10. Dynamic Markets and Price Bubbles

Example 10.1

Time is infinite; so dates are $t = 0, 1, 2, \ldots,$. There is no uncertainty.

There is one security that pays zero dividend at every date.

Two agents ($i = 1, 2$) with the same utility function

$$u^i(c) = \sum_{t=0}^{\infty} \beta^t ln(c_t), \tag{1}$$

where $0 < \beta < 1$.

Agents maximize utility $u^i(c)$ subject to budget constraints

$$c_0 + p_0 h_0 = w_0^i + p_0 \alpha_0^i, \tag{2}$$

$$c_t + p_t h_t = w_t^i + p_t h_{t-1}, \quad for \text{ all } \ t \geq 1, \tag{3}$$

$$h_t \geq -1, \quad for \text{ all } \ t \tag{4}$$

Constraint (4) is a **short-sales constraint**.

Clearly, there is a trivial equilibrium with zero prices, $p_t = 0$ for all t, and no trade.

Claim: There can be an equilibrium with $p_t > 0$ for every t.

If $p_t > 0$, then the price exceeds the "fundamental" value and there is **price bubble**. The fundamental value of zero dividends must be zero.

Suppose that endowments are

$$w_0^1 = B + \eta, \qquad w_0^2 = A - \eta,$$

$$w_t^1 = B, \qquad w_t^2 = A \quad for\ t \text{ even},$$

$$w_t^1 = A, \qquad w_t^2 = B \quad for\ t \text{ odd},$$

where $\beta A > B > 0$.

Initial security holdings are $\alpha_0^1 = 1$ and $\alpha_0^2 = 0$. The total supply is 1.

Let

$$\eta = \frac{\beta A - B}{3(1+\beta)}. \tag{5}$$

There is an equilibrium with strictly positive price

$$p_t = \eta, \quad for \ \ \text{all}\, t \geq 0, \tag{6}$$

and

$$c_t^1 = (B + 3\eta) \ \ for \ \ t \text{ even}, \quad c_t^1 = (A - 3\eta) \ \ for \ \ t \text{ odd}$$

$$c_t^2 = (A - 3\eta) \ \ for \ \ t \text{ even}, \quad c_t^2 = (B + 3\eta) \ \ for \ \ t \text{ odd},$$

and

$$h_t^1 = -1 \ \ for\ t \text{ even}, \quad h_t^1 = 2 \ \ for\ t \text{ odd}$$

$$h_t^2 = 2 \ \ for\ t \text{ even}, \quad h_t^2 = -1 \ \ for\ t \text{ odd}$$

Verifying the equilibrium: (i) markets clear at every date, (ii) budget and short-sales constraints are all satisfied.

(iii) verifying that consumption plans and security holdings are optimal: At each odd t, agent's 1 first-order condition is

$$\frac{\beta^t}{c_t^1} p_t = \frac{\beta^{t+1}}{c_{t+1}^1} p_{t+1}, \tag{7}$$

It holds.

At each even date t, the first-order condition is

$$\frac{\beta^t}{c_t^1} p_t \geq \frac{\beta^{t+1}}{c_{t+1}^1} p_{t+1}, \tag{8}$$

since short-sales constraint is binding. It is satisfied. A suitable transversality condition (see Kocherlakota (1992), Proposition 2) can also be verified to hold. The same for agent 2.

- So there is equilibrium price bubble.

  This security is "fiat money."

General Model.

Time: $t = 0, 1, \ldots.$

Uncertainty: Infinite set of states S. Information about the state at date t is described by a finite partition F_t of S. F_{t+1} is finer than F_t (nondecreasing information). $F_0 = \{S\}$.

This is an infinite **event tree.**

$\xi_t \in F_t$ denotes an event at date t; $\xi_t^- \in F_{t-1}$ is the predecessor of ξ_t at date $t-1$, that is, $\xi_t \subset \xi_t^-$.

Securities are traded at each date - **infinitely-lived securities**. Security j pays dividend $x_j(\xi_t)$ at date-t event ξ_t for every $t \geq 1$.

Price of security j at date t in event ξ_t is $p_j(\xi_t)$. A portfolio in event ξ_t is $h(\xi_t)$; $h = \{h_t\}$ is a portfolio strategy.

Assumption: Dividends are positive, i.e., $x(\xi_t) \geq 0$ for every ξ_t.

Agents.

Consumption plans: $c(\xi_t)$ in event ξ_t at date t; c_t event-contingent consumption plan at date t; $c = (c_0, c_1, \ldots,)$.

Agent i's utility function $u^i : C^i \to \mathcal{R}$, where $C^i \subset \mathcal{R}_+^\infty$. u^i is assumed increasing. Endowment is w^i, and **initial portfolio** is $\alpha^i \in \mathcal{R}^J$.

Consumption-Portfolio Choice.

$$\max_{c,h} u(c) \tag{10}$$

subject to $c(\xi_0) + p(\xi_0)h(\xi_0) = w(\xi_0) + p(\xi_0)\alpha,$

$$c(\xi_t) = w(\xi_t) + (p(\xi_t) + x(\xi_t))h(\xi_t^-) - p(\xi_t)h(\xi_t), \qquad \forall \xi_t \quad \forall t \geq 1, \tag{11}$$

$$c \in C, \qquad h \in \mathcal{H} \tag{12}$$

$\mathcal{H}$ is the set of **feasible portfolio strategies**. The agent is restricted in her choices of portfolio strategies to those that are in $\mathcal{H}$.

Assumption: $\mathcal{H}$ is closed, convex, and $0 \in \mathcal{H}$.

If $\mathcal{H} = \mathcal{R}^\infty$, then there is no restriction. This would make no sense.

Observation: If u is strictly increasing and $p(\xi_t) \neq 0$ for every ξ_t, then there is no solution to the portfolio-consumption choice problem with $\mathcal{H} = \mathcal{R}^\infty$.

Portfolio Constraints.

Examples of constraints that may be used to define feasible portfolio set $\mathcal{H}$:

- **short sales constraint**

$$h_j(\xi_t) \geq -b_j(\xi_t), \qquad \forall \xi_t, \tag{13}$$

- **borrowing constraint**

$$p(\xi_t)h(\xi_t) \geq -B(\xi_t), \qquad \forall \xi_t, \tag{14}$$

for some sequence of positive bounds $\{B(\xi_t)\}$.

- **debt constraint**

$$[p(\xi_t) + x(\xi_t)]h(\xi_t^-) \geq -D(\xi_t), \qquad \forall \xi_t, \ , t \geq 1, \tag{15}$$

for some sequence of positive bounds $\{D(\xi_t)\}$.

There are other possible constraints such as transversality constraint, wealth constraint, solvency constraint, collateral constraint, etc.

First-Order Conditions.

First-order conditions for a solution (c, h) such that (i) $c(\xi_t) > 0$, $\forall \xi_t$, and (ii) portfolio constraint $\mathcal{H}$ is not binding at h, are

$$p_j(\xi_t) = \sum_{\xi_{t+1} \subset \ \xi_t} [p_j(\xi_{t+1}) + x_j(\xi_{t+1})] \frac{\partial_{\xi_{t+1}} u}{\partial_{\xi_t} u}, \qquad \forall j, \ \forall \xi_t \tag{16}$$

The price of security j in event ξ_t equals the sum over immediate successor events ξ_{t+1} of cum-dividend payoffs of security j multiplied by the marginal rate of substitution between consumption in event ξ_{t+1} and consumption in event ξ_t.

Equilibrium in Security Markets with Portfolio Constraints.

An equilibrium is an allocation $\{(c^i, h^i)\}$ and a price system p such that

(i) portfolio strategy h^i and consumption plan c^i are a solution to agent i's choice problem (10) subject to constraints (11), (12),

(ii) markets clear, that is

$$\sum_i h^i(\xi_t) = \bar{\alpha}, \qquad \forall \xi_t \tag{17a}$$

and

$$\sum_i c^i(\xi_t) = \bar{w}(\xi_t) + x(\xi_t)\bar{\alpha}, \quad \forall \xi_t. \tag{17b}$$

Arbitrage and Ponzi Schemes.

Let $z(h,p)$ denote the (net) **payoff** of portfolio strategy h in event ξ_t:

$$z(h,p)(\xi_t) = (p(\xi_t) + x(\xi_t))h(\xi_t^-) - p(\xi_t)h(\xi_t). \tag{18}$$

An **arbitrage** is a portfolio strategy $\hat{h}$ such that

$$z(\hat{h},p)(\xi_t) \geq 0, \quad \forall \xi_t \ \forall \ t \geq 1, \quad \text{and} \quad p_0\hat{h}_0 \leq 0, \tag{19}$$

with at least one strict inequality.

A **finite-time arbitrage** is an arbitrage portfolio strategy $\hat{h}$ such that $\hat{h}_t = 0$ for all $t \geq \tau$ for some τ. An **infinite-time arbitrage** is an arbitrage that is not a finite-time arbitrage.

A **Ponzi scheme** is an infinite-time arbitrage portfolio strategy $\hat{h}$ such that

$$z(\hat{h},p)(\xi_t) = 0, \quad \forall \xi_t \ \forall \ t \geq 1, \quad \text{and} \quad p_0\hat{h}_0 < 0. \tag{20}$$

This is a strategy of rolling over the debt forever.

Arbitrage $\hat{h}$ is **unlimited arbitrage for portfolio constraint** $\mathcal{H}$ if

$$h + \lambda\hat{h} \in \mathcal{H}, \quad \forall \lambda \geq 0, \quad \forall h \in \mathcal{H}. \tag{21}$$

Condition (21) says that adding strategy $\hat{h}$, or an arbitrary multiple thereof, to any portfolio position h does not violate constraint $\mathcal{H}$. Formally, (21) requires that $\hat{h}$ lies in the asymptotic (or recession) cone of $\mathcal{H}$. If $\mathcal{H}$ is a cone, then (21) holds if and only if $\hat{h} \in \mathcal{H}$

Theorem 10.1: *If agents' utility functions are strictly increasing, then, in equilibrium, there is no unlimited arbitrage for $\mathcal{H}$.*

Arbitrage and Event Prices.

- Let $\mathcal{H}$ be defined by the **borrowing constraint** (14).

Portfolio strategy $\hat{h}$ is an unlimited arbitrage under borrowing constraint iff $\hat{h}$ is an arbitrage and

$$p(\xi_t)\hat{h}(\xi_t) \geq 0, \quad \forall \xi_t. \tag{22}$$

Clearly, there is no unlimited Ponzi scheme under borrowing constraint at any p.

Theorem 10.2: *Assume that $p(\xi_t) > 0$ for every ξ_t. Then there is no unlimited arbitrage under borrowing constraint iff there exist strictly positive numbers $q(\xi_t)$ for all ξ_t such that*

$$q(\xi_t)p_j(\xi_t) = \sum_{\xi_{t+1} \subset \xi_t} q(\xi_{t+1})[p_j(\xi_{t+1}) + x_j(\xi_{t+1})] \quad \forall \xi_t \ \forall j. \tag{23}$$

Strictly positive numbers $q(\xi_t)$ satisfying (23) and normalized so that $q(\xi_0) = 1$ are called **event prices**.

• Let $\mathcal{H}$ be defined by the **debt constraint** (6).

Portfolio strategy $\hat{h}$ is an unlimited arbitrage under debt constraint iff $\hat{h}$ is an arbitrage and

$$[p(\xi_t) + x(\xi_t)]\hat{h}(\xi_t^-) \geq 0, \qquad \forall \xi_t, \ , t \geq 1 \tag{24}$$

Theorem 10.3: *There is no unlimited arbitrage under debt constraint iff there exist strictly positive event prices.*

Price Bubbles.

Assume that p is such that there exist event prices $q >> 0$ satisfying (23).

The **present value** at date 0 of security j under q is defined by

$$\sum_{t=1}^{\infty} \sum_{\xi_t \in F_t} q(\xi_t) x_j(\xi_t). \tag{25}$$

Similarly, we can define present value of security j at any event.

Price bubble is the difference between the price and the present value of a security. Price bubble at ξ_t is

$$\sigma_j(\xi_t) \equiv p_j(\xi_t) - \frac{1}{q(\xi_t)} \sum_{\tau > t} \sum_{\xi_\tau \in \xi_t} q(\xi_\tau) x_j(\xi_\tau) \tag{26}$$

Theorem 10.4: *Suppose that there exist strictly positive event prices. Then*

(i) If $p(\xi_t) \geq 0$ for all ξ_t, then

$$0 \leq \sigma_j(\xi_t) \leq p_j(\xi_t), \quad \forall \xi_t \ \forall j.$$

(ii) If security j is of finite maturity (that is, $x_{jt} = 0$ for $t \geq \tau$ for some τ, and that security is not traded after date τ), then $\sigma_j(\xi_t) = 0$ for all ξ_t.

(iii) It holds

$$q(\xi_t)\sigma_j(\xi_t) = \sum_{\xi_{t+1} \subset \xi_t} q(\xi_{t+1})\sigma_j(\xi_{t+1}) \quad \forall \xi_t \ \forall j.$$

Price Bubbles in Equilibrium.

The question is whether price bubbles can be nonzero in equilibrium in security markets.

We say that security markets are **complete** at prices p if the one-period payoff matrix $[p_j(\xi_{t+1}) + x_j(\xi_{t+1})]_{j,\xi_{t+1}}$ has the rank equal to the number of events ξ_{t+1} that are successors of ξ_t, for every ξ_t.

- Let $\mathcal{H}$ be defined by the **borrowing constraint** (14).

Theorem 10.5: *Let $p \geq 0$ be an equilibrium price system under the borrowing constraint. Suppose that security markets are complete at p and there exist strictly positive event prices q. If*

$$\sum_{t=1}^{\infty} \sum_{\xi_t \in F_t} q(\xi_t)\bar{w}(\xi_t) < \infty,$$

and

$$\bar{\alpha}_0 >> 0,$$

then price bubbles are zero.

See Santos and Woodford (1997) (also for incomplete markets), Huang and Werner (2000).

Pareto Optimal Equilibria in Security Markets.

A special case of the borrowing constraint is the **wealth constraint** with

$$B(\xi_t) = -\frac{1}{q(\xi_t)} \sum_{\tau>t} \sum_{\xi_\tau \subset \xi_t} q(\xi_\tau) w(\xi_\tau) \qquad \forall \xi_t, \tag{27}$$

where q is a sequence of event prices.

Theorem 10.6: *Let p and $\{c^i, h^i\}$ be a security market equilibrium under the wealth constraint. If security markets are complete at p and price bubbles are zero, then $\{c^i\}$ and P given by*

$$P(c) \equiv \sum_{\xi_t \in \mathcal{E}} q(\xi_t) c(\xi_t) \tag{28}$$

are an Arrow-Debreu equilibrium. Further, consumption allocation $\{c^i\}$ is Pareto optimal.

Example 10.2: Binomial event-tree with

$$\text{Prob}(up|\xi_t) = \text{Prob}(down|\xi_t) = \frac{1}{2}.$$

Two consumers with

$$u^i(c) = \sum_{t=0}^{\infty} \beta^t E[ln(c_t)]$$

$0 < \beta < 1.$

Two securities with dividends $x_t^1 \equiv 1,\ \forall t$, for security 1, and

$$x^2(\xi_t) = u, \quad \text{or} \quad = d$$

depending on whether $\xi_t = (\xi_{t-1}, up)$ or $\xi_t = (\xi_{t-1}, down)$ for security 2. Assume $u > d$. Initial portfolio holdings are $\alpha^1 = (0, 1)$ and $\alpha^2 = (0, 0)$.

Endowments are

$$w^1(\xi_t) = A - u, \quad w^2(\xi_t) = B,$$

whenever $\xi_t = (\xi_{t-1}, up)$, and

$$w^1(\xi_t) = B - d, \quad w^2(\xi_t) = A,$$

whenever $\xi_t = (\xi_{t-1}, down)$ for all $t \geq 1$, and $w_0^1 = w_0^2 = \frac{A+B}{2}$.

Note that

$$w_t^1 + w_t^2 + (\alpha^1 + \alpha^2)x_t = A + B.$$

This is a no-aggregate risk environment.

Portfolio constraint is the wealth constraint.

Security market equilibrium under the wealth constraint has security prices

$$p_1(\xi_t) = \sum_{\tau=1}^{\infty} \beta^\tau = \frac{\beta}{1-\beta}$$

$$p_2(\xi_t) = \frac{u+d}{2} \sum_{\tau=1}^{\infty} \beta^\tau = \frac{u+d}{2} \frac{\beta}{1-\beta},$$

and consumption allocation

$$c^1(\xi_t) = c^2(\xi_t) = \frac{A+B}{2}, \quad \forall \xi_t.$$

and some portfolio strategies h^1, h^2. Security markets are complete at p and price bubbles are zero. Event prices are $q(\xi_t) = \frac{1}{2^t}\beta^t$.

Equilibrium with bubble has security prices

$$p_1(\xi_t) = \sum_{\tau=1}^{\infty} \beta^\tau + \sigma(\xi_t) = \frac{\beta}{1-\beta} + \sigma(\xi_t),$$

$$p_2(\xi_t) = \frac{u+d}{2} \sum_{\tau=1}^{\infty} \beta^\tau = \frac{u+d}{2} \frac{\beta}{1-\beta},$$

where $\sigma(\xi_t)$ is chosen arbitrarily subject to

$$\sigma(\xi_t) = \beta[\frac{1}{2}\sigma(\xi_t, up) + \frac{1}{2}\sigma(\xi_t, down)],$$

and $\sigma(\xi_t) > 0, \ \forall \xi_t$. Equilibrium consumption allocation $\{c^1, c^2\}$ is as before. Portfolio strategies are different.

At these security prices markets are complete and there is price bubble on security 1 equal to σ. There is no price bubble on security 2.

Speculative Trade.

Example 3 (Harrison-Kreps (1978)

Uncertainty is described by a binomial event tree with the two successor events identified by high and low dividend on a security.

Single security with dividend x_t equal to 0 or 1 at every date $t \geq 1$. Markets are **incomplete.**

Short-sales constraint: $h_t \geq 0$.

The supply of the security is one, i.e., $\bar{\alpha}_0 = 1$. Initial holdings α_0^i are arbitrary. There are no consumption endowments.

Two agents, $i = 1, 2$, with **risk-neutral expected utility** functions

$$u^i(c) = E^i[\sum_{t=0}^{\infty} \gamma^t c_t], \tag{29}$$

where $0 < \gamma < 1$. Negative consumption is permitted. Expectation E^i is taken with respect to agent i's probability beliefs. Agents perceive dividend process $\{x_1, x_2, \ldots\}$ as a Markov chain with different transition matrices – **heterogeneous expectations** –

$$Q^1 = \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{matrix}$$

and

$$Q^2 = \begin{matrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{matrix}$$

Discount factor: $\gamma = 0.75$.

There exist a (recursive) equilibrium with security prices that depend only on current dividend. They are

$$p^*(0) = \frac{24}{13}, \qquad p^*(1) = \frac{27}{13}.$$

Equilibrium security holdings are

$$h^{*1}(0) = 1, \quad h^{*1}(1) = 0, \text{ and } h^{*2}(0) = 0, \quad h^{*2}(1) = 1.$$

Speculation

"We say that investors exhibit speculative behavior if the right to resell the stock makes them willing to pay more for it than they would if obliged to hold forever" Harrison and Kreps, pg 323.

Here, value of the security if obliged to hold it forever is

$$V^1(0) = \frac{4}{3}, \qquad V^1(1) = \frac{11}{9}.$$

$$V^2(0) = \frac{16}{11}, \qquad V^2(1) = \frac{21}{11}.$$

It holds

$$p^*(0) > V^i(0) \text{ and } p^*(1) > V^i(1), \quad \text{for } i = 1, 2.$$

It $x_t = 0$, then agent 1 speculates. He buys it ($h^{*1}(0) = 1$) so as to sell all his holdings the first time that dividend 1 occurs. If $x_t = 1$, agent 2 speculates.

11. Risk-Neutral Probabilities and the Martingale Property

Risk-Neutral Probabilities.

Let $\bar{r}(\xi_t)$ be the one-period risk-free return realized in event ξ_t. By definition, $\bar{r}(\xi_t)$ is the same for all ξ_t that have the same predecessor at $t-1$.

Define the **discount factor** in event ξ_t for $t \geq 1$ by

$$\rho(\xi_t) \equiv \prod_{\tau=1}^{t} [\bar{r}(\xi_\tau)]^{-1}, \tag{30}$$

and $\rho(\xi_0) = 1$.

Suppose that there exist event prices $q >> 0$. For every event ξ_t define

$$\pi^*(\xi_t) \equiv \frac{q(\xi_t)}{\rho(\xi_t)}. \tag{31}$$

Using (12) for the one-period risk-free security with return $\bar{r}(\xi_{t+1})$ we obtain

$$\pi^*(\xi_t) = \sum_{\xi_{t+1} \subset \xi_t} \pi^*(\xi_{t+1}). \tag{32}$$

Applying (16) repeatedly, we have

$$\sum_{\xi_t \subset F_t} \pi^*(\xi_t) = 1.$$

Thus π^* is a probability measure on $\mathcal{F}_t$ – the algebra of subsets of S generated by partition F_t – for every t. Consequently, π^* is a probability measure on the σ-algebra $\mathcal{F} = \cup_{t=0}^{\infty} \mathcal{F}_t$ of all events at all dates.

We call π^* the **risk-neutral probability**.

We can rewrite (12) using π^* as

$$p_{jt} = (\bar{r}_{t+1})^{-1} E_t^*[p_{j,t+1} + x_{j,t+1}] \tag{33}$$

for all j and t.

Date-t price of security j equals the conditional expectation of its one-period payoff discounted by the one-period risk-free return, where the expectation is taken with respect to risk-neutral probabilities.

Martingale in Risk-Neutral Probabilities

The **discounted gain** $d_j(\xi_t)$ on security j in event ξ_t is

$$d_j(\xi_t) = \rho(\xi_t)p_j(\xi_t) + \sum_{\tau=1}^{t} \rho(\xi_\tau)x_j(\xi_\tau), \tag{34}$$

with $d_{j0} = p_{j0}$.

Theorem 11.1: *The discounted gain on any security is a martingale under the risk-neutral probabilities, that is*

$$E_t^*(d_{j\tau}) = d_{jt} \tag{35}$$

for every $\tau \geq t$ and every j.