Mathematical Appendix II

II.1 Theorem of the Maximum

There are two sets $S \subset I\!R^n$ and $T \subset I\!R^m$. Further, there are a correspondence φ mapping S into the sets of subsets of T and a function $f : S \times T \to I\!R$. That is, $\varphi(x)$ is a subset of T for every $x \in S$, and $f(x,t)$ is a real number for every $x \in S$ and $t \in T$.

We are interested in the constrained maximization problem with f as the objective function and φ as the constraint. That is, given $x \in S$,

$$\max_t f(x,t) \tag{4}$$

$$\text{subject to} \qquad t \in \varphi(x).$$

We denote by $g(x)$ the maximized value of function f and by $\mu(x)$ the subset of vectors t in $\varphi(x)$ on which the maximum value is attained. Formally,

$$g(x) = \max_{t \in \varphi(x)} f(x,t) \quad \text{and} \quad \mu(x) = \{t \in \varphi(x) : f(x,t) = g(x)\}. \tag{5}$$

Interpretation: Think about an economic agent whose environment is described by a vector $x \in S$. The agent's set of actions is T, but when the environment is x, she is restricted to choose her action only from the subset $\varphi(x)$. Her utility of action t is $f(x,t)$, when the environment is x. Her objective is to choose an action in $\varphi(x)$ to maximize her utility.

We shall assume that the set T is **compact.**

Correspondence φ is said to be **continuous** if it is lower hemi-continuous and upper hemi-continuous. These are defined as follows:

(LHC) φ is **lower hemi-continuous** at x if for every sequence $\{x_n\}$ in S converging to x and every $t \in \varphi(x)$, there exists a sequence $\{t_n\}$ in T such that $t_n \in \varphi(x_n)$ and $\{t_n\}$ converges to t.

(UHC) φ is **upper hemi-continuous** at x if for every sequence $\{x_n\}$ in S converging to x and every sequence $\{t_n\}$ in T converging to t, with $t_n \in \varphi(x_n)$, it holds that $t \in \varphi(x)$.

Our definition of UHC is the closed graph property. MasColell, Whinston and Green give definitions of LHC and UHC in Appendix M.H, pg. 949-951. Their definition of upper hemi-continuity is different, but if the range of φ (i.e., the set T) is compact as assumed, then their definition is equivalent to the above one. Note that upper hemi-continuous correspondence φ must have compact values $\varphi(x)$.

Theorem II.1: *Suppose that the set T is compact. If correspondence φ is continuous on S and function f is continuous on $S \times T$, then*

(i) g is continuous on S, and

(ii) μ is an upper hemi-continuous correspondence on S.

Proof: (i) Let $\{x_n\}$ be a sequence of vectors in S converging to x. We have to show that $\lim_n g(x_n) = g(x)$. Since $\varphi(x_n)$ is a compact set for every n, there exist $t_n \in \varphi(x_n)$ such that $g(x_n) = f(x_n, t_n)$. Since the set T is compact, sequence $\{t_n\}$ must have a convergent subsequence with a limit $\bar{t} \in T$. We switch to that subsequence of $\{t_n\}$, but we retain the same notation; i.e., we keep $\{t_n\}$ and assume that it converges to $\bar{t}$. Upper hemi-continuity of φ implies that $\bar{t} \in \varphi(x)$. By continuity of f, we have $\lim_n f(x_n, t_n) = f(x, \bar{t})$. Since $f(x, \bar{t}) \leq g(x)$, it follows that

$$\lim_n g(x_n) \leq g(x).$$

To prove the oposite inequality, we note that $g(x) = f(x, t)$ for some $t \in \varphi(x)$ since $\varphi(x)$ is a compact set. Lower hemi-continuity of φ at x implies that there is sequence $\{\tilde{t}_n\}$ converging to t such that $\tilde{t}_n \in \varphi(x_n)$ for every n. We have $f(x_n, \tilde{t}_n) \leq g(x_n)$. Using continuity of f, we obtain $\lim_n f(x_n, \tilde{t}_n) = f(x, t)$. Consequently

$$\lim_n g(x_n) \geq g(x).$$

This concludes the proof of (i)

(ii) Consider two sequences: $\{x_n\}$ in S converging to x, and $\{t_n\}$ in T converging to t such that $t_n \in \mu(x_n)$. We have to show that $t \in \mu(x)$.

We first observe that upper hemi-continuity of φ implies that $t \in \varphi(x)$. Next, consider arbitrary $\tilde{t} \in \varphi(x)$. Lower hemi-continuity of φ at x implies that there is a sequence $\{\tilde{t}_n\}$ converging to $\tilde{t}$ such that $\tilde{t}_n \in \varphi(x_n)$ for every n. Clearly then $f(x_n, t_n) \geq f(x_n, \tilde{t}_n)$. Passing to the limit with n and using continuity of f, we obtain $f(x,t) \geq f(x, \tilde{t})$. Since $\tilde{t}$ was arbitrary in $\varphi(x)$, this implies that $t \in \mu(x)$. This concludes the proof of (ii).

Remarks:

- The assumption that set T is compact can be dropped. Then the MWG definition of upper hemi-continuity has to be used. Note that that definition requires that correspondence φ be compact-valued.

- One application of the Theorem of the Maximum II.1 is in producer theory. We set S as the set of price vectors, T as the production set, i.e., $T = Y$, function f as $f(p, y) = py$, and correspondence φ as $\varphi(p) = Y$. Assuming that Y is compact, Theorem II.1 implies continuity of the profit function and upper hemi-continuity of the supply correspondence (Proposition 4.3 (iii)).

II.2 Monotonicity of a Vector-Valued Function.

Let D be an open convex subset of $I\!R^n$, and let $F : D \to I\!R^n$.

Proposition II.2: *Suppose that F is continuously differentiable. Then the following two conditions are equivalent:*

(i) $[F(x') - F(x)][x' - x] \geq 0$ *for every* $x, x' \in D$,

(ii) the matrix $DF(x)$ *is positive semi-definite for every* $x \in D$.

Proof: Consider arbitrary $x, x' \in D$, and denote $x' - x$ by z. Further, define

$$x(\lambda) = \lambda x' + (1 - \lambda)x, \qquad \text{for } \lambda \in [0, 1]$$

It holds, $x(0) = x$, $x(1) = x'$, and $x(\lambda) = x + \lambda z$. Next, define function $g : [0, 1] \to I\!R$ by

$$g(\lambda) = z[F(x(\lambda)) - F(x)].$$

Note that $g(0) = 0$, $g(1) = [x' - x][F(x') - F(x)]$, and $g'(\lambda) = zDF(x(\lambda))z$. Suppose that (i) holds. Since $g(\lambda) = \frac{1}{\lambda}[x(\lambda) - x][F(x(\lambda)) - F(x)]$ for $\lambda > 0$, it follows that $g(\lambda) \geq 0$. Therefore, g has a minimum at $\lambda = 0$. This implies $g'(0) \geq 0$, which is $zDF(x)z \geq 0$. Since z was arbitrary, we obtain (ii).

Conversely, suppose that (ii) holds. Then $g'(\lambda) \geq 0$ for every $\lambda \in [0, 1]$. So function g is increasing and hence $g(1) \geq g(0) = 0$. This implies (i). QED

Condition (i) of Proposition II.2 may be called Δ-*monotonicity* for it can be imprecisely written as $\Delta F \Delta x \geq 0$. This is different from the usual condition of F being nondecreasing. Function $F : D \to I\!R^n$ is **nondecreasing** on $D \subset I\!R^n$ if

$$x \leq x' \quad \text{implies that } F(x) \leq F(x') \tag{6}$$

for every $x, x' \in D$. Both inequalities in (6) are vectorial inequalities in $I\!R^n$.

Δ-monotonicity of II.2 (i) and property (6) are unrelated (except for when $n = 1$). Neither IV.1 (i) implies (6), nor the opposite.

If function F is differentiable on D, then a necessary and sufficient condition for being nondecreasing in the sense of (6) is that

$$DF(x) \geq 0 \quad \text{for every } x, \tag{7}$$

i.e., that the matrix $DF(x)$ is positive. That is,

$$\frac{\partial F_i}{\partial x_j}(x) \geq 0 \qquad \forall i, j, \quad \text{for every } x.$$

Note that both Δ-monotonicity and nondecreasing imply that $\frac{\partial F_i}{\partial x_i}(x) \geq 0$, but that is about as much as they have in common.