+s)-V(x_0)}{s} \geq \min_{\lambda_0^* \in \Lambda^*(x_0)} \max_{y_0^* \in Y^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)\big]. \tag{83}$$

Similar to (77) and (78), we have

$$V(x_0+s) \leq \mathcal{L}(x_0+s, y_s^*, \lambda_0^*) \tag{84}$$

and

$$V(x_0) \geq \mathcal{L}(x_0, y_s^*, \lambda_0^*), \tag{85}$$

for every $\lambda_s^* \in \Lambda^*(x_0+s)$ and $y_s^* \in Y^*(x_0+s)$. Subtracting side-by-side and dividing by $s > 0$, we obtain

$$\frac{V(x_0+s)-V(x_0)}{s} \leq \Delta f(s, y_s^*) + \lambda_0^* \Delta h(s, y_s^*). \tag{86}$$

Taking the minimum over $\lambda_0^* \in \Lambda^*(x_0)$ on the right-hand side of (86) results in

$$\frac{V(x_0+s)-V(x_0)}{s} \leq \min_{\lambda_0^* \in \Lambda^*(x_0)} [\Delta f(s, y_s^*) + \lambda_0^* \Delta h(s, y_s^*)]. \tag{87}$$

Consider function Φ defined as

$$\Phi(s,y) = \min_{\lambda_0^* \in \Lambda^*(x_0)} [\Delta f(s,y) + \lambda_0^* \Delta h(s,y)]$$

so that the expression on the right-hand side of (87) is $\Phi(s, y_s^*)$. It follows from the Maximum Theorem that Φ is a continuous function of (s, y). Using upper hemi-continuity of correspondence Y^* (see Lemma 1 (iii)), we obtain

$$\limsup_{s\to 0+} \Phi(s, y_s^*) \leq \max_{y_0^* \in Y^*(x_0)} \Phi(0, y_0^*) = \max_{y_0^* \in Y^*(x_0)} \min_{\lambda_0^* \in \Lambda^*(x_0)} [\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)]. \tag{88}$$

It follows now from (88) and (87) that

$$\limsup_{s\to 0+} \frac{V(x_0+s) - V(x_0)}{s} \leq \max_{y_0^* \in Y^*(x_0)} \min_{\lambda_0^* \in \Lambda^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)\big]. \tag{89}$$

It holds (see Lemma 36.1 in Rockafellar (1970)) that

$$\max_{y_0^* \in Y^*(x_0)} \min_{\lambda_0^* \in \Lambda^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)\big] \leq \min_{\lambda_0^* \in \Lambda^*(x_0)} \max_{y_0^* \in Y^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)\big]. \tag{90}$$

It follows from (83), (89) and (90) that the right-hand side derivative $V'(x_0+)$ exists and is given by

$$V'(x_0+) = \max_{y_0^* \in Y^*(x_0)} \min_{\lambda_0^* \in \Lambda^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)\big]$$

where the order of maximum and minimum does not matter. This establishes eq. (5) of Theorem 1. The proof of (6) is similar. □

Proof of Theorem 2: By Theorem 23.2 in Rockafellar (1970), $\phi \in \partial V(x_0)$ if and only if $V'(x_0; \hat{x}) \leq \hat{x}\phi$ for every $\hat{x}$ such that $x_0 + \hat{x} \in X$. Using (7), we obtain that $\phi \in \partial V(x_0)$ if and only if

$$\min_{\lambda_0^* \in \Lambda^*(x_0)} \big[D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\big]\, \hat{x} \leq \phi\hat{x} \quad \text{for every } \hat{x}, \tag{91}$$

for every $y_0^* \in Y^*(x_0)$, where we used the fact that inequality (91) holds for every y_0^* if and only if it holds for the maximum over y_0^*. The left-hand side of (91), as a function of $\hat{x}$, is the negative of the support function of the set

$$\bigcup_{\lambda_0^* \in \Lambda^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}. \tag{92}$$

Since $\Lambda^*(x_0)$ is convex and compact, the set (92) is compact and convex. Theorem 13.1 in Rockafellar (1970) implies that (91) is equivalent to

$$\phi \in \bigcup_{\lambda_0^* \in \Lambda^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}$$

for every $y_0^* \in Y^*(x_0)$. Consequently, $\phi \in \partial V(x_0)$ if and only if

$$\phi \in \bigcap_{y_0^* \in Y^*(x_0)} \bigcup_{\lambda_0^* \in \Lambda^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}.$$

Proof of Theorem 3: The proof is similar to that of Theorem 2. Using (7) and Theorem 23.2 in Rockafellar (1970), we obtain that $\phi \in \partial V(x_0)$ if and only if

$$\max_{y_0^* \in Y^*(x_0)} \left[D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\right] \hat{x} \geq \phi \hat{x} \quad \text{for every } \hat{x}, \tag{93}$$

for every $\lambda_0^* \in \Lambda^*(x_0)$. The left-hand side of (93) is the support function of the compact (but not necessarily convex) set

$$\bigcup_{y_0^* \in Y^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}.$$

Theorem 13.1 in Rockafellar (1970) implies that $\phi \in \partial V(x_0)$ if and only if

$$\phi \in \bigcap_{\lambda_0^* \in \Lambda^*(x_0)} co\left(\bigcup_{y_0^* \in Y^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}\right).$$

B Constrained Qualification condition for the envelope theorem

If f and h_i are differentiable in y, then the Kuhn-Tucker first-order conditions hold for a saddle-point of (3), and the set of saddle-point multipliers $\Lambda^*(x)$ is a subset of the set of Kuhn-Tucker multipliers. Those two sets of multipliers are equal if functions f and h_i are differentiable and concave in y (see Theorem 28.3 in Rockafellar (1970)).

The following Constrained Qualification condition is sufficient for uniqueness of the Kuhn-Tucker multiplier.

CQ (1) f and h_i are continuously differentiable functions of y,
(2) vectors $D_y h_i(x, y^*)$ for $i \in I(x, y^*)$ are linearly independent, where
$I(x, y^*) = \{i : h_i(x, y^*) = 0\}$ is the set of binding constraints.

A weaker form of condition CQ which is necessary and sufficient for uniqueness of Kuhn-Tucker multiplier can be found in Kyparisis (1985).

The CQ can be substituted for assumption A4. With CQ in place of A4, Lemma 1 of Appendix A continues to hold with weakened assumption A3, see Dubeau and Govin (1982).

C Envelope theorem for saddle-point problems

Consider the following parametric saddle-point problem with constraints

$$V(x) \equiv \text{SP} \max_{y \in Y} \min_{z \in Z} f(x, y, z) \tag{94}$$

$$\text{s.t. } h_i(x, y) \geq 0, \quad g_i(x, z) \leq 0, \quad i = 1, \ldots, k \tag{95}$$

where $Y \in \Re^n$, $Z \in \Re^l$ and x is the parameter in $X \subset \Re^m$. The Lagrangian function is $\mathcal{L}(x, y, z, \lambda, \gamma) = f(x, y, z) + \lambda h(x, y) + \gamma g(x, z)$, where $\lambda \in \Re^k_+$ and $\gamma \in \Re^k_+$ are vectors of multipliers. A saddle-point of $\mathcal{L}$ is vector $(y^*, z^*, \lambda^*, \gamma^*)$ where $\mathcal{L}$ is maximized with respect to $y \in Y$ and $\gamma \in \Re^k_+$, and minimized with respect to $z \in Z$ and $\lambda \in \Re^k_+$. If $(y^*, z^*, \lambda^*, \gamma^*)$ is a saddle-point, then (y^*, z^*) is a solution to (94–95). The set of saddle-points of $\mathcal{L}$ at x is a product of two sets $M^*(x)$ and $N^*(x)$ so that $(y^*, \gamma^*) \in M^*(x)$ and $(z^*, \lambda^*) \in N^*(x)$ for every saddle-point $(y^*, z^*, \lambda^*, \gamma^*)$, see Appendix A or Lemma 36.2 in Rockafellar (1970).

Consider first a single-dimensional parameter set $X \subset \Re$. We have

Theorem 4. *Suppose that Y and Z are convex, functions f, h_i and g_i are continuous, the constraint set given by (95) is a compact-valued and continuous correspondence of x, the weak Slater condition*[23] *holds, and the set of saddle-points is non-empty for every x. If partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial h}{\partial x}$ and $\frac{\partial g}{\partial x}$ are continuous*

[23] That is, for every x and every i there exist $\hat{y}_i \in Y$ such that $h_i(x, \hat{y}_i) > 0$ and $h_j(x, \hat{y}_i) \geq 0$ for $j \neq i$, and $\hat{z}_i \in Z$ such that $g_i(x, \hat{z}_i) < 0$ and $g_j(x, \hat{y}_i) \leq 0$ for $j \neq i$.

functions of (x, y, z), *then the directional derivatives of the value function* V *at* $x \in intX$ *are*

$$V'(x+) = \max_{(y^*,\gamma^*)\in M^*(x)} \min_{(z^*,\lambda^*)\in N^*(x)} \left[\frac{\partial f}{\partial x}(x, y^*, z^*) + \lambda^* \frac{\partial h}{\partial x}(x, y^*) + \gamma^* \frac{\partial g}{\partial x}(x, z^*)\right] \tag{96}$$

and

$$V'(x-) = \min_{(y^*,\gamma^*)\in M^*(x)} \max_{(z^*,\lambda^*)\in N^*(x)} \left[\frac{\partial f}{\partial x}(x, y^*, z^*) + \lambda^* \frac{\partial h}{\partial x}(x, y^*) + \gamma^* \frac{\partial g}{\partial x}(x, z^*)\right] \tag{97}$$

where the order of maximum and minimum does not matter.[24]

Milgrom and Segal (2002; Theorem 4) prove an envelope theorem for saddle-point problems without constraints and assuming that sets Y and Z are compact. As in Section 2, Theorem 4 can be used to derive sufficient conditions for differentiability of the value function in (94–95). Further, it can be used to derive directional derivatives of V for a multidimensional parameter set X.

In applications in Section 3, the following case with convex value function is particularly relevant. Suppose that the multi-dimensional parameter x can be decomposed in $x = (x^1, x^2)$, and the constraints (95) can be written as $h_i(x^1, y) \geq 0$ and $g_i(x^2, z) \leq 0$, for $i = 1, \ldots, k$. If function f is concave in x^1 and convex in x^2, while the functions h_i are concave in $x^1$and y and the functions g_i are convex in x^2 and z, then $V(x^1, x^2)$ is concave in x^1 and convex in x^2. The subdifferential of value function V with respect to x^2 can then be written in a similar way as in Theorem 3 as

$$\partial V_{x^2}(x) = \bigcap_{(z^*,\lambda^*)\in N^*} co \left\{ \bigcup_{(y^*,\gamma^*)\in M^*} \{D_{x^2} f(x, y^*, z^*) + \lambda^* D_{x^2} h(x, y^*) + \gamma^* D_{x^2} g(x, z^*)\} \right\}. \tag{98}$$

D Proof of Proposition 1

Proof: We show first that $\{\mu_t^*, \gamma_t^*\}$ minimizes the Lagrangian in (17) when $\{c_t^*, \lambda_t^*\}$ are fixed. Consider any sequence $\{\mu_t, \gamma_t\}$ such that $\{\mu_t\} \in \ell_\infty^m$ and $\mu_t \geq 0, \gamma_t \geq 0$ and $\mu_0 = \mu$. Let D_T be the difference between date-T partial sums of the Lagrangians for $\{\mu_t^*, \gamma_t^*\}$ and $\{\mu_t, \gamma_t\}$. We have

$$\begin{aligned} D_T = \sum_{t=0}^{T} \beta^t \Bigg\{ \sum_{i=1}^{m} \big[& \Delta\mu_{i,t+1} \left(u(c_{i,t}^*) - v_i(y_{i,t})\right) + \Delta\mu_{i,t} v_i(y_{i,t}) \\ & - \lambda_{i,t+1}^* (\Delta\mu_{i,t+1} - \Delta\mu_{i,t}) + (\gamma_{t+1}^* - \gamma_{t+1})(y_{i,t} - c_{i,t}^*) \big] \Bigg\}, \end{aligned}$$

[24]The proof of Theorem 4 is essentially the same as the proof of Theorem 1 in Appendix A.

where $\Delta\mu_t = \mu_t^* - \mu_t$. It follows from the Euler equation (18) and complementary slackness that

$$D_T = \beta^{T+1} \sum_{i=1}^{m} \left\{ -[\mu_{i,T+1}^* - \mu_{i,T+1}][v_i(y_{i,T+1}) + \lambda_{i,T+2}^*] - \gamma_{t+1}(y_{i,t} - c_{i,t}^*) \right\}. \tag{99}$$

Since $\gamma_{t+1} \geq 0$ and $c_{i,t}^* \leq y_{i,t}$, it follows from (99) that

$$D_T \leq -\beta^{T+1} \sum_{i=1}^{m} [\mu_{i,T+1}^* - \mu_{i,T+1}][v_i(y_{i,T+1}) + \lambda_{i,T+2}^*]. \tag{100}$$

Since $\mu_{i,T}$ and $\mu_{i,T}^*$ are bounded, the transversality condition (19) implies that the limit on the RHS of (100) is zero. Therefore $\lim_{T\to\infty} D_T \leq 0$.

Next we prove that $\{c_t^*, \lambda_t^*\}$ maximizes the Lagrangian in (17) when $\{\mu_t^*, \gamma_t^*\}$ are fixed. Let

$$\hat{D}_T = \sum_{t=0}^{T} \beta^t \left\{ \sum_{i=1}^{m} \left[\mu_{i,t+1}^*(u(c_{i,t}^*) - u(c_{i,t})) + \gamma_{t+1}^*(c_{i,t} - c_{i,t}^*) \right] \right\}.$$

Using concavity of u, we have

$$\hat{D}_T \geq \sum_{t=0}^{T} \beta^t \left\{ \sum_{i=1}^{m} \left[\mu_{i,t+1}^* u'(c_{i,t}^*)(c_{i,t}^* - c_{i,t}) + \gamma_{t+1}^*(c_{i,t} - c_{i,t}^*) \right] \right\}. \tag{101}$$

Using the first-order condition $\gamma_{t+1}^* = \mu_{i,t+1}^* u'(c_{i,t}^*)$, it follows from (101) that $\hat{D}_T \geq 0$ and consequently $\lim_{T\to\infty} \hat{D}_T \geq 0$. □

E On recursive contracts with recursive preferences

In this appendix, we show how recursive contract theory (MM19) can encompass recursive preferences. In doing this, we provide the guidelines for further generalizations to stochastic problems. Optimization problem (42) is a recursive contract problem with one-period forward-looking constraints (43), with the special feature that the constraints include next period value instead of next period action. Under R1 and R2, assumptions A2-A6 and A7s of (MM19) hold and A1 is vacuous with no uncertainty. With a minor reformulation,[25] Theorems 1 and 2 apply as well. The latter because it assumes that the value function is differentiable in μ and this is also key here, while it is not in (MM19) if there are only one-period forward

[25] It requires to add constraints (43) for date 0 with $v_{i,0} = 0$ which are not binding at a solution.

constraints. However, when the one-period constraints contain future values, solutions to the saddle-point Bellman equation (44) may fail, as we show, to be solutions to the planner's problem (42) in the absence of differentiability. Nevertheless, they are solutions to (42) if the *envelope selection* condition is satisfied, see Proposition 4. In other words, (MM19) Corollary to Theorem 2 extends to recursive preferences. An extension of (MM19) Theorem 3 establishing existence of the saddle-point value function needs a small clarification: (MM19) use Blackwells' sufficiency conditions of *monotonicity* and *discounting* (Stokey *et al.* (1989)Theorem 3.3) to prove the contraction property. For the optimization problem (42), the former follows from R2. The latter follows from R3, as the following simple lemma shows.

LEMMA 4.1. *If F_i satisfies R3, then $F_i(c, v+r) \leq F_i(c, v)+\beta r$ for every $c \in R_+, v \in R_+$, and $r \in R_+$.*

Proof. It follows from R3 that $F_i(c, v+r)-F_i(c, v) \leq |F_i(c, v+r)-F_i(c, v)| \leq \beta r$ for $c \in R_+, v \in R_+$, and $r \in R_+$. □

F The envelope selection condition in standard dynamic programming

Consider the following dynamic constrained maximization problem studied in Stokey et al. (1989):

$$\begin{aligned} V(x_0) \quad \equiv \quad & \max_{\{x_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ & \text{s.t. } \; h_i(x_t, x_{t+1}) \geq 0, \;\; i = 1, ..., k, \; t \geq 0, \end{aligned} \tag{102}$$

for given $x_0 \in X$, where $\{x_t\}_{t=1}^{\infty}$ is a bounded sequence (i.e., $\{x_t\} \in \ell_\infty^n$) such that $x_t \in X \subset \Re^n$ for every t. We impose the following conditions:

D1. X is convex, F is bounded, and $\beta \in (0, 1)$.

D2. F and h_i are concave and differentiable functions of (x, y) on $X \times X$.

D3. The constraint correspondence $\Gamma(x) = \{y \in X : h(x, y) \geq 0\}$ is uniformly compact near x for every $x \in X$.[26]

D4. For every $x \in X$ there exists $\hat{y} \in X$ such that $h_i(x, \hat{y}) > 0$ for every i.

The saddle-point problem associated with (102) is

$$\text{SP} \max_{\{x_t\}_{t=1}^{\infty}} \min_{\{\lambda_t\}_{t=1}^{\infty}, \lambda_t \geq 0} \sum_{t=0}^{\infty} \beta^t \big[F(x_t, x_{t+1}) + \lambda_{t+1} h(x_t, x_{t+1})\big], \tag{103}$$

[26] Γ is uniformly compact near x if there is a neighborhood N_x of x such that the set $\bigcup_{x' \in N_x} \Gamma(x')$ is compact. Together, D2-D4 imply that Γ is continuous.

where $\lambda_t \in \Re_+^k$ are Lagrange multipliers. It has the same value function $V(x_0)$. If $\{x_t^*\}$ is a solution to (102), then under D1-D2 and D4 there exists a summable sequence of multipliers $\{\lambda_t^*\}$ such that $\{x_t^*, \lambda_t^*\}$ is a saddle-point of (103). Conversely, if $\{x_t^*, \lambda_t^*\}$ is a saddle-point of (103), then $\{x_t^*\}$ is a solution to (102). The first-order *necessary conditions* for saddle-point $\{x_t^*, \lambda_t^*\}_{t=1}^{\infty}$ of (103) are the following *intertemporal Euler equations*:

$$D_y F(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* D_y h(x_t^*, x_{t+1}^*) + \beta \left[D_x F(x_{t+1}^*, x_{t+2}^*) + \lambda_{t+2}^* D_x h(x_{t+1}^*, x_{t+2}^*) \right] = 0 \tag{104}$$

for every $t \geq 0$, with $x_0^* = x_0$. Under D1-D2, the Euler equations together with complementary slackness conditions and a transversality condition, are sufficient for a saddle-point of (103) .

The value function V satisfies the standard Bellman equation

$$V(x) = \max_y \quad \{F(x,y) + \beta V(y)\} \qquad \text{s.t. } \; h_i(x,y) \geq 0, \;\; i = 1, ..., k \tag{105}$$

for every $x \in X$. Function V is concave and bounded under D1-D4. By Theorem 2, the superdifferential of V is

$$\partial V(x) = \bigcap_{y^* \in Y^*(x)} \bigcup_{\lambda^* \in \Lambda^*(x)} \{D_x F(x, y^*) + \lambda^* D_x h(x, y^*)\}. \tag{106}$$

If the saddle-point multiplier is unique, then V is differentiable. The *saddle-point Bellman equation* is

$$V(x) = \text{SP} \min_{\lambda \geq 0} \max_y \{F(x,y) + \lambda h(x,y) + \beta V(y)\}. \tag{107}$$

If (x_t^*, λ_t^*) and $(x_{t+1}^*, \lambda_{t+1}^*)$ are consecutive elements of a saddle-point sequence of (103), then $(x_{t+1}^*, \lambda_{t+1}^*)$ is a saddle-point of (107) at x_t^*, for every $t \geq 0$. As in Propositions 2 and 4, the converse implication requires that an *envelope selection condition* is satisfied. The first-order condition for saddle-point $(x_{t+1}^*, \lambda_{t+1}^*)$ at x_t^* states that there exists a subgradient vector $\phi_{t+1}^* \in \partial V(x_{t+1}^*)$ such that

$$D_y F(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* D_y h(x_t^*, x_{t+1}^*) + \beta \phi_{t+1}^* = 0. \tag{108}$$

The *envelope selection condition* is

$$\phi_t^* = D_x F(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* D_x h(x_t^*, x_{t+1}^*). \quad (109)$$

Equations (108) and (109) imply that the Euler equations (104) hold. Therefore $\{x_t^*, \lambda_t^*\}_{t=1}^{\infty}$ is a saddle-point of (103) and $\{x_t^*\}$ a solution to (102).

The *envelope selection condition* (109) guarantees *consistency* of multipliers generated by the saddle-point Bellman equation. It can be dispensed with if the saddle-point multiplier is unique (which is sufficient for value function V to be differentiable) but not if there are multiple multipliers. On the other hand, the solutions generated by the saddle-point Bellman equation are solutions to the dynamic optimization problem (102) regardless of whether the *envelope selection* condition is imposed or not. This is so because the principle of recursive optimality – through a sequence $\{x_t^*\}$ – holds in the standard dynamic programming.