

Course Handouts

ECON 8101
MICROECONOMIC THEORY

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PART I: Producer Theory

1. Production Set

Production set is a subset Y of commodity space \mathbb{R}^L , where L is the number of commodities. Vectors in Y represent **production plans** that are technologically feasible.

Negative coordinates of production plan $y = (y_1, \dots, y_L) \in Y$ are understood as input quantities; positive coordinates of y are output quantities.

Production plan $y \in Y$ is **efficient** if there is no alternative production plan $y' \in Y$, $y' \neq y$, such that $y' \geq y$.

Example (Activity analysis):

If two activities $a^1, a^2 \in \mathbb{R}^L$ can be combined together at arbitrary scale, then the production set is $Y = \{y \in \mathbb{R}^L : y = \lambda_1 a^1 + \lambda_2 a^2, \lambda_1 \geq 0, \lambda_2 \geq 0\}$.

Some **properties of production sets**:

- (i) Y closed; $0 \in Y$.
 - (ii) *no free production*: $Y \cap \mathbb{R}_+^L = \{0\}$.
 - (iii) *free disposal*: $Y - \mathbb{R}_+^L \subset Y$.
 - (iv) Y convex,
- Property (i) will be assumed throughout.

A convenient specification of a production set is in the form

$$Y = \{y \in \mathbb{R}^L : T(y) \leq 0\} \quad (1)$$

for some function $T : \mathbb{R}^L \rightarrow \mathbb{R}$, called **transformation function**. Typically, function T is increasing, continuous, and such that $T(0) = 0$. Such specification permits the use of marginal rates of transformation $\frac{\partial T}{\partial y_i} / \frac{\partial T}{\partial y_j}$.

Production function:

Often in applied work and in examples, production technology is specified by a **production function**. In the simple case of single output, production function is $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ that associates a quantity of single output with a vector of some n inputs. We write $f(x) = z$, where $x = (x_1, \dots, x_n)$ is a vector of inputs (here with positive sign!).

Examples: Cobb-Douglas, Leontief, CES, etc.

Some properties of production functions:

(i) $f(0) = 0$; f continuous (or differentiable) function.

(ii) f concave or quasi-concave.

Production function f gives rise to production set Y_f given by

$$Y_f = \{(x, z) \in \mathbb{R}^{n+1} : x \leq 0, 0 \leq z \leq f(-x)\}. \quad (2)$$

2. Returns to Scale in Production

Properties of returns to scale for production set are defined as follows:

constant – if $y \in Y$, then $\lambda y \in Y$ for every $\lambda \geq 0$,

nonincreasing – if $y \in Y$, then $\lambda y \in Y$ for every $0 \leq \lambda \leq 1$,

nondecreasing – if $y \in Y$, then $\lambda y \in Y$ for every $\lambda \geq 1$,

Actually, returns to scale can be more crisply defined for production function.

These definitions are

constant: $f(\lambda x) = \lambda f(x)$, for every $\lambda \geq 0$ and $x \geq 0$.

decreasing: $f(\lambda x) < \lambda f(x)$, for every $\lambda > 1$ and $x \geq 0$ such that $f(x) \neq 0$.

increasing: $f(\lambda x) > \lambda f(x)$, for every $\lambda > 1$ and $x \geq 0$ such that $f(x) \neq 0$.

One can show (Exercise) that constant, decreasing or increasing returns to scale for f imply that the production set Y_f of (2) exhibits constant, nonincreasing or nondecreasing returns to scale, respectively.

3. Profit Maximization

Profit maximization at price vector $p \in \mathbb{R}^L$ is

$$\text{maximize } py \quad \text{over } y \in Y. \quad (3)$$

The solutions (there could be many) are the **supply** of the firm at p , denoted by $s^*(p)$. We can write

$$s^*(p) = \{y^* \in Y : py^* \geq py, \forall y \in Y\}. \quad (4)$$

The (maximum) **profit** is

$$\pi^*(p) = \sup_{y \in Y} py. \quad (5)$$

π^* is a function of p while s^* is, in general, a correspondence.

If supply s^* is a differentiable function, then the $L \times L$ -matrix $Ds^*(p)$ is called **the supply substitution matrix**.

Profit function π^* is the **support function** of production set Y (see Math Appendix). Unless set Y is compact, there may exist positive price vectors for which maximum profit is infinite and supply does not exist. The set of price vectors for which profit function takes finite values is the *domain* of π^* . It is a convex set in \mathbb{R}^L . The domain of s^* is a subset of the domain of π^* .

4. Supply and Profit

Fundamental properties of the supply and the profit function of a profit-maximizing firm are:

Theorem 4.1: *Suppose that Y is closed. Then the following properties hold:*

- (i) π^* is homogeneous of deg. 1;
- (ii) π^* is a convex function;
- (iii) s^* is homogeneous of deg. 0;
- (iv) If π^* is differentiable at p (this holds iff s is single-valued at p), then
$$D\pi^*(p) = s^*(p).$$

Proof: (i) and (iii) are easy; (ii) do this one on your own! (iv) see Math Appendix.

Using (ii) and (iv) of Theorem 4.1, we obtain

Corollary 4.2: *If π^* is twice-differentiable, then*

$$D^2\pi^*(p) = Ds^*(p). \tag{7}$$

The substitution matrix $Ds^(p)$ is positive semi-definite and symmetric.*

Corollary 4.2 implies the following **comparative statics** property of supply:

$$\frac{\partial s_i^*}{\partial p_i} \geq 0. \tag{8}$$

Some **extra properties** of supply and profit of profit-maximizing firm:

Proposition 4.3:

- (i) *if Y exhibits constant returns to scale, then $\pi^*(p) = 0$ whenever it is well-defined.*
- (ii) *if Y is convex, then $s^*(p)$ is a convex set.*
- (iii) *if Y is compact, then π^* is a continuous function and s^* is an upper hemi-continuous correspondence.*

Proof: (i) and (ii) left as exercises, (iii) follows from the Maximum Theorem (in Math Appendix)

Supply and profit for production functions:

Profit maximization for production function is typically written as

$$\max_{x_1 \geq 0, \dots, x_n \geq 0} [qf(x_1, \dots, x_n) - \sum_{i=1}^n w_i x_i]$$

or, for short,

$$\max_{x \geq 0} [qf(x) - wx],$$

where q denotes the price of (single) output and $w = (w_1, \dots, w_n)$ input prices.

The solutions (again, there could be many) are the **input (or factor) demand** at prices (w, q) , denoted by

$$x^*(w, q).$$

The (maximum) **profit** is

$$\pi^*(w, q) = \sup_{x \geq 0} [qf(x) - wx].$$

There is also the profit-maximizing **output supply**

$$z^*(w, q) = f(x^*(w, q)).$$

If $q > 0$, then these solutions for f coincide with the supply and profit of the production set Y_f given by (2). More precisely, supply s^* of Y_f is $(-x^*, z^*)$ and the maximum profit π^* of f and of Y_f are the same function.

Hence, the properties of Theorem 4.1 and Corollary 4.2 apply to production function.

5. Profit-rationalizability

Consider a function $\pi : \mathbb{R}^L \rightarrow \mathbb{R}$ that assigns profit to each price vector p in \mathbb{R}^L . ($\pi(p)$ can take infinite value $+\infty$, but not for every p .) Call π a profit function, but it is not known whether or not π results from maximizing profit on some production set, that is, whether π is a maximum profit function. Production set Y **profit-rationalizes** function π if $\pi(p) = \max\{py : y \in Y\}$ for every p . Properties (i) and (ii) of Theorem 4.1 together with a continuity condition turn out to be sufficient for profit-rationalizability.

Theorem 5.1: *If π is (i) homogeneous of deg 1, (ii) convex, and (iii) lower semi-continuous, then there exists a closed and convex set Y that profit-rationalizes π .*

Proof – This result follows from Corollary 13.2.1 in Rockafellar’s “Convex Analysis,” see Math Appendix. The set Y that profit-rationalizes function π is

$$Y = \{y \in \mathbb{R}^L : py \leq \pi(p), \forall p\}.$$

Function $\pi : \mathbb{R}^L \rightarrow \mathbb{R}$ is *lower semi-continuous* at p , if

$$\pi(p) \leq \liminf_{n \rightarrow \infty} \pi(p_n).$$

for every sequence $\{p_n\}$ such that p_n converges to p and the limit of $\pi(p_n)$ exists (possibly equal to $+\infty$).

6. Weak Axiom of Profit Maximization

Suppose that we have several observations of prices and production plans of a firm. They are

$$\begin{aligned} y^1 & \text{ at } p^1, \\ \dots & \quad \dots, \\ y^T & \text{ at } p^T. \end{aligned}$$

If the firm maximizes its profit, it must hold that

$$p^t y^t \geq p^t y^s, \tag{9}$$

for all $s = 1, \dots, T$, for each t .

Property (9) is called the **Weak Axiom of Profit Maximization**.

Production set Y **profit-rationalizes** observations $(p^1, y^1), \dots, (p^T, y^T)$ if $y^t \in Y$ and $p^t y^t = \max\{p^t y : y \in Y\}$ for every t .

Proposition 6.1: *Observations $(p^1, y^1), \dots, (p^T, y^T)$ satisfy WAPM if and only if there exists a closed, convex production set Y that profit-rationalizes these observations.*

Write WPAM twice as

$$p^s y^s \geq p^s y^t$$

and

$$p^t y^t \geq p^t y^s,$$

and add side by side. We obtain

$$[p^t - p^s][y^t - y^s] \geq 0.$$

One can write this as

$$\Delta p \Delta y \geq 0 \tag{10}$$

Property (10) can be viewed as algebraic comparative statics of profit maximization. For instance, if all prices except for the price of good i are the same at two observations s and t , i.e., $p_j^t = p_j^s$ for $j \neq i$, and if good i is an output good, then (10) indicates that the output supply of a profit maximizing firm increases when the price of the output increases. The calculus comparative-statics expression for this is $\frac{\partial s_i^*}{\partial p_i} \geq 0$.

PART II: Consumer Theory

7. Preferences and Utility Functions

Consumption set is a subset $X \subset \mathcal{R}^L$. Vectors in X represent consumption bundles that the consumer considers possible for consumption.

Often, it is assumed that consumption set X is closed and convex, or more specifically that $X = \mathbb{R}_+^L$.

The consumer's preferences over commodity bundles in X are specified by a **preference relation** \succeq .

Properties that a preference relation may have:

- (i) reflexive, transitive and complete,
- (ii) continuous,
- (iii) nonsatiated, or locally nonsatiated,
- (iv) increasing, or strictly increasing (also called weakly monotone, or strongly monotone),
- (v) convex, or strictly convex.

Other special properties: homothetic, quasi-linear, etc.

Examples of preferences: lexicographic; Leontief; etc.

Function $u : X \rightarrow \mathcal{R}$ is a **utility representation** of \succeq if, for every $x, x' \in X$,

$$u(x) \geq u(x') \quad \text{if and only if } x \succeq x'. \quad (11)$$

Theorem 7.1: *If preference relation \succeq on X is complete, reflexive, transitive, and continuous, then it has a (continuous) utility representation.*

Proof: See Hildenbrand and Kirman (1976). An easy proof is available if two additional assumptions are imposed: $X = \mathcal{R}_+^L$, and \succeq strictly increasing.

This proof can be found in MWG and in Varian.

8. Utility Maximization

The problem of utility maximization for a price vector $p \in \mathcal{R}_+^L$ and an income $w > 0$ is written as

$$\text{maximize } u(x) \tag{12}$$

$$\text{subject to } px \leq w \quad \text{and } x \geq 0.$$

The solutions (there could be many) are denoted by $x^*(p, w)$ – the **demand** at prices p and income w . Demand correspondence $x^*(p, w)$ is often called **Walrasian** or **Marshallian**.

$u^*(p, w) \equiv u(x^*(p, w))$ is the **indirect utility function**.

9. Expenditure Minimization

The expenditure minimization problem for $p \in \mathcal{R}_+^L$ and utility level \bar{u} in the image of u is:

$$\text{minimize } px \tag{13}$$

$$\text{subject to } u(x) \geq \bar{u} \quad \text{and } x \geq 0.$$

The solutions are $h(p, \bar{u})$ – **Hicksian demand** correspondence, or function whenever single-valued.

Also $e(p, \bar{u}) \equiv ph(p, \bar{u})$ is the **expenditure function**.

The fundamental properties of Hicksian demand and expenditure function are:

Theorem 9.1: *Suppose that u is a continuous and locally non-satiated utility function on $X = \mathbb{R}_+^L$. Then, on the domain of strictly positive prices,*

- (i) e is homogeneous of deg. 1 in prices;
- (ii) e is a concave function of prices;
- (iii) h is homogeneous of deg. 0 in prices.
- (iv) If e is differentiable at (p, \bar{u}) and h is single-valued at (p, \bar{u}) , then

$$D_p e(p, \bar{u}) = h(p, \bar{u}). \quad (14)$$

Using (ii) and (iv) of Theorem 9.1, we obtain

Corollary 9.2: *If e is twice-differentiable with respect to prices, then $D_p^2 e(p, \bar{u}) = D_p h(p, \bar{u})$. The matrix $D_p h(p, \bar{u})$ is negative semi-definite and symmetric.*

Corollary 9.2 implies the following **comparative statics** property of Hicksian demand:

$$\frac{\partial h_i}{\partial p_i} \leq 0. \quad (15)$$

Remark: The matrix $D_p h(p, \bar{u})$ is singular. This is so because $p D_p h(p, \bar{u}) = 0$ as follows from (iii).

Digression on Cost Minimization.

The problem of cost minimization for a producer with production function is formally equivalent to expenditure minimization.

Using the setup as in Section 4 (page 7), the problem of minimizing cost is written as

$$\text{minimize } wx \tag{16}$$

$$\text{subject to } f(x) \geq z \text{ and } x \geq 0,$$

where $w = (w_1, \dots, w_n)$ is a vector of input prices (*not income!*).

Solution is $x^*(w, z)$ – **factor demand** correspondence, or function whenever single-valued. Also $C^*(w, z) \equiv wx^*(w, z)$ is the **cost function**.

Theorem 9.1 when applied to cost minimization (15) says that the cost function C^* is a concave and homogeneous of deg 1 function of input prices.

Further

$$D_w C^*(w, z) = x^*(w, z), \tag{17}$$

and

$$\frac{\partial x_i^*}{\partial w_i} \leq 0. \tag{18}$$

Results of Theorem 9.1 say nothing about how the cost function C^* depends on the output quantity z . This is clearly an interesting question in the context of cost minimization. We will have more to say about this in Part III.

10. Walrasian Demand and Hicksian Demand

Let $h(p, \bar{u})$ be the Hicksian demand and $x^*(p, w)$ be the Walrasian demand correspondences of utility function u on consumption set $X = \mathcal{R}_+^L$. Let $w > 0$, $\bar{u} > u(0)$ and $p \gg 0$.

Proposition 10.1: *If u is continuous and locally non-satiated, then*

$$h(p, \bar{u}) = x^*(p, e(p, \bar{u})), \quad (18)$$

and

$$x^*(p, w) = h(p, u^*(p, w)). \quad (19)$$

Proof (Outline): We first have the following

Lemma 10.2: (1) If u is locally non-satiated, then $px^*(p, w) = w$.

(2) If u is continuous, then $u(h(p, \bar{u})) = \bar{u}$.

Step 1: Next we prove the following two relations:

$$(i') \quad h(p, \bar{u}) \subset x^*(p, e(p, \bar{u}))$$

$$(ii') \quad x^*(p, w) \subset h(p, u^*(p, w))$$

Step 2: From (i') it follows that $u^*(p, e(p, \bar{u})) = \bar{u}$. From (ii') it follows that $e(p, u^*(p, w)) = w$.

Step 3: Since $u^*(p, e(p, \bar{u})) = \bar{u}$, relation reverse to (i') follows from (ii').

Similarly, relation reverse to (ii') follows from (i') and $e(p, u^*(p, w)) = w$.

11. The Slutsky Equation and the Slutsky Matrix

Suppose that (18) holds and h and x^* are single-valued and differentiable. It follows that

$$D_p h(p, \bar{u}) = D_p x^*(p, w) + D_w x^*(p, w) \cdot x^*(p, w) \quad (20)$$

where $w = e(p, \bar{u})$, or equivalently $\bar{u} = u^*(p, w)$

More specifically (and rearranging)

$$\frac{\partial x_l^*(p, w)}{\partial p_k} = \frac{\partial h_l(p, \bar{u})}{\partial p_k} - \frac{\partial x_l^*(p, w)}{\partial w} x_k^*(p, w), \quad (21)$$

where $\bar{u} = u^*(p, w)$.

Equation (21) is the **Slutsky equation**. It provides decomposition of the effect of change in price of good k on Walrasian demand for good l into the **pure substitution effect** and the **income effect**.

Define the $L \times L$ matrix $S = [s_{kl}]$ by

$$s_{kl} = \frac{\partial x_l^*(p, w)}{\partial p_k} + \frac{\partial x_l^*(p, w)}{\partial w} x_k^*(p, w). \quad (22)$$

Matrix S is the **Slutsky matrix** associated with Walrasian demand x^* . It follows from (20) that

$$S(p, w) = D_p h(p, \bar{u}),$$

for $\bar{u} = u^*(p, w)$. Corollary 9.1 implies that S is **negative semi-definite, symmetric and satisfies** $S(p, w)p = 0$. Negative semi-definiteness of Slutsky matrix S is the comparative statics of Walrasian demand.

12. Integrability

We found in Section 11 that Walrasian demand function $x^*(p, w)$ of a utility-maximizing consumer with l.n.s. utility function necessarily has the following three properties: (1) homogeneous of deg. 0, (2) negative semi-definite and symmetric Slutsky matrix, (3) budget equation $px^*(p, w) = w$.

Question: Are these *all* properties of Walrasian demand functions?

One way to answer this question is to verify whether, for every demand function d that satisfies (1-3), a utility function u can be found such that the Walrasian demand function of u is the function d .

The answer is **yes**.

Theorem 12.1: *Let $d : \mathcal{R}_{++}^L \times \mathcal{R}_+ \rightarrow \mathcal{R}_+^L$ be a C^1 demand function such that*

- (1) d is homogeneous of deg. 0,*
- (2) the Slutsky matrix associated with d is negative semi-definite and symmetric,*
- (3) $pd(p, w) = w$.*

Then there exists a strictly increasing, strictly quasi-concave utility function u such that d is the Walrasian demand of utility function u .

Proof: see MWG, Section 3.H.

13. Revealed Preference

Suppose that we have several observations of price vectors and consumption plans of a consumer. They are

$$\begin{array}{ccc} x^1 & \text{at} & p^1, \\ \dots & & \dots, \\ x^T & \text{at} & p^T. \end{array}$$

We assume that $x^t \in \mathcal{R}_+^L$ and $p^t \in \mathcal{R}_{++}^L$ for all t .

If the consumer maximizes locally non-satiated utility u (defined on \mathcal{R}_+^L) subject to the budget constraint, then these observations indicate the following:

- (1) her income in situation t is $p^t x^t$,
- (2) $u(x^t) \geq u(x)$ for every x such that $p^t x \leq p^t x^t$,
- (3) $u(x^t) > u(x)$ for every x such that $p^t x < p^t x^t$.

Note that local nonsatiation is crucial for (3). (2) and (3) imply that

$$\text{if } p^t x^s \leq p^t x^t, \quad \text{then } p^s x^t \geq p^s x^s \quad (23)$$

for all $s, t = 1, \dots, T$.

Property (23) is called the **Generalized Weak Axiom of Revealed Preference**.

Utility function u **rationalizes** observations $\{(p^1, x^1), \dots, (p^T, x^T)\}$ if, for every t , $u(x^t) \geq u(x)$ for every $x \in \mathcal{R}_+^L$ such that $p^t x \leq p^t x^t$.

We have just shown that GARP necessarily holds for a set of observations rationalized by locally nonsatiated utility function. Is GARP also a sufficient condition for rationalizability? The answer is **no**. To understand why, we take another look at what follows from utility maximization.

Let us define relations R and P between an observation x^t and a bundle $x \in \mathcal{R}_+^L$ as follows:

$$x^t R x, \quad \text{if } p^t x \leq p^t x^t, \quad (24)$$

$$x^t P x, \quad \text{if } p^t x < p^t x^t. \quad (25)$$

If $x^t R x$, we say that x^t is (directly) **weakly revealed preferred to** x .

If $x^t P x$, we say that x^t is (directly) **strictly revealed preferred to** x .

Again, if the consumer maximizes locally non-satiated utility u subject to the budget constraint, i.e., if the observations are rationalized by u , then

$$x^t R x \text{ implies } u(x^t) \geq u(x), \text{ and}$$

$$x^t P x \text{ implies } u(x^t) > u(x).$$

Consequently, if $x^t R x^s$ then not $x^s P x^t$. This is the GARP restated. But there is more. For every subset of observations $(p^{t_1}, x^{t_1}), \dots, (p^{t_n}, x^{t_n})$,

$$\text{if } x^{t_1} R x^{t_2}, x^{t_2} R x^{t_3}, \dots, x^{t_{n-1}} R x^{t_n}, \text{ then not } x^{t_n} P x^{t_1}. \quad (26)$$

Property (26) is called the Generalized Strong Axiom of Revealed Preference, or simply **Generalized Axiom of Revealed Preference, GARP**.

Theorem 13.1 (Afriat): *Observations $(p^1, x^1), \dots, (p^T, x^T)$ satisfy GARP if and only if there exists a locally nonsatiated utility function u that rationalizes these observations.*

Proof: See Varian, Ch. 8, also Varian (1982). The utility function u is defined as follows: First, it is proved that the system of inequalities

$$u^t \leq u^s + \lambda^s p^s (x^t - x^s), \quad \forall t, s.$$

has solution u^t, λ^t . Then, function u is defined by

$$u(x) = \min_t \{u^t + \lambda^t p^t (x^t - x)\}.$$

It holds $u(x^t) = u^t$. This function u is continuous, concave, and increasing.

GARP is a generalization of two standard “axioms.” The **Weak Axiom of Revealed Preference** is

$$\text{if } x^t R x^s \text{ and } x^t \neq x^s, \text{ then not } x^s R x^t. \quad (27)$$

The **Strong Axiom of Revealed Preference** is

$$\text{if } x^{t_1} R x^{t_2}, x^{t_2} R x^{t_3}, \dots, x^{t_{n-1}} R x^{t_n} \text{ and } x^{t_1} \neq x^{t_n}, \text{ then not } x^{t_n} R x^{t_1}. \quad (28)$$

These axioms hold for observations **strictly rationalized** by utility function u , i.e, if $u(x^t) > u(x)$ for every $x \in \mathcal{R}_+^L, x \neq x^t$, such that $p^t x \leq p^t x^t$, for every t . SARP is strictly stronger (as long as $L > 2$) than WARP.

Remarks:

- Axioms of revealed preference can be used to derive algebraic comparative statics of Walrasian demand, see Varian, Ch.8.
- Axioms of revealed preference can also be used in the context of a demand function, instead of finite demand observations. MWG prove in Section 3.J that the Strong Axiom of Revealed Preference holds for a demand function satisfying budget equation if and only if there exists a rational preference relation \succeq that rationalizes the demand function.
- There is a strong relationship between negative semi-definiteness of Slutsky matrix and axioms of revealed preference for a demand function. MWG prove in Section 2.F that if the WARP holds for a differentiable demand function d satisfying budget equation, then the Slutsky matrix associated with d is negative semi-definite.