Mathematical Appendix I.

Kuhn-Tucker Theorems

I.1 Constrained Maximization: Necessary Conditions.

Function $F : \mathbb{R}^n_+ \to \mathbb{R}$ is the objective function; functions $g^j : \mathbb{R}^n_+ \to \mathbb{R}$, for $j = 1, \ldots, k$, are constraint functions. Assume that F and g^j are differentiable, with partial derivatives $\frac{\partial F}{\partial x_i}$ and $\frac{\partial g^j}{\partial x_i}$ denoted by $\partial_i F$ and $\partial_i g^j$, respectively.

The constrained maximization problem (with nonnegativity constraints) is

$$\max_x F(x) \tag{1}$$

$$\text{subject to} \qquad g^1(x) \geq 0,$$

$$\ldots\ldots,$$

$$g^k(x) \geq 0,$$

$$x_1 \geq 0, \ldots, x_n \geq 0.$$

We write the Lagrangian as

$$\mathcal{L}(\lambda^1, \ldots, \lambda^k, x) = F(x) + \sum_{j=1}^{k} \lambda_j g^j(x),$$

where $\lambda_j \geq 0$, for $j = 1, \ldots, k$, are the Lagrange multipliers. We use λ to denote the k-vector of multipliers.

Kuhn-Tucker conditions for $x^* \geq 0$ and $\lambda^* \geq 0$ are:

$$\text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, k,$$

$$\partial_i F(x^*) + \sum_{j=1}^{k} \lambda_j^* \partial_i g^j(x^*) \leq 0, \quad \text{and if } x_i^* > 0, \text{ then “} = 0\text{”}, \tag{2a}$$

$$g^j(x^*) \geq 0, \quad \text{and if } \lambda_j^* > 0, \text{ then “} = 0\text{”}. \tag{2b}$$

Where do these conditions come from? Think about *maximizing* Lagrangian $\mathcal{L}(\lambda, x)$ with respect to x and *minimizing* it with respect to λ, unconstrained, except for $x \geq 0$ and $\lambda \geq 0$. This is the saddle-point. K-T conditions (2) are FOCs for such max-min (or saddle-point) problem.

Theorem (Kuhn-Tucker): *If $x^* \geq 0$ is a solution to the constrained maximization problem, and the Constraint Qualification Condition holds, then x^* and some $\lambda^* \geq 0$ satisfy K-T conditions (2).*

Constraint Qualification Condition:

(i) Kuhn-Tucker original – don't touch it.

(ii) g^j concave for all j, and **Slater's condition**, that is, there is some $x^0 \geq 0$ with $g^j(x^0) > 0$ for all j.

(iii) rank condition (see Takayama 1.D.4, or Varian, ch 27),

(iv) g^j linear for all j, (Arrow-Hurwicz-Uzawa, see Takayama 1.D.4)

I.2 Sufficiency of Kuhn-Tucker Conditions.

The most standard theorem is:

Theorem S1: *Suppose that F and $g^1, \ldots, g^k$ are all concave functions. If $x^* \geq 0$ and $\lambda^* \geq 0$ satisfy K-T conditions (2), then x^* is a solution to the constrained maximization problem.*

A better theorem is due to Arrow and Enthoven (1961).

Theorem S2: *Suppose that F and $g^1, \ldots, g^k$ are all quasi-concave functions and some "mild" condition holds. If $x^* \geq 0$ and $\lambda^* \geq 0$ satisfy K-T conditions (2), then x^* is a solution to the constrained maximization problem.*

The extra ("mild") condition is not needed if F is concave (and $g^1, \ldots, g^k$ are quasi-concave). See Takayama 1.E for three versions of the condition.

Quasi-concavity (and therefore also concavity) of functions g^j implies that the constraint set, i.e. the set of $x \geq 0$ satisfying $g^1(x) \geq 0, \ldots, g^k(x) \geq 0$, is convex.

I.3 Constrained Minimization

The constrained minimization problem (with nonnegativity constraints) is

$$\min_x F(x) \tag{3}$$

$$\text{subject to} \qquad g^1(x) \le 0, \ \ldots, \ \ g^k(x) \le 0,$$

$$x_1 \ge 0, \ldots, x_n \ge 0.$$

The Lagrangian is

$$\mathcal{L}(\lambda, x) = F(x) + \sum_{j=1}^{k} \lambda_j g^j(x).$$

Kuhn-Tucker conditions for $x^* \ge 0$ and $\lambda^* \ge 0$ are,

for all $i = 1, \ldots, n$ and $j = 1, \ldots, k$,

$$\partial_i F(x^*) + \sum_{j=1}^{k} \lambda_j^* \partial_i g^j(x^*) \ge 0, \quad \text{and if } x_i^* > 0, \text{ then " } = 0\text{"}, \tag{4a}$$

$$g^j(x^*) \le 0, \quad \text{and if } \lambda_j^* > 0, \text{ then " } = 0\text{"}. \tag{4b}$$

The corresponding saddle-point problem is to *minimize* Lagrangian $\mathcal{L}(\lambda, x)$ with respect to x and *maximize* it with respect to λ for $x \ge 0$ and $\lambda \ge 0$.

The Kuhn-Tucker Theorem holds with no change for the constrained minimization problem. However, in constraint qualification conditions concavity of functions g^j, if present, has to be replaced by their convexity. This guarantees convexity of the constraint set described here by inequalities $g^j(x) \le 0$.

Theorems S1 and S2 continue to hold with concavity (quasi-concavity) of functions F and g^j replaced by their convexity (quasi-convexity, respectively).

I.4 Remarks:

• **Applications** of K-T theorems in microeconomics:

(i) Consumer theory: utility maximization subject to budget constraint, and expenditure minimization.

(ii) Welfare economics: Characterization of Pareto optimal allocations as solutions to maximization of a welfare function subject to resource constraints, or maximization of one agent's utility subject to constraints on other agents' utilities and resource constraints.

(iii) Producer theory: cost minimization.

• There are versions of K-T theorems for maximization and minimization with mixed constraints, i.e., when some constraints are of the equality form, $g^j(x) = 0$. See Sundaram [2], Section 6.4.

• K-T theorems hold for *local* maxima (minima) as well.

References: [**1**] Takayama (1995), 1.D and 1.E. [**2**] Sundaram (1999), Chapter 6. [**3**] Varian, ch 27. [**4**] MasColell et al. *Warning:* Takayama [1] and Sundaram [2] do not explicitly write nonnegativity constraints $x \geq 0$. Varian [3] writes constraints in the maximization problem as $g^j(x) \leq 0$.

I.5 Example: Consider the following constrained maximization problem:

$$\text{maximize} \quad \ln(x_1 + 1) + \ln(x_2 + 1)$$

$$\text{subject to} \quad p_1 x_1 + p_2 x_2 \leq m$$

$$x_1 \geq 0, \quad x_2 \geq 0,$$

where $p_1 > 0$, $p_2 > 0$ and $m > 0$.

In order to derive the solution (as a function of parameters p_1, p_2 and m) we write the Kuhn-Tucker first-order conditions (2) as

$$(1) \qquad \frac{1}{x_1^* + 1} - \lambda^* p_1 \leq 0, \quad \text{and if} \quad x_1^* > 0, \text{ then "} = 0\text{".}$$

$$(2) \qquad \frac{1}{x_2^* + 1} - \lambda^* p_2 \leq 0, \quad \text{and if} \quad x_2^* > 0, \text{ then "} = 0\text{".}$$

$$(3) \qquad p_1 x_1^* + p_2 x_2^* \leq m, \quad \text{and if } \lambda^* > 0, \text{ then "} = 0\text{".}$$

with $x^* \geq 0$ and $\lambda^* \geq 0$.

Note that (3) holds with equality since it follows from (1) that $\lambda^* > 0$.

We solve inequalities (1-3) by considering cases:

Case 1. $x_1^* > 0$, $x_2^* > 0$.

Then (1) and (2) hold with equalities. Solving (1), (2) and (3) we find $x_1^* = \dfrac{m + p_2 - p_1}{2p_1}$ and $x_2^* = \dfrac{m + p_1 - p_2}{2p_2}$ and $\lambda^* = \dfrac{2}{m + p_1 + p_2}$. For x_1^* and x_2^* to be strictly positive, it has to be that $m + p_2 > p_1$ and $m + p_1 > p_2$. Thus Case 1 applies with x_1^* and x_2^* as listed above if $m + p_2 > p_1$ and $m + p_1 > p_2$.

Case 2. $x_1^* > 0$, $x_2^* = 0$.

(3) implies that $x_1^* = \frac{m}{p_1}$. Since (1) holds with equality, we solve it for $\lambda^* = \frac{1}{m + p_1}$. Next we need to verify inequality (2). It states

$$1 - \frac{p_2}{m + p_1} \leq 0,$$

and it holds if $p_2 \geq m + p_1$. Thus Case 2 applies (with $x_1^* = \frac{m}{p_1}$, $x_2^* = 0$) if $p_2 \geq m + p_1$.

Case 3. $x_1^* = 0$, $x_2^* > 0$.

This case is very similar to Case 2. From (3) and (2) we obtain $x_1^* = \frac{m}{p_2}$, $\lambda^* = \frac{1}{m + p_2}$. Verifying inequality (1), we obtain $p_1 \geq m + p_2$. Thus Case 3 applies (with $x_1^* = 0$, $x_2^* = \frac{m}{p_2}$) if $p_1 \geq m + p_2$.

The case $x_1^* = x_2^* = 0$ cannot hold since it violates equation (3). This concludes our solution to the K-T conditions.

Since utility function is concave and the constraint function is concave (in fact, it is linear) $K - T$ conditions are sufficient (Theorem S1). Hence, the solution to K-T conditions is a constrained maximizer. Further, since the Slater's condition holds, every constrained maximizer has to satisfy $K - T$ conditions.