

Course Handouts

ECON 8101
MICROECONOMIC THEORY

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PART I: Producer Theory and Convex Analysis

1. Production Set

Production set is a subset Y of commodity space \mathbb{R}^L , where L is the number of commodities. Vectors in Y represent **production plans** that are technologically feasible.

Negative coordinates of production plan $y = (y_1, \dots, y_L) \in Y$ are understood as input quantities; positive coordinates of y are output quantities.

Production plan $y \in Y$ is **efficient** if there is no alternative production plan $y' \in Y$, $y' \neq y$, such that $y' \geq y$.

Example (*Activity analysis*):

If two activities $a^1, a^2 \in \mathbb{R}^L$ can be combined together at arbitrary scale, then the production set is $Y = \{y \in \mathbb{R}^L : y = \lambda_1 a^1 + \lambda_2 a^2, \lambda_1 \geq 0, \lambda_2 \geq 0\}$.

Some **properties of production sets**:

- (i) Y closed; $0 \in Y$.
 - (ii) *no free production*: $Y \cap \mathbb{R}_+^L = \{0\}$.
 - (iii) *free disposal*: $Y - \mathbb{R}_+^L \subset Y$.
 - (iv) Y convex,
- Property (i) will be assumed throughout.

A convenient specification of a production set is in the form

$$Y = \{y \in \mathbb{R}^L : T(y) \leq 0\} \quad (1)$$

for some function $T : \mathbb{R}^L \rightarrow \mathbb{R}$, called **transformation function**. Typically, function T is increasing, continuous, and such that $T(0) = 0$. Such specification permits the use of marginal rates of transformation $\frac{\partial T}{\partial y_i} / \frac{\partial T}{\partial y_j}$.

Production function:

Often in applied work and in examples, production technology is specified by a **production function**. In the simple case of single output, production function is $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ that associates a quantity of single output with a vector of some n inputs. We write $f(x) = z$, where $x = (x_1, \dots, x_n)$ is a vector of inputs (here with positive sign!).

Examples: Cobb-Douglas, Leontief, CES, etc.

Some properties of production functions:

(i) $f(0) = 0$; f continuous (or differentiable) function.

(ii) f concave or quasi-concave.

Production function f gives rise to production set Y_f given by

$$Y_f = \{(x, z) \in \mathbb{R}^{n+1} : x \leq 0, 0 \leq z \leq f(-x)\}. \quad (2)$$

2. Returns to Scale in Production

Properties of returns to scale for production set are defined as follows:

constant – if $y \in Y$, then $\lambda y \in Y$ for every $\lambda \geq 0$,

nonincreasing – if $y \in Y$, then $\lambda y \in Y$ for every $0 \leq \lambda \leq 1$,

nondecreasing – if $y \in Y$, then $\lambda y \in Y$ for every $\lambda \geq 1$,

Actually, returns to scale can be more crisply defined for production function.

These definitions are

constant: $f(\lambda x) = \lambda f(x)$, for every $\lambda \geq 0$ and $x \geq 0$.

decreasing: $f(\lambda x) < \lambda f(x)$, for every $\lambda > 1$ and $x \geq 0$ such that $f(x) \neq 0$.

increasing: $f(\lambda x) > \lambda f(x)$, for every $\lambda > 1$ and $x \geq 0$ such that $f(x) \neq 0$.

One can show (Exercise) that constant, decreasing or increasing returns to scale for f imply that the production set Y_f of (2) exhibits constant, nonincreasing or nondecreasing returns to scale, respectively.

3. Profit Maximization

Profit maximization at price vector $p \in \mathbb{R}^L$ is

$$\text{maximize } py \quad \text{over } y \in Y. \quad (3)$$

The solutions (there could be many) are the **supply** of the firm at p , denoted by $s^*(p)$. We can write

$$s^*(p) = \{y^* \in Y : py^* \geq py, \forall y \in Y\}. \quad (4)$$

The (maximum) **profit** is

$$\pi^*(p) = \sup_{y \in Y} py. \quad (5)$$

π^* is a function of p while s^* is, in general, a correspondence.

If supply s^* is a differentiable function, then the $L \times L$ -matrix $Ds^*(p)$ is called **the supply substitution matrix**.

Unless set Y is compact, there may exist positive price vectors for which maximum profit is infinite and supply does not exist. The set of price vectors for which profit function takes finite values is the *domain* of π^* . It is a convex set in \mathbb{R}^L . The domain of s^* is a subset of the domain of π^* .

4. Convex Analysis and Duality

Profit function π^* is the **support function** of production set Y .

Extensive discussion of properties of support functions can be found in Rockafellar (1970), ch 13. Here we present the most useful results and definitions.

For a closed set $K \in \mathbb{R}^n$, the **support function** δ_K^* is defined by

$$\delta_K^*(p) = \sup_{x \in K} px. \quad (6)$$

for every $p \in \mathbb{R}^n$.

- Support function δ_K^* is a convex function.
- A **duality** property holds for a closed and convex set $K \in \mathbb{R}^n$:

$$K = \{x \in \mathbb{R}^n : px \leq \delta_K^*(p), \quad \forall p\} \quad (7)$$

See Corollary 13.2.1. in Rockafellar (1970).

Theorem 4.1 (Duality): *Suppose that K is nonempty and closed. There exists a unique $\bar{x} \in K$ such that $p\bar{x} = \delta_K^*(p)$ if and only if δ_K^* is differentiable at p . Moreover, in this case*

$$D\delta_K^*(p) = \bar{x}. \quad (8)$$

See Theorem 23.5 in Rockafellar (1970), or MWG, Section 3.F.

5. Supply and Profit

Fundamental properties of the supply and the profit function of a profit-maximizing firm are:

Theorem 5.1: *Suppose that Y is closed. Then the following properties hold:*

- (i) π^* is homogeneous of deg. 1;
- (ii) π^* is a convex function;
- (iii) s^* is homogeneous of deg. 0;
- (iv) If π^* is differentiable at p (this holds iff s is single-valued at p), then

$$D\pi^*(p) = s^*(p). \tag{9}$$

Proof: (i) and (iii) are easy;

(ii) and (iv) are properties of support functions, Section 4.

Using (ii) and (iv) of Theorem 5.1, we obtain

Corollary 5.2: *If π^* is twice-differentiable, then $D^2\pi^*(p) = Ds^*(p)$. The substitution matrix $Ds^*(p)$ is positive semi-definite and symmetric.*

Corollary 5.2 implies the following **comparative statics** property of supply:

$$\frac{\partial s_i^*}{\partial p_i} \geq 0. \tag{10}$$

Some **extra properties** of supply and profit of profit-maximizing firm:

Proposition 5.3:

- (i) *if Y exhibits constant returns to scale, then $\pi^*(p) = 0$ whenever it is well-defined.*
- (ii) *if Y is convex, then $s^*(p)$ is a convex set.*
- (iii) *if Y is compact, then π^* is a continuous function and s^* is an upper hemi-continuous correspondence.*

Proof: (i) and (ii) left as exercises, (iii) follows from the Maximum Theorem (in Math Appendix)

6. Profit-rationalizability

Consider a function $\pi : \mathbb{R}^L \rightarrow \mathbb{R}$ that assigns profit to each price vector p in \mathbb{R}^L . ($\pi(p)$ can take infinite value $+\infty$, but not for every p .) Call π a profit function, but it is not known whether or not π results from maximizing profit on some production set, that is, whether π is a maximum profit function. Production set Y **profit-rationalizes** function π if $\pi(p) = \max\{py : y \in Y\}$ for every p . Properties (i) and (ii) of Theorem 5.1 together with a continuity condition turn out to be sufficient for profit-rationalizability.

Theorem 6.1: *If π is (i) homogeneous of deg 1, (ii) convex, and (iii) lower semi-continuous, then there exists a closed and convex set Y that profit-rationalizes π .*

Proof: This follows from (7).

The set Y that profit-rationalizes function π is

$$Y = \{y \in \mathbb{R}^L : py \leq \pi(p), \forall p\}.$$

Function $\pi : \mathbb{R}^L \rightarrow \mathbb{R}$ is *lower semi-continuous* at p , if

$$\pi(p) \leq \liminf_{n \rightarrow \infty} \pi(p_n).$$

for every sequence $\{p_n\}$ such that p_n converges to p and the limit of $\pi(p_n)$ exists (possibly equal to $+\infty$).

7. Cost Minimization

The problem of minimizing cost for a producer with production function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is

$$\mathbb{R}_+^n \rightarrow \mathbb{R}_+ \text{ is}$$

$$\text{minimize } wx \tag{11}$$

$$\text{subject to } f(x) \geq z \text{ and } x \geq 0,$$

where $w = (w_1, \dots, w_n)$ is a vector of input prices.

Solution is $x^*(w, z)$ – **(conditional) factor demand** correspondence, or function whenever single-valued. Also $C^*(w, z) \equiv wx^*(w, z)$ is the **cost function**.

Cost minimization is a constrained optimization problem and could be analyzed using the Kuhn-Tucker approach. We can also use Section 4 since $C^*(\cdot, z)$ is the support function of the **input requirement set**

$$V(z) = \{x \in \mathbb{R}_+^n : f(x) \geq z\}. \tag{12}$$

The fundamental properties of factor demand and cost function are:

Theorem 7.1: *Suppose that f is continuous. Then the following properties hold:*

(i) C^* is homogeneous of degree 1 in factor prices w ,

(ii) C^* is a concave function of w ,

(iii) x^* is homogeneous of degree 0 in w ;

(iv) If C^* is differentiable at w (this holds iff x^* is single-valued) then

$$D_w C^* = x^*. \quad (13)$$

Proof: (ii) and (iv) are properties of support functions, Section 4.

Equation (13) is the **Shepard's Lemma**. The cost function, is **nondecreasing** in factor prices w .

Using (ii) and (iv) of Theorem 7.1, we obtain

Corollary 7.2: *If C^* is twice-differentiable with respect to prices, then $D_w^2 C^*(w, z) = D_w x^*(w, z)$. The matrix $D_w x^*$ is negative semi-definite and symmetric.*

Corollary 7.2 implies the following comparative statics property of factor demand:

$$\frac{\partial x_i^*}{\partial w_i} \leq 0 \quad (14)$$

Remark: The matrix $D_w x^*$ is singular. This is so because $D_w x^*(w, z)w = 0$ as follows from (iii) and the Euler's Theorem (see MWG, Appendix).

PART II: Consumer Theory

8. Preferences and Utility Functions

Consumption set is a subset $X \subset \mathcal{R}^L$. Vectors in X represent consumption bundles that the consumer considers possible for consumption.

Often, it is assumed that consumption set X is closed and convex, or more specifically that $X = \mathbb{R}_+^L$.

The consumer's preferences over commodity bundles in X are specified by a **preference relation** \succeq .

Properties that a preference relation may have:

- (i) reflexive, transitive and complete,
- (ii) continuous,
- (iii) nonsatiated, or locally nonsatiated,
- (iv) increasing, or strictly increasing (also called weakly monotone, or strongly monotone),
- (v) convex, or strictly convex.

Other special properties: homothetic, quasi-linear, etc.

Examples of preferences: lexicographic; Leontief; etc.

Function $u : X \rightarrow \mathcal{R}$ is a **utility representation** of \succeq if, for every $x, x' \in X$,

$$u(x) \geq u(x') \quad \text{if and only if } x \succeq x'. \quad (16)$$

Theorem 8.1: *If preference relation \succeq on X is complete, reflexive, transitive, and continuous, then it has a (continuous) utility representation.*

Proof: See Hildenbrand and Kirman (1976). An easy proof is available if two additional assumptions are imposed: $X = \mathcal{R}_+^L$, and \succeq strictly increasing. This proof can be found in MWG and in Varian.

9. Utility Maximization

The problem of utility maximization for a price vector $p \in \mathcal{R}_+^L$ and an income $w > 0$ is written as

$$\text{maximize } u(x) \quad (17)$$

$$\text{subject to } px \leq w \quad \text{and } x \geq 0.$$

The solutions (there could be many) are denoted by $x^*(p, w)$ – the **Walrasian demand** (or **Marshallian**) at prices p and income w .

$u^*(p, w) \equiv u(x^*(p, w))$ is the **indirect utility function**.

First-Order Conditions for maximization (17) obtain from Kuhn-Tucker Theorems (see Math Appendix).

10. Expenditure Minimization

The expenditure minimization problem for $p \in \mathcal{R}_+^L$ and utility level \bar{u} in the image of u is:

$$\text{minimize } px \tag{18}$$

$$\text{subject to } u(x) \geq \bar{u} \quad \text{and } x \geq 0.$$

The solutions are $h(p, \bar{u})$ – **Hicksian demand** correspondence, or function whenever single-valued. $e(p, \bar{u}) \equiv ph(p, \bar{u})$ is the **expenditure function**.

First-Order Conditions for (18) obtain from Kuhn-Tucker Theorems.

The problem of expenditure minimization is formally equivalent cost minimization for producer. The fundamental properties of Hicksian demand and expenditure function are the same as those of conditional factor demand and cost function in Section 7:

Theorem 10.1: *Suppose that u is a continuous utility function on $X = \mathbb{R}_+^L$.*

Then

- (i) e is homogeneous of deg. 1 in prices;*
- (ii) e is a concave function of prices;*
- (iii) h is homogeneous of deg. 0 in prices.*
- (iv) If e is differentiable at (p, \bar{u}) and h is single-valued at (p, \bar{u}) , then*

$$D_p e(p, \bar{u}) = h(p, \bar{u}). \tag{19}$$

Using (ii) and (iv) of Theorem 10.1, we obtain

Corollary 10.2: *If e is twice-differentiable with respect to prices, then $D_p^2 e(p, \bar{u}) = D_p h(p, \bar{u})$. The matrix $D_p h(p, \bar{u})$ is negative semi-definite and symmetric.*

Corollary 10.2 implies the following **comparative statics** of Hicksian demand:

$$\frac{\partial h_i}{\partial p_i} \leq 0. \tag{20}$$

The matrix $D_p h(p, \bar{u})$ is singular.

10. Walrasian Demand and Hicksian Demand

Let $h(p, \bar{u})$ be the Hicksian demand and $x^*(p, w)$ be the Walrasian demand correspondences of utility function u on consumption set $X = \mathcal{R}_+^L$. Let $w > 0$, $\bar{u} > u(0)$ and $p \gg 0$.

Proposition 10.1: *If u is continuous and locally non-satiated, then*

$$h(p, \bar{u}) = x^*(p, e(p, \bar{u})), \quad (21)$$

and

$$x^*(p, w) = h(p, u^*(p, w)). \quad (22)$$

Proof (Outline): We first have the following

Lemma 10.2: (1) If u is locally non-satiated, then $px^*(p, w) = w$.

(2) If u is continuous, then $u(h(p, \bar{u})) = \bar{u}$.

Step 1: Next we prove the following two relations:

$$(i') \quad h(p, \bar{u}) \subset x^*(p, e(p, \bar{u}))$$

$$(ii') \quad x^*(p, w) \subset h(p, u^*(p, w))$$

Step 2: From (i') it follows that $u^*(p, e(p, \bar{u})) = \bar{u}$. From (ii') it follows that $e(p, u^*(p, w)) = w$.

Step 3: Since $u^*(p, e(p, \bar{u})) = \bar{u}$, relation reverse to (i') follows from (ii').

Similarly, relation reverse to (ii') follows from (i') and $e(p, u^*(p, w)) = w$.

11. The Slutsky Equation and the Slutsky Matrix

Suppose that (21) holds and h and x^* are single-valued and differentiable. It follows that

$$D_p h(p, \bar{u}) = D_p x^*(p, w) + D_w x^*(p, w) \cdot x^*(p, w) \quad (23)$$

where $w = e(p, \bar{u})$, or equivalently $\bar{u} = u^*(p, w)$

More specifically (and rearranging)

$$\frac{\partial x_l^*(p, w)}{\partial p_k} = \frac{\partial h_l(p, \bar{u})}{\partial p_k} - \frac{\partial x_l^*(p, w)}{\partial w} x_k^*(p, w), \quad (24)$$

where $\bar{u} = u^*(p, w)$.

Equation (24) is the **Slutsky equation**. It provides decomposition of the effect of change in price of good k on Walrasian demand for good l into the **pure substitution effect** and the **income effect**.

Define the $L \times L$ matrix $S = [s_{kl}]$ by

$$s_{kl} = \frac{\partial x_k^*(p, w)}{\partial p_l} + \frac{\partial x_k^*(p, w)}{\partial w} x_l^*(p, w). \quad (25)$$

Matrix S is the **Slutsky matrix** associated with Walrasian demand x^* . It follows from (20) that

$$S(p, w) = D_p h(p, \bar{u}),$$

for $\bar{u} = u^*(p, w)$. Corollary 9.1 implies that S is **negative semi-definite, symmetric and satisfies** $S(p, w)p = 0$. Negative semi-definiteness of Slutsky matrix S is the comparative statics of Walrasian demand.

12. Integrability

We found in Section 11 that Walrasian demand function $x^*(p, w)$ of a utility-maximizing consumer with l.n.s. utility function necessarily has the following three properties: (1) homogeneous of deg. 0, (2) negative semi-definite and symmetric Slutsky matrix, (3) budget equation $px^*(p, w) = w$.

Question: Are these *all* properties of Walrasian demand functions?

One way to answer this question is to verify whether, for every demand function d that satisfies (1-3), a utility function u can be found such that the Walrasian demand function of u is the function d .

The answer is **yes**.

Theorem 12.1: *Let $d : \mathcal{R}_{++}^L \times \mathcal{R}_+ \rightarrow \mathcal{R}_+^L$ be a C^1 demand function such that*

- (1) d is homogeneous of deg. 0,*
- (2) the Slutsky matrix associated with d is negative semi-definite and symmetric,*
- (3) $pd(p, w) = w$.*

Then there exists a strictly increasing, strictly quasi-concave utility function u such that d is the Walrasian demand of utility function u .

Proof: see MWG, Section 3.H.

PART III: Algebraic Methods of Comparative Statics

13. Weak Axiom of Profit Maximization

Suppose that we have several observations of price vectors and production plans of a firm. They are

$$\begin{aligned} y^1 & \text{ at } p^1, \\ \dots & \quad \dots, \\ y^T & \text{ at } p^T. \end{aligned}$$

If the firm maximizes its profit at given prices, it follows that

$$p^t y^t \geq p^t y^s, \tag{26}$$

for all $s = 1, \dots, T$, for each t .

Property (26) is called the **Weak Axiom of Profit Maximization**.

Production set Y **profit-rationalizes** observations $(p^1, y^1), \dots, (p^T, y^T)$ if $y^t \in Y$ and $p^t y^t = \max\{p^t y : y \in Y\}$ for every t .

Proposition 13.1: *Observations $(p^1, y^1), \dots, (p^T, y^T)$ satisfy WAPM if and only if there exists a closed, convex production set Y that profit-rationalizes these observations.*

WAPM (26) can be considered as algebraic comparative statics property. It has the following simple implication:

$$[p^t - p^s][y^t - y^s] \geq 0, \quad (27).$$

(24) obtains by adding two inequalities $p^s y^s \geq p^s y^t$ and $p^t y^t \geq p^t y^s$ side by side.

Property (27), when it holds for supply function $y^t = s(p^t)$ and for arbitrary pair of vectors p^t, p^s , says that (vector-valued) function s is Δ -monotone (see Math Appendix II) for (27) can then be loosely written as $\Delta p \Delta s \geq 0$

If s is differentiable, a necessary and sufficient condition for s being Δ -monotone is that $Ds(p)$ is positive semi-definite for every p (see Math Appendix II).

14. Revealed Preference

Suppose that we have several observations of price vectors and consumption plans of a consumer. They are

$$\begin{array}{ccc} x^1 & \text{at} & p^1, \\ \dots & & \dots, \\ x^T & \text{at} & p^T, \end{array}$$

where $x^t \in \mathcal{R}_+^L$ and $p^t \in \mathcal{R}_{++}^L$ for all t .

Utility function u on \mathcal{R}_+^L **rationalizes** observations $\{(p^1, x^1), \dots, (p^T, x^T)\}$ if, for every t , $u(x^t) \geq u(x)$ for every $x \in \mathcal{R}_+^L$ such that $p^t x \leq p^t x^t$.

If observations $\{(p^1, x^1), \dots, (p^T, x^T)\}$ are rationalized by locally non-satiated utility function u , then the following must hold:

- (1) the consumer's income in situation t is $p^t x^t$,
- (2) $u(x^t) \geq u(x)$ for every x such that $p^t x \leq p^t x^t$,
- (3) $u(x^t) > u(x)$ for every x such that $p^t x < p^t x^t$.

Note that local nonsatiation is crucial for (3). (2) and (3) imply that

$$\text{if } p^t x^s \leq p^t x^t, \quad \text{then } p^s x^t \geq p^s x^s \tag{29}$$

for all $s, t = 1, \dots, T$.

Property (29) is called the **Generalized Weak Axiom of Revealed Preference**.

We have just shown that GARP necessarily holds for a set of observations rationalized by locally nonsatiated utility function. Is GARP also a sufficient condition for rationalizability? The answer is **no**. To understand why, we take another look at what follows from utility maximization.

Let us define relations R and P between an observation x^t and a bundle $x \in \mathcal{R}_+^L$ as follows:

$$x^t R x, \quad \text{if } p^t x \leq p^t x^t, \quad (30)$$

$$x^t P x, \quad \text{if } p^t x < p^t x^t. \quad (31)$$

If $x^t R x$, we say that x^t is (directly) **weakly revealed preferred to** x .

If $x^t P x$, we say that x^t is (directly) **strictly revealed preferred to** x .

Again, if the consumer maximizes locally non-satiated utility u subject to the budget constraint, i.e., if the observations are rationalized by u , then

$$x^t R x \text{ implies } u(x^t) \geq u(x), \text{ and}$$

$$x^t P x \text{ implies } u(x^t) > u(x).$$

Consequently, if $x^t R x^s$ then not $x^s P x^t$. This is the GARP restated. But there is more. For every subset of observations $(p^{t_1}, x^{t_1}), \dots, (p^{t_n}, x^{t_n})$,

$$\text{if } x^{t_1} R x^{t_2}, x^{t_2} R x^{t_3}, \dots, x^{t_{n-1}} R x^{t_n}, \text{ then not } x^{t_n} P x^{t_1}. \quad (32)$$

Property (32) is called the Generalized Strong Axiom of Revealed Preference, or simply **Generalized Axiom of Revealed Preference**, GARP.

Theorem 14.1 (Afriat): *Observations $(p^1, x^1), \dots, (p^T, x^T)$ satisfy GARP if and only if there exists a locally nonsatiated utility function u that rationalizes these observations.*

Proof: See Varian, Ch. 8, also Varian (1982). The utility function u is defined as follows: First, it is proved that the system of inequalities

$$u^t \leq u^s + \lambda^s p^s (x^t - x^s), \quad \forall t, s.$$

has solution u^t, λ^t . Then, function u is defined by

$$u(x) = \min_t \{u^t + \lambda^t p^t (x^t - x)\}.$$

It holds $u(x^t) = u^t$. This function u is continuous, concave, and increasing.

Remarks:

- GARP is a generalization of two standard “axioms.” The **Weak Axiom of Revealed Preference** is

$$\text{if } x^t R x^s \text{ and } x^t \neq x^s, \text{ then not } x^s R x^t. \quad (33)$$

The **Strong Axiom of Revealed Preference** is

$$\text{if } x^{t_1} R x^{t_2}, x^{t_2} R x^{t_3}, \dots, x^{t_{n-1}} R x^{t_n} \text{ and } x^{t_1} \neq x^{t_n}, \text{ then not } x^{t_n} R x^{t_1}. \quad (34)$$

These axioms hold for observations **strictly rationalized** by utility function u , i.e, if $u(x^t) > u(x)$ for every $x \in \mathcal{R}_+^L, x \neq x^t$, such that $p^t x \leq p^t x^t$, for every t . SARP is strictly stronger (as long as $L > 2$) than WARP.

- Axioms of revealed preference can be used in the context of demand function, too. There is a relationship between negative semi-definiteness of Slutsky matrix and the axioms of revealed preference. MWG prove in Section 2.F that if the WARP holds for a differentiable demand function d satisfying budget equation, then the Slutsky matrix associated with d is negative semi-definite.

Further, the Strong Axiom of Revealed Preference holds for a demand function satisfying budget equation if and only if there exists a rational preference relation \succeq that rationalizes the demand function (see Section 3.J).

- See Varian, Ch.8, for more on comparative statics of Walrasian demand implied by the axioms of revealed preference.

PART IV: Monotone Comparative Statics

15. The Theorem of Topkis

Monotone comparative statics is based on mathematical theories of supermodularity and vector lattices developed by D.M. Topkis and others (see the book by Topkis (1998)).

Lattice Operations

For two vectors $x, y \in \mathbb{R}^n$, we use $x \vee y$ to denote the **supremum** and $x \wedge y$ to denote the **infimum** of x and y . That is,

$$x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \dots, \max\{x_n, y_n\}),$$

$$x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \dots, \min\{x_n, y_n\}),$$

Operations \vee and \wedge are called **lattice operations**.

Note that

$$x + y = x \vee y + x \wedge y, \tag{35}$$

A set $X \subset \mathbb{R}^n$ is said to be a **lattice** if

$$x \wedge x' \in X \quad \text{and} \quad x \vee x' \in X, \tag{36}$$

for every $x, x' \in X$.

Examples: Interval $[a, b] \subset \mathbb{R}^n$ is a lattice; \mathbb{R}_+^n is a lattice.

Supermodular Functions

Let $X \subset \mathbb{R}^n$ be a lattice. A function $f : X \rightarrow \mathbb{R}$ is **supermodular** on X if

$$f(x \vee y) - f(x) \geq f(y) - f(x \wedge y), \quad (37)$$

for every $x, y \in X$.

Figure 1 illustrates definition (37) for $n = 2$. For a production function or a utility function f , supermodularity is a form of complementarity among goods.

Supermodularity can be characterized using second-order cross derivatives.

Proposition 15.1: *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be twice differentiable on an interval $(a, b) \subset \mathbb{R}^n$. Then f is supermodular on (a, b) if and only if*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \quad (38)$$

for every i, j , $i \neq j$ and every $x \in (a, b)$.

Nondecreasing Maximizers and the Theorem of Topkis.

Assume now that X is either the entire space \mathbb{R}^n , or the positive orthant \mathbb{R}_+^n .

Let T be a subset of \mathbb{R}^m .

For a function $f : X \times T \rightarrow \mathbb{R}$ and a set $S \subset X$, consider the following maximization problem

$$\max_x f(x, t) \tag{39}$$

subject to $x \in S$.

We denote the **set of solutions** by $\varphi^*(t)$. That is,

$$\varphi^*(t) = \operatorname{argmax}_{x \in S} f(x, t). \tag{40}$$

Monotone comparative statics is concerned with conditions on function f and set S so that correspondence φ^* is monotone nondecreasing in t .

Correspondence φ^* is **monotone nondecreasing** in t if

$$\varphi^*(t) \leq \varphi^*(t'), \tag{41}$$

for every $t \leq t'$. Inequality (41) between sets means the **strong set order**: for every $x \in \varphi^*(t)$ and $x' \in \varphi^*(t')$, it holds $x \wedge x' \in \varphi^*(t)$ and $x \vee x' \in \varphi^*(t')$.

If $\varphi^*(t)$ and $\varphi^*(t')$ are singleton sets, then the strong set order coincides with the usual order on vectors, so the inequality (41) is inequality between two vectors.

Theorem 15.2 (Topkis): *If S is a lattice, f is supermodular in x , and has nondecreasing differences in $(x; t)$, then φ^* is monotone nondecreasing in t .*

Function $f : X \times T \rightarrow \mathbb{R}$ has **nondecreasing differences in $(x; t)$** if the difference $f(x', t) - f(x, t)$ is monotone nondecreasing in t for every $x' \geq x$,

That is,

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t) \quad (42)$$

for every $x' \geq x$ and $t' \geq t$,

The condition of f having nondecreasing differences in $(x; t)$ can be characterized using second-order cross derivatives.

Proposition 15.3: *Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ be twice differentiable on an interval $(a, b) \subset \mathbb{R}^n \times \mathbb{R}^m$. Then f has nondecreasing differences in $(x; t)$ if and only if*

$$\frac{\partial^2 f}{\partial x_i \partial t_k}(x, t) \geq 0 \quad (43)$$

for every i, k and every $(x, t) \in (a, b)$

Proof of Theorem 15.2: Let $x \in \varphi^*(t)$ and $x' \in \varphi^*(t')$. First, we prove that $x \vee x' \in \varphi^*(t')$. Supermodularity in x implies that

$$f(x \vee x', t') \geq f(x', t') + f(x, t') - f(x \wedge x', t') \quad (44)$$

Nondecreasing differences (42) imply that

$$f(x, t') - f(x \wedge x', t') \geq f(x, t) - f(x \wedge x', t) \quad (45)$$

By the lattice property of S , we have $x \wedge x' \in S$. This and $x \in \varphi^*(t)$ imply

$$f(x, t) \geq f(x \wedge x', t). \quad (46)$$

Combining (44), (45) and (46) we obtain

$$f(x \vee x', t') \geq f(x', t') \quad (47)$$

Since $x \vee x' \in S$ and $x' \in \varphi^*(t')$, (47) implies that $x \vee x' \in \varphi^*(t')$.

The argument for $x \wedge x' \in \varphi^*(t)$ is similar:

$$\begin{aligned} f(x \wedge x', t) &\geq f(x, t) + f(x', t) - f(x \vee x', t) \geq \\ &\geq f(x, t) + f(x', t') - f(x \vee x', t') \geq f(x, t). \end{aligned}$$

Since $x \wedge x' \in S$, it follows that $x \wedge x' \in \varphi^*(t)$.

Nonincreasing Maximizers.

φ^* is **monotone nonincreasing** in t if $\varphi^*(t) \geq \varphi^*(t')$ for every $t \leq t'$.

A counterpart of Theorem 15.2 for monotone nonincreasing solutions to maximization problem (39) is

Theorem 15.4: *If S is a lattice, f is supermodular in x and has nonincreasing differences in $(x; t)$, then φ^* is monotone nonincreasing in t .*

Note that only monotonicity of differences gets reversed. The assumption of supermodularity remains unchanged.

Function $f : X \times T \rightarrow \mathbb{R}$ has **nonincreasing differences in $(x; t)$** if

$$f(x', t') - f(x, t') \leq f(x', t) - f(x, t), \quad (48)$$

for every $x' \geq x$ and $t' \geq t$. For twice differentiable function f , (48) is equivalent to

$$\frac{\partial^2 f}{\partial x_i \partial t_k}(x, t) \leq 0 \quad (49)$$

for every i, k and every (x, t) .

16. Examples of Monotone Comparative Statics

16.1 Profit Maximization

Consider the problem of profit maximization for a firm with production function f (see Section 2):

$$\max_{x \geq 0} [qf(x) - wx]$$

where q is the price of output and $w \in \mathbb{R}^n$ is a vector of prices of n inputs. Function f is assumed strictly increasing.

If f is nondecreasing, then the objective function $F(x, q) = qf(x) - wx$ has nondecreasing differences in $(x; q)$. If f is supermodular, then $F(x, q)$ is supermodular in x . Theorem 15.2 implies that input demand $x^*(q)$ is monotone nondecreasing in output price q .

16.2 Matching

There are m firms and n types of workers. The quality of a worker is described by a real number. For each type i , let $X_i \subset \mathbb{R}$ be the finite set of qualities of type- i workers.

Each firm j needs to hire one worker of every type. For a vector $x^j = (x_1^j, \dots, x_n^j)$ of qualities of a team of n workers, one from each type, the firm's profit is $f(x^j, j)$. The firm has profit function $f(\cdot, j) : \mathbb{R}^n \rightarrow \mathbb{R}$.

We first consider firm's j optimal hiring problem:

$$\max_x f(x, j) \tag{50}$$

$$\text{subject to } x^j \in \times_{i=1}^n X_i.$$

The set $\times_{i=1}^n X_i$ is a lattice. It follows from Theorem 15.2 that if $f(x, j)$ is supermodular in x and has nondecreasing differences in $(x; j)$, then optimal hiring decisions are nondecreasing, that is,

$$x^j \leq x^{j+1}.$$

Matching is an assignment of workers to firms (x^1, \dots, x^m) such that $\{x_i^1, \dots, x_i^m\}$ is a subset of X_i for every type i . A matching (x^1, \dots, x^m) is **nondecreasing** if $x^j \leq x^{j+1}$ for every $j = 1, \dots, m - 1$.

The **optimal matching problem** is stated as follows:

$$\max_{(x^1, \dots, x^m)} \sum_{j=1}^m f(x^j, j) \tag{51}$$

$$\text{subject to } \{x_i^1, \dots, x_i^m\} \subset X_i.$$

Topkis (1998) shows that if $f(x, j)$ is supermodular in x and has nondecreasing differences in $(x; j)$, then there exists a nondecreasing optimal matching.

16.3 Normal Demand for Supermodular Concave Utility

Theorem 16.3: *Suppose that utility function $u : \mathbb{R}_+^L \rightarrow \mathbb{R}$ is supermodular, strictly concave and locally nonsatiated. Then the Walrasian demand function x^* is a nondecreasing function of income, that is*

$$x^*(p, w') \geq x^*(p, w) \quad (51)$$

for every $w' \geq w > 0$ and every $p \gg 0$. In other words, the demand for every good is normal.

Theorem 16.3 does not follow from Theorem 15.2. A proof can be found in Appendix III.

A version of Theorem 16.3 was first proved by Professor John Chipman in 1977 under the assumption $\frac{\partial^2 u}{\partial x_i \partial x_j} \geq 0, \forall i \neq j$, instead of supermodularity.

References:

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PART V: Choice Under Uncertainty

17. Expected Utility under Uncertainty

Uncertainty is described by a set $S = \{1, \dots, S\}$ of **states of nature**. State-contingent consumption plan specifies consumption conditional on each state. We assume that there is a single commodity. Consumption plan is a vector $c = (c_1, \dots, c_S) \in \mathcal{R}_+^S$.

We consider a preference relation \succeq on the set \mathcal{R}_+^S of state-contingent consumption plans. Assume that \succeq is **strictly increasing and continuous**.

We say that \succeq has **state-separable utility representation** if there exist utility functions $v_s : \mathcal{R}_+ \rightarrow \mathcal{R}$ for all s , such that

$$c \succeq c' \text{ iff } \sum_{s=1}^S v_s(c_s) \geq \sum_{s=1}^S v_s(c'_s) \quad (52)$$

for every $c, c' \in \mathcal{R}_+^S$.

We say that \succeq has **expected utility representation** with respect to probabilities $\{\pi_s\}$ if there exists function $v : \mathcal{R}_+ \rightarrow \mathcal{R}$ such that

$$c \succeq c' \text{ iff } \sum_{s=1}^S \pi_s v(c_s) \geq \sum_{s=1}^S \pi_s v(c'_s), \quad (53)$$

for every $c, c' \in \mathcal{R}_+^S$.

Utility function v in the expected utility representation is the von Neumann-Morgenstern (or Bernoulli) utility. Expected utility is written as $E[v(c)]$.

Axiomatization of State-Separable Utility

For $c \in \mathcal{R}_+^S$ and $y \in \mathcal{R}_+$, let $c_{-s}y$ denote the consumption plan c with consumption c_s in state s replaced by y , that is, $(c_1, \dots, c_{s-1}, y, c_{s+1}, \dots, c_S)$.

The **independence axiom** (*sure-thing principle*):

$$c_{-s}y \succeq d_{-s}y \quad \text{iff} \quad c_{-s}w \succeq d_{-s}w \tag{54}$$

for all $c, d \in \mathcal{R}_+^S$ and $y, w \in \mathcal{R}_+$.

Theorem 17.1: *Assume that $S \geq 3$, and that preference relation \succeq is strictly increasing and continuous. Then \succeq has a state-separable utility representation iff it obeys the independence axiom.*

Proof: see Debreu (1959), “Topological methods in cardinal utility theory”.

Remark: This theorem does not hold for $S = 2$. With two states, the independence axiom is trivially satisfied by every strictly increasing preference relation regardless of whether it has a state-separable representation or not.

Axiomatization of Risk-Averse Expected Utility

For probabilities $\{\pi_s\}$ of states such that $\pi_s > 0$ for each s , let $E(c) = \sum_s \pi_s c_s$ be the expected value of $c = (c_1, \dots, c_S)$ and let $\mathbf{E}(c)$ denote the deterministic consumption plan $(E(c), \dots, E(c))$.

Preference relation \succeq is **risk averse** (with respect to $\{\pi_s\}$) if

$$\mathbf{E}(c) \succeq c \tag{55}$$

for every c . That is, if deterministic consumption plan equal to $E(c)$ is preferred to c . Expected utility $E[v(\cdot)]$ is risk averse if and only if v is concave. (This will be proved later.)

Theorem 17.2: *Assume that $S \geq 3$, and that \succeq is strictly increasing and continuous. Then \succeq satisfies the independence axiom and is risk averse with respect to probabilities $\{\pi_s\}$ if and only if it has a concave expected utility representation with respect to $\{\pi_s\}$.*

Proof: Theorem 17.1 implies that \succeq has a state-separable representation $\sum_s v_s(c_s)$. Suppose that each function v_s is differentiable.

For each $x \in \mathcal{R}$, consider the problem

$$\max_c \sum_s v_s(c_s) \tag{56}$$

subject to

$$E(c) = x.$$

By risk aversion, $c = (x, \dots, x)$ must be a solution to (56). FOCs for this solution are

$$v'_s(x) = \lambda \pi_s, \quad s = 1, \dots, S. \quad (57)$$

It follows from (57) that

$$v'_s(x) = \frac{\pi_s}{\pi_1} v'_1(x). \quad (58)$$

Equation (58) holds for every x . Therefore

$$v_s(x) = \frac{\pi_s}{\pi_1} v_1(x) + A_s$$

for some $A_s \in \mathcal{R}$. Consequently, $\sum_s \pi_s v(c_s)$ with $v \equiv v_1$ represents \succeq . Since \succeq is risk averse, v is concave (again, this will be proved later).

Proof without the extra assumption that functions v_s are differentiable can be found in Werner (2005).

Remarks: There are two possible interpretations of the probabilities in the above theorem. One is that probabilities $\{\pi_s\}$ are objectively given. The other is that they are consumer's subjective probabilities revealed by his or her choices. That is, when faced with a choice between state-contingent plan c and deterministic plan $\mathbf{E}(c)$, where $\mathbf{E}(c)$ has been calculated using $\{\pi_s\}$, he or she prefers $\mathbf{E}(c)$ for every c . Then, the theorem says, probabilities $\{\pi_s\}$ are her "subjective" probabilities.

Ellsberg paradox

An urn has 90 balls of which 30 are red and the rest are blue and yellow. Exact numbers of blue balls and yellow balls are not known. Consider bets of \$ 1 on a ball of a certain color (or colors) drawn from the urn. Denote bets by 1_R , 1_B , $1_{R\vee Y}$, etc. Typical preferences over bets are

$$1_R \succ 1_B, \quad 1_{B\vee Y} \succ 1_{R\vee Y}$$

This pattern of preferences is incompatible with expected utility: it cannot be that $\pi(R) > \pi(B)$ and $\pi(B\vee Y) > \pi(R\vee Y)$, because $\pi(B\vee Y) = \pi(B) + \pi(Y)$ holds for any probability measure π .

Multiple-Prior Expected Utility

An alternative to expected utility and one that can explain the Ellsberg paradox is the **multiple-prior expected utility**.

It takes the form

$$\min_{P \in \mathcal{P}} E_P[v(c)], \tag{57}$$

where $v : \mathcal{R}_+ \rightarrow \mathcal{R}$ is von Neumann-Morgenstern utility (with no date-0 consumption) and \mathcal{P} is a convex and closed set of probability measures on \mathcal{S} .

Set of probability measures (priors) \mathcal{P} reflects agent's **ambiguous beliefs**.

Examples of sets of priors:

- The set Δ of all probabilities on S . Then

$$\min_{P \in \Delta} E_P[v(c)] = \min_s v(c_s).$$

This is the maxmin utility of Hurwicz (1952).

- Bounds on probabilities:

$$\mathcal{P} = \{P \in \Delta : \lambda_s \leq P(s) \leq \gamma_s, \forall s\},$$

where $\lambda_s, \gamma_s \in [0, 1]$ are lower and upper bounds on probability of state s , respectively, and such that $\sum_s \lambda_s \leq 1$ and $\sum_s \gamma_s \geq 1$.

Expected Utility on Lotteries with Objective Probabilities.

Let Z be a (finite) set of **outcomes**, say $Z = \{z_1, \dots, z_K\}$. A **lottery** is a probability distribution on Z , that is, an assignment of probabilities $\{p_i\}_{i=1}^K$ to outcomes so that p_i is the probability of winning outcome z_i . Lottery with probabilities $\{p_i\}_{i=1}^K$ is denoted by L . Let \mathcal{L} be the set of all lotteries on Z . Since probabilities add up to one and are positive, the set \mathcal{L} can be identified with the unit simplex Δ in \mathcal{R}^K .

Preference relation \succeq on the set of lotteries \mathcal{L} has an **expected utility representation** if there exists function $v : Z \rightarrow \mathcal{R}$ such that

$$L \succeq L' \quad \text{if and only if} \quad \sum_{i=1}^K p_i v(z_i) \geq \sum_{i=1}^K p'_i v(z_i).$$

Axiomatization of expected utility on lotteries is due to **von Neumann and Morgenstern (1954)**.

See MWG, Chapter 6.

18. Risk Aversion and the Pratt's Theorem

A consumer with expected utility function $E[v(\cdot)]$ on \mathcal{R}_+^S is **risk averse** if

$$E[v(c)] \leq v(E(c)), \quad (59)$$

for every consumption plan $c \in \mathcal{R}_+^S$.

The consumer is **strictly risk averse** if

$$E[v(c)] < v(E(c)) \quad (60)$$

for every consumption plan $c \in \mathcal{R}_+^S$ such that $c \neq E(c)$.

The consumer is **risk neutral** if

$$E[v(c)] = v(E(c)) \quad (61)$$

for every $c \in \mathcal{R}_+^S$.

Measures of Risk Aversion

The **risk compensation** for additional state-contingent consumption plan $z \in \mathcal{R}^S$ with $E(z) = 0$ at deterministic “initial” consumption $x \in \mathcal{R}$ is $\rho(x, z)$ that solves

$$E[v(x + z)] = v(x - \rho(x, z)). \quad (62)$$

If v is twice-differentiable and strictly increasing (so that $v'(x) > 0$ for every x), we also have:

– the Arrow-Pratt measure of **absolute risk-aversion**

$$A(x) \equiv -\frac{v''(x)}{v'(x)}, \quad (63)$$

– the Arrow-Pratt measure of **relative risk aversion**

$$R(x) \equiv -\frac{v''(x)}{v'(x)}x. \quad (63)$$

The Theorem of Pratt

Let v_1, v_2 be two C^2 , strictly increasing vN-M. utility functions with ρ_1, ρ_2 , and A_1 and A_2 , respectively.

Theorem 18.1 (Pratt): *The following conditions are equivalent:*

- (i) $A_1(x) \geq A_2(x)$ for every $x \in \mathcal{R}$.
- (ii) $\rho_1(x, z) \geq \rho_2(x, z)$ for every $x \in \mathcal{R}$ and every $z \in \mathcal{R}^S$ with $E(z) = 0$.
- (iii) v_1 is a concave transformation of v_2 , i.e. $v_1(x) = f(v_2(x))$ for every x , for f concave and strictly increasing.

Risk Aversion and Concavity

Let v be twice-differentiable and strictly increasing.

Corollary 18.2:

(i) *A consumer is risk averse iff his von Neumann-Morgenstern utility function v is concave.*

(ii) *A consumer is risk neutral iff his von Neumann-Morgenstern utility function v is linear.*

(iii) *A consumer is strictly risk averse iff his von Neumann-Morgenstern utility function v is strictly concave.*

Note: “*iff*” means “if and only if.”

This corollary holds true even without the assumption of differentiability of v , see LeRoy and Werner(2001).

Decreasing, Constant and Increasing Risk Aversion

Corollary 18.3: Let v be C^2 and strictly increasing. Then

(i) $\rho(x, z)$ is increasing in x for every z with $E(z) = 0$, iff $A(x)$ is increasing in x .

(ii) $\rho(x, z)$ is constant in x for every z with $E(z) = 0$, iff $A(x)$ is constant in x .

(iii) $\rho(x, z)$ is decreasing in x for every z with $E(z) = 0$, iff $A(x)$ is decreasing in x .

Some Common Utility Functions

The functions most often used as von Neumann-Morgenstern utility functions in applied work and as examples are:

Linear utility:

$$v(x) = x$$

has zero absolute risk aversion, so the consumer is risk-neutral.

Negative Exponential Utility:

$$v(x) = -e^{-\alpha x},$$

where $\alpha > 0$, has constant absolute risk-aversion (CARA) equal to α .

Quadratic utility:

$$v(x) = -(\alpha - x)^2, \quad \text{for } x < \alpha,$$

has absolute risk aversion equal to $1/(\alpha - x)$.

Logarithmic utility:

$$v(x) = \ln(x + \alpha), \quad \text{for } x > -\alpha.$$

If $\alpha = 0$, then relative risk-aversion is constant (CRRA).

Power utility:

$$v(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \text{for } x \geq 0,$$

where $\gamma \geq 0, \gamma \neq 1$, has constant relative risk-aversion equal (CRRA) to γ .

Linear Risk Tolerance

The **risk tolerance**:

$$T(x) \equiv \frac{1}{A(x)}.$$

The negative exponential utility function, the quadratic utility function, the logarithmic utility function, the power utility function — all have linear risk tolerance (**LRT** or **HARA**).

Proof of Pratt's Theorem 18.1:

(i) implies (iii): Define f by $f(t) = v_1(v_2^{-1}(t))$ for every t . The first derivative of f is

$$f'(t) = \frac{v_1'(v_2^{-1}(t))}{v_2'(v_2^{-1}(t))} \quad (43)$$

and is strictly positive since $v_i' > 0$ for $i = 1, 2$. The second derivative is

$$f''(t) = \frac{v_1''(x) - (v_2''(x)v_1'(x))/v_2'(x)}{[v_2'(x)]^2}, \quad (44)$$

where we used $x = v_2^{-1}(t)$. Equation (44) can be rewritten as

$$f''(t) = (A_2(x) - A_1(x)) \frac{v_1'(x)}{[v_2'(x)]^2}.$$

Thus $f''(t) \leq 0$ for every t , and hence f is concave.

(iii) implies (ii): By the definition of ρ_1 (see (40))

$$E[v_1(x + z)] = v_1(x - \rho_1(x, z)). \quad (45)$$

Since $v_1 = f(v_2)$ and f is concave, Jensen's inequality yields

$$E[v_1(x + z)] = E[f(v_2(x + z))] \leq f(E[v_2(x + z)]). \quad (46)$$

The right-hand side of (46) equals $f(v_2(x - \rho_2(x, z)))$ or $v_1(x - \rho_2(x, z))$. Using (45) and (46) we obtain

$$v_1(x - \rho_1(x, z)) \leq v_1(x - \rho_2(x, z)). \quad (47)$$

Since v_1 is strictly increasing, (47) implies that $\rho_1(x, z) \geq \rho_2(x, z)$.

(ii) implies (i): (... in class)

19. Stochastic Dominance and Risk

For a consumer whose preferences over state-contingent consumption plans in \mathcal{R}_+^S have an expected utility representation, it is only the *probability distribution* of consumption that matters. That is, any two consumption plans that have the same probability distribution have the same expected utility. For instance, if there are two states with equal probabilities, then the expected utility of consumption plans (1, 2) and (2, 1) is the same.

Stochastic dominance is a ranking of random variables based on their distributions. Random variables, such as y and z , could be two state-contingent consumption plans on a finite set of states S equipped with probabilities $\{\pi_s\}$, or random variables with continuous distributions on an infinite probability space. All that matters are the cumulative distribution functions of y and z .

For simplicity, we assume that y and z take values in a bounded interval $[a, b]$.

Let F_z and F_y be their **cumulative distribution functions**. That is,

$$F_z(t) = \text{Prob}(z \leq t)$$

for $t \in [a, b]$.

The expected utility of z and the expected value of z can be written as

$$E(z) = \int_a^b t dF_z(t) \quad \text{and} \quad E[v(z)] = \int_a^b v(t) dF_z(t).$$

First-Order Stochastic Dominance

Definition 19.1: z *first-order stochastically dominates* y if

$$F_z(t) \leq F_y(t), \quad \forall t \in [a, b]. \quad (64)$$

We have

Theorem 19.2: z *first-order stochastically dominates* y if and only if

$$E[v(z)] \geq E[v(y)]$$

for every nondecreasing continuous v .

That is, z FSD y if and only if every expected-utility maximizing agent with nondecreasing utility prefers z to y .

Example 19.1: Let y take values 1 and 3 with probabilities $1/2$, and z take value 1 with probability $1/4$, value 3 with probability $1/4$, and value 4 with probability $1/2$. Then z FSD y .

Second-Order Stochastic Dominance and Risk

Definition 19.3: z second-order stochastically dominates y if

$$\int_a^w F_z(t)dt \leq \int_a^w F_y(t)dt, \quad \forall w \in [a, b]. \quad (65)$$

Since $E(z) = b - \int_a^b F_z(t)dt$, (65) for $w = b$ implies that $E(z) \geq E(y)$.

Thus, if z SSD y , then $E(z) \geq E(y)$. Also, if z FSD y , then z SSD y .

Theorem 19.4: z second-order stochastically dominates y if and only if

$$E[v(z)] \geq E[v(y)]$$

for every nondecreasing concave continuous v .

That is, z SSD y if and only if every agent with risk-averse nondecreasing expected utility prefers z to y .

If z SSD y and z and y have the same expectation $E(z) = E(y)$, then we say that y is **more risky**.

Proposition 19.5: If $E(z) = 0$, then $2z$ is more risky than z .

Proof: It suffices to prove that $E[v(z)] \geq E[v(2z)]$ for every nondecreasing concave v . Since $z = \frac{1}{2}(2z) + \frac{1}{2}(0)$, we have

$$\frac{1}{2}v(2z) + \frac{1}{2}v(0) \leq v(z). \quad (66)$$

Taking expectations on both sides of (66), we obtain

$$\frac{1}{2}E[v(2z)] + \frac{1}{2}v(0) \leq E[v(z)]. \quad (67)$$

Concavity of v and $E(z) = 0$ imply that

$$E[v(z)] \leq v(0). \quad (68)$$

Combining (66) with (67), we obtain $E[v(z)] \geq E[v(2z)]$.

Risk and Variance

For z and y with $E(z) = E(y)$, if y is more risky than z , then $\text{var}(y) \geq \text{var}(z)$.

[This follows from $E[v(z)] \geq E[v(y)]$ applied to the quadratic utility $v(x) = -(\alpha - x)^2$.] The converse is not true!

Example 19.2: Let z take on the values 1, 3, 4, 6 with equal probabilities, and let y take value 2 with probability 1/2 and values 3 and 7, each with probability 1/4. We have $E(z) = E(y) = 3.5$, and

$$\text{var}(y) = 4.25, \quad \text{var}(z) = 3.25.$$

Thus $\text{var}(y) > \text{var}(z)$. For the logarithmic utility $v(x) = \ln(x)$, we have

$$E[v(z)] = \frac{1}{4} \ln(72), \quad E[v(y)] = \frac{1}{4} \ln(84).$$

Thus, $E[v(z)] < E[v(y)]$. Since v is concave, it follows that y is not more risky than z . [In fact, neither y is more risky than z , nor z is more risky than y .]

Proof of Theorem 19.2 on First-Order Stochastic Dominance:

First, let $E[v(z)] \geq E[v(y)]$ for every nondecreasing continuous v . We want to show that $F_z(t) \leq F_y(t)$, $\forall t \in [a, b]$. Suppose, by contradiction, that $F_z(t_0) > F_y(t_0)$ for some $t_0 \in [a, b]$. Define the following utility function

$$v(t) = \begin{cases} 0, & \text{if } t \leq t_0 \\ 1, & \text{if } t > t_0 \end{cases}$$

We have

$$E[v(z)] - E[v(y)] = F_y(t_0) - F_z(t_0) < 0.$$

Function v is nondecreasing, but it is not continuous. However, it can be approximated by a nondecreasing continuous function so that the expression $E[v(z)] - E[v(y)]$ remains strictly negative. This is a contradiction.

Second, let $F_z(t) \leq F_y(t)$, $\forall t \in [a, b]$. We want to show that $E[v(z)] \geq E[v(y)]$ for every nondecreasing continuous v . Suppose first that v is differentiable. We use integration by parts:

$$\begin{aligned} E[v(z)] - E[v(y)] &= \int_a^b v(t) dF_z(t) - \int_a^b v(t) dF_y(t) = \\ &[v(b)F_z(b) - v(a)F_z(a)] - \int_a^b F_z(t)v'(t)dt - [v(b)F_y(b) - v(a)F_y(a)] \\ &\quad + \int_a^b F_y(t)v'(t)dt = \int_a^b (F_y(t) - F_z(t))v'(t)dt \geq 0. \end{aligned}$$

The same argument holds without differentiability: see Tesfatsion (1976).

Proof of Theorem 19.4 on Second-Order Stochastic Dominance:

First, let $E[v(z)] \geq E[v(y)]$ for every nondecreasing continuous and concave v . We want to show that $\int_a^w F_z(t)dt \leq \int_a^w F_y(t)dt$ for all $w \in [a, b]$. Suppose, by contradiction, that $\int_a^{w_0} F_z(t)dt > \int_a^{w_0} F_y(t)dt$ for some w_0 . Define the following utility function

$$v(t) = \begin{cases} t - w_0, & \text{if } t \leq w_0 \\ 0, & \text{if } t > w_0 \end{cases}$$

We have

$$\begin{aligned} E[v(z)] - E[v(y)] &= \int_a^{w_0} (t - w_0)dF_z(t) - \int_a^{w_0} (t - w_0)dF_y(t) = \\ &= - \int_a^{w_0} F_z(t)dt + \int_a^{w_0} F_y(t)dt < 0, \end{aligned}$$

where we used integration by parts. Function v is nondecreasing, continuous and concave. This is a contradiction.

Second, let $\int_a^w F_z(t)dt \leq \int_a^w F_y(t)dt$ for all $w \in [a, b]$. We want to show that $E[v(z)] \geq E[v(y)]$ for every nondecreasing continuous and concave v . Suppose first that v is twice-differentiable. We use the derivation from the proof of FSD and apply integration by parts one more time:

$$\begin{aligned} E[v(z)] - E[v(y)] &= \int_a^b (F_y(t) - F_z(t))v'(t)dt = \\ &= v'(b) \left[\int_a^b (F_y(t) - F_z(t))dt \right] - \int_a^b \left[\int_a^w (F_y(t) - F_z(t))dt \right] v''(w)dw \geq 0. \end{aligned}$$

The same argument holds without differentiability: see Tesfatsion (1976).