Ordinal Representations and Properties of Recursive Utilities*

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Abstract: Recursive utilities which represent preferences as solutions to non-linear difference equations are used extensively in macroeconomics and asset pricing. They offer great flexibility in modeling time preference, intertemporal risk aversion and attitudes to timing of resolution of uncertainty. We revisit the issues of existence and properties of recursive utilities in regard to time preference, risk and timing attitudes in a Markov setting. As these properties are of ordinal nature, we focus on ordinal representations of recursive utilities which involve joint transformation of the aggregator and the certainty equivalent. Taking into account ordinal representations delivers weaker conditions for existence, discounting, risk aversion, and preference for early resolution of recursive utilities than the conditions previously known in the literature.

We examine Epstein-Zin and risk-sensitive recursive utilities that are often used in applications. Further, we introduce a novel class of Koopmans recursive utilities that are generated by an additively separable aggregator with non-linear discount function and the expectation operator as a certainty equivalent. Koopmans recursive utilities feature non-exponential discounting and a clear-cut separation between timing and risk attitudes, and offer great flexibility in these dimensions.

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1. Introduction

Recursive utilities play an important role in macroeconomics and finance. They offer great flexibility in modeling time preferences and intertemporal risk aversion.[1] Their main advantage over the standard discounted expected utility is that they discard time separability. In particular, the popular Epstein and Zin (1989) and Weil (1990) recursive utility has been extensively used in asset pricing to explain long standing empirical puzzles such as the equity premium puzzle and the risk-free rate puzzle. The Epstein-Zin recursive utility allows for separate parametrization of risk aversion and elasticity of intertemporal substitution. Recursive utilities represent preferences as solutions to non-linear difference equations and therefore can be easily used in dynamic programming.

The idea of recursive utility dates back to Koopmans (1960) and Kreps and Porteus (1978). It postulates that the utility of a plan of current and future consumption is an aggregate according to some specific function of a utility of current consumption and a certainty equivalent of next period continuation utility which is stochastic. The standard discounted expected utility has such recursive structure with quasi-linear aggregator (linear in continuation utility, but not in current consumption) and expectation as the certainty equivalent. Other recursive utilities involve different specifications of an aggregator and a certainty equivalent. In particular, Epstein-Zin utility has a non-linear aggregator and a utility-based certainty equivalent.

Recursive utility is merely a representation of underlying preferences. Every strictly increasing transformation of recursive utility represents the same preferences. We show that every transformation of recursive utility function is recursive as well under a joint transformation of the aggregator and the certainty equivalent. Many important properties of recursive utilities are properties of the underlying preferences, that is, they are ordinal. Examples are time preference, attitudes toward risk and timing of resolution of uncertainty, and the basic property of the existence of recursive utility as a solution to the difference equation. While these properties are invariant to transformations of recursive utility, the corresponding sufficient conditions on the aggregator and the certainty equivalent are not invari-

[1] See Backus et al. (2004).

ant to transformations. For example, contraction conditions of the aggregator and the certainty equivalent, or concavity/convexity of the aggregator are not invariant to transformations. The former conditions are critical for the existence and time preference of recursive utility while the latter is critical for attitudes toward risk and timing of resolution. This implies that, for the said properties of recursive utilities, it is sufficient that the relevant conditions hold for what we call an equivalent ordinal representation, that is, a suitably chosen transformation of the aggregator and the certainty equivalent.[2]

We revisit in this paper the issues of existence and properties of recursive utilities taking into account their equivalent ordinal representations. Doing so delivers significantly weaker conditions for the existence of recursive utility than the conditions previously known in the literature. Further, it uncovers new sufficient conditions for risk aversion, time preference, and preference for timing of resolution of uncertainty.

We consider Epstein-Zin and risk-sensitive recursive utilities of Hansen and Sargent, and propose a novel class of Koopmans recursive utilities under uncertainty. Epstein-Zin utility is specified by constant-elasticity-of-substitution (CES) aggregator $F(c,z) = (c^\alpha + \beta z^\alpha)^{\frac{1}{\alpha}}$ with elasticity parameter $\alpha > 0$, and a utility-based (i.e., quasi-arithmetic) certainty equivalent with CRRA utility index with risk-aversion parameter ρ. Risk-sensitive utility has quasi-linear aggregator $F(c,z) = u(c)+\beta z$ with period-utility function u and a CARA utility-based certainty equivalent with risk-aversion parameter σ. Koopmans utility is generated by an additively separable aggregator $F(c,z) = u(c) + f(z)$ with (non-linear) discount function f and a certainty equivalent given by the expectation operator. Additively separable aggregators have been advocated in the work of Koopmans (1960, 1972) in settings without uncertainty.

We first establish existence of recursive utility functions relying on contraction properties of the aggregator and the certainty equivalent. The Contraction Theorem implies that there exists a solution to the difference equation defining a recursive utility in the space of norm-continuous utility functions on the space of non-negative and bounded consumption sequences. We restrict attention to bounded consumption sequences in order to focus on the method of ordinal repre-

[2]Potential usefulness of equivalent representations has been pointed out by Ma (1993).

sentations. We apply the Contraction Theorem to three classes of recursive utilities under consideration taking into account their ordinal representations. Epstein-Zin utility has two alternative representations: one with quasi-linear aggregator and transformed CRRA certainty equivalent, the other with transformed CES aggregator and the expectation operator as the certainty equivalent. Risk-sensitive utility has an alternative representation with Cobb-Douglas aggregator and the expectation operator. A simple shift transformation gives rise to a useful equivalent representation of the Koopmans utility. Using these ordinal representations when applying the general existence result delivers weaker than previously known sufficient conditions for existence for the three classes of utility functions. Epstein-Zin utility is shown to be well-defined for every $\alpha > 0$ and $\rho < 1$. Risk-sensitive utility is well-defined for every $\sigma \neq 0$. Koopmans utility exists under weak restrictions on the discount function.

Contraction conditions are of critical importance not only for the existence of recursive utility function but also for time preference. There are two qualitative aspects of time preference that can be expressed and characterized for recursive utilities: The first is preference for early consumption introduced by Koopmans et al (1960) in the setting of no uncertainty. It postulates that and extra amount of consumption is preferred when offered at an earlier date to it offered at a later date. The second is tail insensitivity that has been extensively studied by Streufert (1990) under no uncertainty. It postulates that consumption in very distant future does not asymptotically matter for current utility. It gives rise to a sequential representation of recursive utility. We show that contraction conditions on the aggregator and the certainty equivalent together with additive separability of the aggregator are sufficient for preference for early consumption. Contraction conditions alone are sufficient for tail insensitivity. When applying these results to Epstein-Zin and risk-sensitive utilities, it turns out important yet again to take ordinal representations into account. Risk-sensitive and Koopmans utilities exhibit preference for early consumption and tail insensitivity under the same conditions found earlier to be sufficient for their existence. For Epstein-Zin utility, preference for early consumption involves an additional restriction that $\frac{\alpha}{1-\rho} \leq 1$.

The property of risk aversion of a utility function has strong implication in many economic setting. In particular, it has profound implication for risk sharing among multiple agents. We distinguish two concepts of risk aversion of recursive utility. The first is *atemporal risk aversion* considered by Ma (1993) which states that replacing risky consumption at all future dates by their expected value conditional on current date and state is preferred under the current-date utility. The second is *temporal risk aversion* which states that replacing next period risky consumption by its expected value conditional on current date and state is preferred under the current-date utility. Sufficient conditions for atemporal risk aversion are concavity of the aggregator and a risk-aversion condition on the certainty equivalent for some ordinal representation. For Epstein-Zin utility, these conditions hold if $\alpha \leq 1$ and $\rho \geq 0$. For risk-sensitive utility with logarithmic period-utility they hold if $\sigma \leq 1$.[3]

For Koopmas utility, the sufficient condition is concavity of period-utility and discount functions. Temporal risk aversion requires separability of the aggregator and holds only for Koopmans utility with concave period-utility function.

Attitudes to timing of resolution of uncertainty have been brought to consideration for recursive utility functions by Kreps and Porteus (1978) in the setting of temporal lotteries. It is well known that discounted time-separable expected utility exhibits indifference to timing of resolution of uncertainty. That is, a comparison of two consumption plans that differ only by the date of resolution of uncertainty, say $t+1$ versus $t+2$, results in indifference according to date-t (and all prior dates) utility function. Recursive utilities offer flexibility in modeling different attitudes to timing of resolution. Epstein-Zin and risk-sensitive utilities can exhibit preference for early or late resolution of uncertainty depending on parameters. Extending earlier results of Kreps and Porteus (1978) and Strzalecki (2013), we provide sufficient conditions for preference for early and late resolution of uncertainty taking into account ordinal representations of recursive utilities. We identify conditions on parameters of Epstein-Zin and risk-sensitive utilities corresponding to different timing attitudes. For Koopmans utility, preference for early or late resolution corresponds to the discount function being convex or concave, respectively.

Summing up, our results show that while all three classes of Epstein-Zin, risk-sensitive and Koopmans recursive utilities offer flexibility in regard to time preference and attitudes to risk and time of resolution, the Koopmans recursive utilities provide the most clear-cut separation between timing and risk attitudes. Period-

[3]Parameter σ is the negative of the coefficient of risk aversion of the CARA utility index.

utility function expresses (temporal) risk attitudes through its concavity/convexity properties while discount function expresses attitudes to timing of resolution of uncertainty through its concavity/convexity properties and preference for early consumption through its contraction properties.

The paper is organized as follows. In Section 2 we introduce the concept of ordinal representation of a recursive utility function and provide a characterization for quasi-arithmetic certainty equivalents. Epstein-Zin and risk-sensitive utilities are introduced in Section 3 along with their alternative ordinal representations. The novel class of Koopmans recursive utilities is presented in Section 3 as well. Section 4 is about the existence of recursive utilities and the contraction conditions on the aggregator and the certainty equivalent. The importance of the contraction conditions is on display in Section 5, too, where we study time preference of recursive utilities. Attitudes toward risk and timing of resolution of uncertainty are discussed in Sections 6 and 7, respectively. Section 8 concludes the paper with, among other things, some results on recursive utilities under ambiguity.

Related literature: An important application of the Epstein-Zin recursive utility is in the asset pricing theory. Stochastic discount factor derived from the Epstein-Zin utility combined with the long-run risk specification of the representative agent's consumption provide an explanation of the equity premium puzzle. The parameters of the utility function estimated to fit the data in the leading studies are $\alpha = \frac{1}{3}$ and $\rho = 10$ (see Bansal and Yaron (2004)), or $\alpha \approx \frac{1}{2}$ and $\rho \approx 9$ (see Schorfheide et al. (2017)). These values do not satisfy the conditions of Corollaries 1-3 for well-definiteness, preference for early consumption, or tail insensitivity of Epstein-Zin utilities. They do, however, satisfy sufficient conditions for atemporal risk aversion and preference for late resolution of uncertainty of Corollaries 5 and 6. F Their analysis concerns consumption sequences with bounded rates of growth.

The long-run risk specification of a consumption sequence in Bansal and Yaron (2004) and others involves unbounded growth rates, and hence is beyond the scope of the analysis in this paper which is limited to bounded sequences. Hansen and Scheinkman (2009, 2012), Borovicka and Stachurski (2020), and Christensen (2022) study the existence of Epstein-Zin recursive utility for long-run risk models of consumption. The objective of these studies is to prove that the recursive utility is well-defined as a solution to the difference equation for a particular consumption sequence, or a narrow class of sequences. This should be contrasted with the approach of this paper that seeks a utility function on the consumption set of non-negative sequences which is of relevance, for example, to welfare analysis of risk sharing among many agents (see Werner (2023b)).

There is a strand of literature proving existence of recursive utility as a solution to an optimal investment problem and applying methods of dynamic programming. Examples are Ozaki and Streufert (1996), Bloise and Vailakis (2018), Bloise et al. (2024), and Balbus (2016). References to earlier work on the existence of recursive utilities include Epstein and Zin (1989) in the setting of consumption lotteries and Ma (1993). The recent book by Sargent and Stachurski (2023) provides an excellent exposition of the issues related to existence of a recursive utility.[4]

2. Recursive Utilities and Ordinal Representations

The set of possible states at each date is a finite set S. The product set S^∞ represents all sequences of states over infinite time. For a sequence (or path) of states $(s_0, \dots, s_t, \dots)$, we use s^t the denote the partial history $(s_0, \dots, s_t)$ through date t. Partial histories are date-t events. The set S^∞ together with the σ-field Σ of products of subsets of S is the measurable space describing the uncertainty.

Consumption plans are non-negative bounded sequences adapted to Σ. Consumption set denoted by C is thus isomorphic to ℓ^∞_+. A generic consumption plan is $c = (c_0, c_1, \dots)$. We use c_t to denote event-dependent consumption at date t, and c^t for a consumption plan for infinite future starting at t, event-dependent.

There is reference probability measure on (S^∞, Σ) generated by transition kernel (or matrix) $Q : S \times S \to [0, 1]$. We use $Q(s)$ to denote the vector of conditional probabilities of next-period states if the current state is s. Expectation operator with respect to $Q(s)$ on R^S is denoted by E_s.

Definition 1: *State-dependent utility function $U : S \times C \to R$ is recursive if it satisfies*

$$U(s_t, c^t(s^t)) = F\big(c_t(s^t), \mu_{s_t}(U(c^{t+1}))\big), \quad \forall s_t, \ \forall t. \tag{1}$$

[4]We omit all references to the literature on existence of recursive utilities in deterministic settings.

for every t and every c for some aggregator function $F : R_+ \times Z \to R$ and state-dependent certainty equivalent $\mu_s : Z^S \to Z$ for every $s \in S$, where the domain Z is either R_+ or R.

Function U in this definition is the *stationary stochastic recursive utility* introduced by Koopmans (1960). Aggregator function F and certainty equivalent μ will be referred to as a *representation* of recursive utility U. Equation (1) is often written more concisely as

$$U_t(c^t) = F\big(c_t, \mu_t(U_{t+1}(c^{t+1})\big), \quad \forall t. \tag{2}$$

In this notation, subscript t in U_t indicates conditioning on state at date t, rather than time-dependence of recursive utility U.

Aggregator function $F : R_+ \times Z \to R$ is assumed throughout the paper to satisfy the following condition:

(AG) F is strictly increasing and continuous.

An example is the *quasi-linear* aggregator

$$F(y, z) = u(y) + \beta z \tag{3}$$

for $y \in R_+$ and $z \in R$, with strictly increasing and continuous period-utility function $u : R_+ \to R$ and discount factor $0 < \beta < 1$.

Certainty equivalent μ, where $\mu_s : Z^S \to Z$, is assumed to satisfy

(CE) μ_s is increasing, i.e., if $\tilde{z} \geq \tilde{z}'$, then $\mu_s(\tilde{z}) \geq \mu_s(\tilde{z}')$ for $\tilde{z}, \tilde{z}' \in Z^S$

for every $s \in S$. Two simple and intuitive properties of certainty equivalent μ that will be needed in some results are *risk aversion* meaning that $\mu_s(\tilde{z}) \leq E_s(\tilde{z})$ for every $\tilde{z} \in Z^S$ and $s \in S$, and *normalization* defined by $\mu_s(k) = k$ for every risk-free $k \in Z$ and every s.

An example of a certainty equivalent is the *expectation operator*

$$\mu_s(\tilde{z}) = E_s[\tilde{z}], \tag{4}$$

for every $\tilde{z} \in R^S$. When put together, the quasi-linear aggregator and the expectation operator are a representation of the discounted expected utility

$$U_t(c^t) = E_t[\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau)], \tag{5}$$

for every t.

An important class of certainty equivalents that includes the expectation operator are *quasi-arithmetic* operators of the form

$$\mu_s(\tilde{z}) = g^{-1}(E_s[g(\tilde{z})]) \tag{6}$$

for $\tilde{z} \in Z^S$, where utility index $g : Z \to R$ is strictly increasing and continuous.[5] Every quasi-arithmetic certainty equivalent is normalized and every quasi-arithmetic certainty equivalent with concave utility index is risk averse.

Utility functions are, of course, merely representations of preference relations. Every strictly increasing transformation of a utility function represents the same preference relation. We introduce the notion of ordinal representation of a recursive utility function.

Definition 2: *Aggregator $\hat{F}$ and certainty equivalent $\hat{\mu}$ are an ordinal representation of recursive utility function U if there exists a strictly increasing, continuous function $h : R \to R$ such that $\hat{F}$ and $\hat{\mu}$ are a representation of utility function $\hat{U} = h \circ U$.*

If certainty equivalent $\hat{\mu}$ in Definition 2 is quasi-arithmetic with utility index $\hat{g}$, then we say that $\hat{F}$ and $\hat{g}$ are an ordinal representation of U. If two pairs (F, g) and $(\hat{F}, \hat{g})$ are ordinal representations of the same recursive utility U, then we say that (F, g) and $(\hat{F}, \hat{g})$ are *ordinally equivalent*. We have the following

Proposition 1: *If (F, g) is a representation of recursive utility function U, then for every strictly increasing and continuous function h, the pair $(\hat{F}, \hat{g})$, where*

$$\hat{F}(y, z) = h\big(F(y, h^{-1}(z))\big), \quad \text{and} \quad \hat{g} \equiv g \cdot h^{-1} \tag{7}$$

is an ordinal representation of U.

Proof: The proof is elementary and is left to the reader.

3. Epstein-Zin, Risk-Sensitive, and Koopmans Recursive Utilities

The *Epstein-Zin* recursive utility is a solution to the equation

$$U_t(c^t) = \left(c_t^\alpha + \beta[E_t(U_{t+1}(c^{t+1}))^{1-\rho}]^{\alpha/(1-\rho)}\right)^{1/\alpha} \tag{8}$$

[5]Note that any linear transformation of g gives rise to the same certainty equivalent.

where $0 < \beta < 1$, $\alpha > 0$, and $\rho \neq 1$, or a strictly increasing transformation thereof. The aggregator in (8) is the constant elasticity of substitution (CES) function

$$F(y,z) = (y^\alpha + \beta z^\alpha)^{1/\alpha} \tag{9}$$

for $y \in R_+$ and $z \in R_+$ while the certainty equivalent is quasi-arithmetic with CRRA (or power) index

$$g(x) = \frac{1}{1-\rho} x^{1-\rho}, \tag{10}$$

for $x \in R_+$, with $x \neq 0$ if $1-\rho < 0$, and where ρ is the coefficient of relative risk aversion.[6]

There are two other ordinal representations of the Epstein-Zin recursive utility (8) that will be useful in the analysis to follow. The first is with quasi-linear aggregator

$$\hat{F}(y,z) = y^\alpha + \beta z, \tag{11}$$

for $y \in R_+$ and $z \in R_+$, and a quasi-arithmetic certainty equivalent with transformed CRRA utility index

$$\hat{g}(x) = \frac{\alpha}{1-\rho} x^{(1-\rho)/\alpha}. \tag{12}$$

It obtains via transformation $h(x) = x^\alpha$. The second representation is with the expectation certainty equivalent. It obtains via transformation $h(x) = g(x)$ and has the CES-like aggregator

$$\hat{F}(y,z) = \frac{1}{1-\rho}\big(y^\alpha + \beta[(1-\rho)z]^{\alpha/(1-\rho)}\big)^{(1-\rho)/\alpha} \tag{13}$$

for $y \in R_+$ and $z \in R_+$ if $1-\rho > 0$, and $z \in R_{--}$ if $1-\rho < 0$. Recursive utility with this representation takes negative values if $1-\rho < 0$, in which case the aggregator is not defined for $z = 0$.

The *risk-sensitive recursive utility* of Hansen and Sargent (see Sargent and Stachurski (2023)) is a solution to

$$U_t(c^t) = u(c_t) + \beta \frac{1}{\sigma} \ln\left(E_t[e^{\sigma U_{t+1}(c^{t+1})}]\right), \tag{14}$$

[6] Often, the aggregator is written in a slightly different form as $\bar{F}(y,z) = \left((1-\beta)y^\alpha + \beta z^\alpha\right)^{1/\alpha}$. It can be easily seen that the pairs (F,g) and $(\bar{F},g)$ are ordinally equivalent via transformation $h(x) = (1-\beta)^{1/\alpha}x$.

where $0 < \beta < 1$ and $\sigma \neq 0$, or a strictly increasing transformation thereof. Period-utility function $u : R_+ \to R$ is strictly increasing and continuous. The aggregator in (14) is quasi-linear while the certainty equivalent is quasi-arithmetic with CARA (exponential) utility

$$g(x) = \frac{1}{\sigma} e^{\sigma x} \tag{15}$$

for $x \in R$, where σ is the negative of the coefficient of absolute risk aversion. A frequently used specification of period-utility function in the risk-sensitive recursive utility is $u(y) = \ln(y)$. This logarithmic function is not well-defined at zero which makes our existence results in Section 4 not applicable to this specification. We provide further comments in Section 4.

An equivalent ordinal representation of recursive utility (14) with the expectation certainty equivalent obtains via transformation $h(x) = g(x)$. It features the aggregator

$$\hat{F}(y, z) = \frac{1}{\sigma} e^{\sigma u(y)} (\sigma z)^{\beta}, \tag{16}$$

for $y \in R_+$ and $z \in R_+$ if $\sigma > 0$, and $z \in R_-$ if $\sigma < 0$. The recursive utility with this representation takes negative values if $\sigma < 0$. For the logarithmic period-utility function, aggregator (16) is the Cobb-Douglas-like function[7]

$$\hat{F}(y, z) = \frac{1}{\sigma} y^{\sigma(1-\beta)} (\sigma z)^{\beta}. \tag{17}$$

We introduce a third class of recursive utilities that we refer to as *Koopmans recursive utilities.* As we will see, they retain some but not all properties of discounted expected utilities in regard to efficient risk sharing. Further, when compared with Epstein-Zin and risk-sensitive utilities, they have superior properties of attitudes toward risk and timing of resolution of uncertainty, and time preference.

The Koopmans recursive utility is a solution to the equation

$$U_t(c^t) = u(c_t) + f(E_t[U_{t+1}(c^{t+1})]), \tag{18}$$

[7]A slightly different representation with Cobb-Douglas aggregator $\hat{F}(y, z) = y^{(1-\beta)} z^{\beta}$ and CRRA utility index $\hat{g}(x) = \frac{1}{\sigma} x^{\sigma/(1-\beta)}$ obtains by transformation $h(x) = e^{(1-\beta)x}$. This shows that risk-sensitive utility can be considered a special (or limit) case of Epstein-Zin utility as the Cobb-Douglas aggregator can be considered CES with $\alpha = 0$.

for some period-utility function $u : R_+ \to R$ and a strictly increasing and continuous *discount function* $f : Z \to R$, or a strictly increasing transformation thereof. The aggregator in (18) is an *additively separable* function

$$F(y,z) = u(y) + f(z) \tag{19}$$

for $y \in R_+$ and $z \in Z$, while the certainty equivalent is the expectation operator.[8] An example of aggregator (19) with non-linear discount function f is

$$F(y,z) = u(y) + \beta \ln(z+1), \tag{20}$$

for $y \in R_+$ and $z \in R_+$, considered by Koopmans et al. (1968).

4. Existence, Uniqueness, and the Blackwell Contraction Condition

We establish existence and uniqueness of recursive utility functions in this section relying on contraction properties of the aggregator and the certainty equivalent. Contraction conditions are of critical importance for discounting and tail insensitivity of recursive utility as well (see Section 5).

An aggregator F satisfies the *Blackwell contraction condition* if

$$|F(y,z) - F(y,z')| \le \delta |z - z'| \quad \forall y \in R_+,\ \forall z, z' \in Z, \quad \text{for some } 0 < \delta < 1. \tag{21}$$

Clearly, the quasi-linear aggregator (3) satisfies the Blackwell contraction condition with $\delta = \beta$. The CES aggregator (9) satisfies it if $\alpha \ge 1$, with $\delta = \beta^{1/\alpha}$.

The (weak) contraction condition for certainty equivalent μ is

$$|\mu_s(\tilde{z}) - \mu_s(\tilde{z}')| \le ||\tilde{z} - \tilde{z}'||, \quad \forall\ \tilde{z}, \tilde{z}' \in Z^S, \tag{22}$$

for every s. Contraction condition (22) is equivalent[9] to *constant subadditivity* of μ, i.e.,

$$\mu_s(\tilde{z} + k) \le \mu_s(\tilde{z}) + k, \quad \forall\ \tilde{z} \in Z^S \text{ and } \forall k \ge 0. \tag{23}$$

Marinacci and Montrucchio (2010, Theorem 12) show that quasi-arithmetic certainty equivalent with twice-differentiable utility index g is constant subadditive

[8]Period utility function u in representation (18) of Koopmans utility is unique up to strictly increasing linear transformation.

[9]This can be demonstrated using the arguments of the proof of the Contraction Lemma in Sundaram (1996).

if and only if index g is IARA, i.e., it exhibits increasing absolute risk aversion. In particular, the expectation operator and the CARA certainty equivalent are constant additive, hence subadditive, and therefore satisfy contraction condition (22).

We consider in this section a subset of the consumption set C consisting of all consumption plans c such that $0 \leq c \leq \omega$ for some arbitrary upper-bound $\omega \in C$. This set is the order-interval $[0, \omega]$. The space of utility functions $\mathcal{U}_\omega$ consists of all norm continuous state-dependent utility functions on $[0, \omega]$, with the sup-norm of function U being $||U|| = \sup_{s \in S} \sup_{c \in C} |U(s, c)|$.

We say that recursive utility is *well-defined* if there exists a continuous utility function $U \in \mathcal{U}_\omega$ solving equation (1). The solution is a fixed point of an operator T on $\mathcal{U}_\omega$ defined by the operation seen on the right-hand side of (1), that is, $T(U)(s, c) = F(c_s, \mu_s(U(c^1)))$. Operator T maps $\mathcal{U}_\omega$ into itself. If conditions (21) and (22) hold, then T is a contraction. The Contraction Theorem implies that there exists a fixed point. Since the existence of recursive utility is invariant to ordinal representations, we have

Proposition 2: *Recursive utility is well-defined if there exists an ordinal representation $(\hat{F}, \hat{\mu})$ such that $\hat{F}$ satisfies the Blackwell condition and $\hat{\mu}$ is a contraction.*

The Contraction Theorem implies further that the operator $\hat{T}$ associated with $(\hat{F}, \hat{\mu})$ is globally stable and the fixed-point utility function is unique. Of course, recursive utility function is not unique in the ordinal sense of representing the preference relation. As discussed in Section 2, every strictly increasing and continuous transformation of the fixed-point utility function is a recursive utility representation as well.

When applying Proposition 2 to Epstein-Zin recursive utility, we consider the equivalent ordinal representations introduced in Section 3. We first consider the case when $\rho < 1$. The representation with quasi-linear aggregator (11) and transformed CRRA index (12) satisfies conditions of Proposition 2 if the CRRA index is IARA. This is so if $\frac{1-\rho}{\alpha} \geq 1$. The second representation with CES-like aggregator (13) and expectation operator satisfies conditions of Proposition 2 if the former satisfies the Blackwell condition which holds if $\frac{\alpha}{1-\rho} \geq 1$. One of these two conditions holds for every $\alpha > 0$ and $\rho < 1$. If $\rho > 1$, then either the certainty equivalent or the aggregator of the ordinal representations in Section 3 is not defined at zero. This makes Proposition 2 not applicable. For risk-sensitive recursive utility, the representation with quasi-linear aggregator (3) and CARA certainty equivalent satisfies the conditions of Proposition 2 for every $\sigma \neq 0$. Summing up, we have[10]

Corollary 1:

(i) Epstein-Zin recursive utility is well-defined for every $\alpha > 0$ and $\rho < 1$.

(ii) Risk-sensitive recursive utility is well-defined for every $\sigma \neq 0$.

Marinacci and Montrucchio (2010) proved that Epstein-Zin utility is well-defined for every $\alpha \leq 1$ and $\rho > 1$ on a truncated set of consumption plans bounded away from zero (see also Sargent and Stachurski (2023)). As mentioned in Section 3, Corollary 1 (ii) can not be applied to the risk-sensitive utility with logarithmic period-utility function. However, it applies to period-utility $u(y) = \ln(y + \epsilon)$ for arbitrary $\epsilon > 0$, which can be used to approximate $\ln(y)$.

Koopmans recursive utility is obviously well-defined if discount function f satisfies the Blackwell condition. However, many concave functions which are of special importance (see Section 6) do not satisfy this condition. For example, function $f(z) = z^r$, for $0 < r < 1$, is not a contraction. The following lemma shows that a linear transformation of a concave function on R_+ satisfies the Blackwell condition.

Lemma 1: *If $f : R_+ \to R_+$ is a strictly increasing and concave function and there exists unique $z^* > 0$ such that $f(z^*) = z^*$, then the shift-function $\hat{f}$ defined by $\hat{f}(z) = f(z + z^*) - z^*$ satisfies the Blackwell condition.*

Proof: We present a proof assuming that function f is differentiable. Consider a function $q(z) = f(z) - z$ for $z \in R_+$. Function q is concave and it holds $q(0) \geq 0$ and $q(z^*) = 0$. Therefore q is decreasing on $[z^*, \infty)$ and $q'(z^*) < 0$. Let $\delta = f'(z^*)$. We have $0 < \delta < 1$. Since $f(z') - f(z) \leq f'(z)[z' - z]$ for $z', z \geq z^*$ and $f'(z) \leq f'(z^*)$, we have $f(z') - f(z) \leq \delta[z' - z]$ for $z', z \geq z^*$. For shift-function $\hat{f}$, this is $\hat{f}(z') - \hat{f}(z) \leq \delta[z' - z]$ for $z', z \geq 0$. Thus $\hat{f}$ satisfies the Blackwell condition on R_+ □

[10] Parameter β is assumed to satisfy $0 < \beta < 1$ and period-utility function u is assumed strictly increasing and continuous on R_+ throughout the rest of the paper. These conditions will not be mentioned in any results.

If F is the aggregator of a Koopmans recursive utility with period-utility function u and discount function f, then the pair (F, E) of F and the expectation operator E is ordinally equivalent to $(\hat{F}, E)$, where $\hat{F}(y, z) = u(y) + \hat{f}(z)$ for the shift-function $\hat{f}$. The transformation function is linear, $h(x) = x - z^*$, and leaves the expectation operator unchanged. Using Lemma 1 we obtain

Corollary 2: *Koopmans recursive utility is well-defined if the discount function f satisfies either (i) the Blackwell condition, or (ii) $f : R_+ \to R_+$ is concave and there exists unique $z^* > 0$ s.t. $f(z^*) = z^*$.*

Discount function $f(z) = z^r$, for $0 < r < 1$, satisfies condition (ii). The shift-function for $z^* = 1$ is $\hat{f}(z) = (z+1)^r - 1$, and it satisfies the Blackwell condition. Condition (ii) is slightly stronger than the Thompson condition which requires, beside concavity, that there exists $\hat{z}$ such that $f(\hat{z}) < \hat{z}$ (see Marinacci and Montrucchio (2010) and Bloise and Vailakis (2018)). The Thompson condition is not sufficient to guarantee that there is a shift-function satisfying the Blackwell condition.[11]

5. Time Preference

Discounted time-separable expected utility (5) features exponential discounting with discount factor β per period. It is well known that exponential discounting fits rather poorly most of empirical and experimental evidence. Recursive utilities offer great generality of alternative forms of discounting while maintaining dynamic consistency of preferences. We consider two aspects of time preference (or discounting) of recursive utilities in this section: preference for early consumption, and tail-insensitivity.

The definition of preference for early consumption in the stochastic setting is as follows.

Definition 3: *Recursive utility function U exhibits preference for early consump-*

[11] An example is function f defined by $f(z) = z$ for $0 \leq z \leq 1$ and $f(z) = 1 + \ln(z)$ for $z \geq 1$. None of fix-points of this f makes the shift-function satisfy the Blackwell condition. A discount function $f(z) = \min\{z, \alpha + \beta z\}$, for $\alpha > 0$ considered by Bloise and Vailakis (2018) does not satisfy condition (ii), but the shift-function with $z^* = \frac{\alpha}{(1-\beta)}$, which is the greatest fix-point of f, satisfies the Blackwell condition.

tion if for every $c \in C$, $w, v \in R_+$ such that $w > v$, and every s^t

$$U(s_t, \mathring{c}^t(s^t)) > U(s_t, (\bar{c}^t(s^t)) \tag{24}$$

where $\mathring{c}_\tau = \bar{c}_\tau = c_\tau$ for every $\tau \neq t, t+1$, $\mathring{c}_\tau(\{-s^t\}) = \bar{c}_\tau(\{-s^t\}) = c_\tau(\{-s^t\})$ for $\tau = t, t+1$, and $\mathring{c}_t(s^t) = w, \mathring{c}_{t+1}(s^{t+1}) = v, \bar{c}_t(s^t) = v, \bar{c}_{t+1}(s^{t+1}) = w$, for every $s^{t+1} \subset s^t$.

The plan $\mathring{c}$ in Definition 3 has a greater amount of consumption offered at date t than at $t+1$, while the plan $\bar{c}$ has the same amounts offered in reversed order. Note that because $\mathring{c}$ and $\bar{c}$ are equal for all dates before t, the preference of $\mathring{c}$ over $\bar{c}$ holds at all dates from 0 to t as well.

We have the following

Proposition 3: *Recursive utility displays preference for early consumption if there exists an ordinal representation $(\hat{F}, \hat{\mu})$ such that $\hat{F}$ is additively separable and satisfies the Blackwell condition and $\hat{\mu}$ is a contraction.*

Proof: If $\hat{F}(y, z) = u(y) + f(z)$, then

$$\hat{U}(s_t, \mathring{c}^t(s^t)) = u(w) + f\big(\hat{\mu}_{s_t}[u(v) + W(s^{t+1})]\big)$$

and

$$\hat{U}(s_t, \bar{c}^t(s^t)) = u(v) + f\big(\hat{\mu}_{s_t}[u(w) + W(s^{t+1})]\big),$$

where we used $W(s^{t+1}) = f(\hat{\mu}_{s_{t+1}}[\hat{U}_{t+2}(c^{t+2})])$. Using the Blackwell condition (21) for f and the contraction condition (22) for $\hat{\mu}_s$, and keeping in mind that $u(w) > u(v)$, we obtain

$$f\big(\hat{\mu}_{s_t}[u(w) + W(s^{t+1})]\big) - f\big(\hat{\mu}_{s_t}[u(v) + W(s^{t+1})]\big) \leq \delta\big(\hat{\mu}_{s_t}[u(w) + W(s^{t+1})] - \hat{\mu}_{s_t}[u(v) + W(s^{t+1})]\big) < u(w) - u(v).$$

Hence, (24) holds. □

Proposition 3 can be applied to Epstein-Zin recursive utilities via the representation with quasi-linear aggregator (11) and certainty equivalent with CRRA index (12). The contraction condition for the certainty equivalent holds if $\frac{1-\rho}{\alpha} \geq 1$, see Section 4. For risk-sensitive recursive utilities, the representation with quasi-linear aggregator (3) and CARA certainty equivalent satisfies conditions of Proposition 3 for every $\sigma \neq 0$. Lastly for Koopmans utilities, Proposition 3 applies under any of the conditions of Corollary 2. Summing up, we have

Corollary 3:

(i) Epstein-Zin recursive utility displays preference for early consumption for $\alpha > 0, \rho < 1$ *and* $\frac{\alpha}{1-\rho} \leq 1$.

(ii) Risk-sensitive recursive utility displays preference for early consumption for every $\sigma \neq 0$.

(iii) Koopmans recursive utility displays preference for early consumption if discount function f *satisfies one of the conditions of Corollary 2.*

Since Epstein-Zin and risk-sensitive recursive utilities have ordinal representations with quasi-linear aggregators, they are time-separable when restricted to deterministic consumption plans. In contrast, Koopmans recursive utilities are not time-separable on deterministic consumption plans, reflecting the original motivation of Koopmans (1960) for studying recursive utilities.

The second aspect of discounting is tail insensitivity. A definition in the stochastic setting is as follows.

Definition 4: *Recursive utility function* U *is tail insensitive if*

$$U_0(c) = \lim_{t \to \infty} U_0(c^{-t}e) \tag{25}$$

for every $c \in C$ *and* $e \in C$, *where* $c^{-t}e$ *denotes the consumption plan equal to* c_τ *for* $\tau \leq t$ *and to* e_τ *for* $\tau > t$.

This definition says that consumption in distant future does not asymptotically matter for current utility. Equation (25) can be equivalently stated as

$$U_0(c) = \lim_{t \to \infty} F\big(c_0, \mu_0(F(c_1, \mu_1(F(c_2, \ldots, \mu_{t-1}(F(c_t, z_t)))))))\big) \tag{26}$$

where $z_t = \mu_t(U_{t+1}(e^{t+1}))$ for some $e \in C$. Equation (26) with $e = 0$, that is $z_t = U(0)$, provides a sequential representation of recursive utility U, akin to the infinite sum (5) for discounted expected utility.

The property of tail insensitivity is invariant to transformations of recursive utility function. Contraction conditions for some ordinal representation are sufficient for tail insensitivity.

Proposition 4: *Recursive utility function U is tail insensitive if there exists an ordinal representation $(\hat{F}, \hat{\mu})$ such that $\hat{F}$ satisfies the Blackwell contraction condition and $\hat{\mu}$ is a contraction.*

Proof: Let us denote

$$\Delta_t \equiv |\hat{U}_0(c) - \hat{U}_0(c^{-t}e)| \tag{27}$$

for $c, e \in C$. Using equation (2) recursively from 0 to t, we obtain

$$\hat{U}_0(c) = \hat{F}\big(c_0, \hat{\mu}_0(\hat{F}(c_1, \hat{\mu}_1(\hat{F}(c_2, \ldots, \hat{\mu}_{t-1}(\hat{F}(c_t, \hat{\mu}_t(\hat{U}_{t+1}(c^{t+1})))))))\big). \tag{28}$$

Writing $\hat{U}_0(c^{-t}e)$ in equation (27) in the way as in (26) (with $\hat{\mu}_t(\hat{U}_{t+1}(e^{t+1}))$ substituted for z_t) and repeatedly using contraction conditions (21) and (22), there results

$$\Delta_t \le \delta^t ||\hat{U}_{t+1}(c^{t+1}) - \hat{U}_{t+1}(e^{t+1})|| \tag{29}$$

Since $\hat{U}_t(c^t)$ and $\hat{U}_t(e^t)$ are bounded sequences, the right-hand side of inequality (29) converges to zero. Therefore $\lim_t \Delta_t = 0$ and (26) follows. □

We have

Corollary 4: *Epstein-Zin, risk-sensitive, and Koopmans recursive utilities are tail insensitive under the conditions of Corollary 1 and Corollary 2, respectively.*

We illustrate results of this section and Section 4 with an example of Koopmans recursive utility function with a discount function that does not satisfy the Blackwell condition, but there is an equivalent representation which is a contraction.

Example 1: Let the discount function be $f(z) = \sqrt{z}$ and the period-utility be linear $u(y) = y$. The shift-function $\hat{f}(z) = \sqrt{z+1} - 1$ is a contraction and provides an equivalent representation of this Koopmans recursive utility, see Corollary 2. Recursive utility $\hat{U}$ obtains as the fixed point of a contraction operator. Function $\hat{U}$ has a sequential representation from the expression of tail insensitivity. For a constant, deterministic plan d, utility $\hat{U}(d)$ is the unique positive solution to the equation $\hat{U}(d) = d + \sqrt{\hat{U}(d) + 1} - 1$. For example, $\hat{U}(0) = 0$. Recursive utility function U solving the recursive equation with f is the shift-transformation of $\hat{U}$, that is $U(c) = \hat{U}(c) + 1$ for every $c \in C$. Note that $U(0) = 1$. This value is one of two positive solutions to the equation $U(0) = \sqrt{U(0)}$ of the recursion for U. It is selected so that the resulting function be continuous and tail insensitive.

The utility of a deterministic consumption plan of one unit at date t and zero at all other dates is $U(1_t) = (1+1)^{\frac{1}{2^t}}$, where the first term in $1+1$ is the period-utility of one unit of consumption while the second is the continuation utility of zero future consumption. We see that $U(1_t)$ is decreasing in t as implied by the property of preference for early consumption, and $\lim_{t\to\infty} U(1_t) = U(0)$. □

6. Risk Aversion

Discounted time-separable expected utility with concave period-utility displays risk aversion in several ways. First is *temporal risk aversion* which states that replacing next period risky consumption by its expected value conditional on current state is preferred under the current-state preferences. Second is *atemporal risk aversion* which states that replacing risky consumption for all future periods by their expected value conditional on current state is preferred under the current-state preferences. Further, there is a multi-period form of temporal risk aversion which we will discuss later. All these forms of risk aversion have implications on risk sharing under discounted time-separable expected utility (see Werner (2023b)). We explore in this section extensions of temporal and atemporal risk aversion to general recursive utilities.

Recursive utility function U is *temporaly risk averse* if

$$U_t(c^t) \leq U_t(c_t, E_t[c_{t+1}], c^{t+2}), \tag{30}$$

for every $c \in C$ and every t. This is more precisely written as

$$U(s_t, c^t(s^t)) \leq U(s_t, (c_t(s^t), E_{s_t}[c_{t+1}], c^{t+2}(s^t))),$$

for every s^t. Temporal risk aversion holds for discounted time-separable expected utility if (and only if) period-utility function is concave. This result extends to general recursive utilities in the following way.

Proposition 5: *Recursive utility is temporaly risk averse if there exists an ordinal representation $(\hat{F}, \hat{\mu})$ such that $\hat{F}$ is additively separable with concave period-utility function and $\hat{\mu}$ is the expectation operator.*

Proof: We prove (30) for $t = 0$. If $\hat{F}(y, z) = u(y) + f(z)$ and u is concave, then

$$\begin{aligned}\hat{U}_0(c) &= u(c_0) + f\big(E_0[u(c_1) + f(E_1(\hat{U}_2(c^2)))]\big)\\ &\leq u(c_0) + f\big(u(E_0(c_1)) + E_0[f(E_1(\hat{U}_2(c^2)))]\big)\\ &= \hat{U}_0(c_0, E_0(c_1), c^2).\end{aligned}$$

The proof for arbitrary $t \geq 1$ is the same. □

The key argument of the proof of Proposition 5 is that there is no hedging effect between next-period consumption and continuation utility. More precisely, the absence of hedging appears in the preference for risk-free consumption in $E_0[u(c_1)+f(\tilde{z})] \leq E_0[u(E_0(c_1))+f(\tilde{z})]$, where $\tilde{z}$ is state-dependent continuation utility. This would not hold for non-separable aggregator or non-linear certainty equivalent.

Proposition 5 states that Koopmans recursive utility with concave period-utility function is temporaly risk averse. Epstein-Zin and risk-sensitive recursive utilities do not a have representation with an additively separable aggregator and the expectation certainty equivalent.

The second notion of risk aversion is atemporal risk aversion studied in Ma (1993). Recursive utility function U is *atemporaly risk averse* if

$$U_t(c^t) \leq U_t(E_t(c^t)), \tag{31}$$

for every $c \in C$ and every t. The following proposition is due to Ma (1993).

Proposition 6: *Recursive utility is atemporaly risk averse if it is tail-insensitive and there exists an ordinal representation $(\hat{F}, \hat{\mu})$ such that $\hat{F}$ is concave and $\hat{\mu}$ is risk averse.*

Proof: We first prove that (31) holds for consumption plans that are risk-free and constant over time from some date T on. For any $c \in C$, we consider the consumption plan $c^{-T}d$ equal to risk-free $d \in R_+$ for every $t > T$.

For $t = 0$ in (31), we have[12]

$$\begin{aligned}
&\hat{U}_0(c^{-T}d) = \hat{F}\big(c_0, \hat{\mu}_0(\hat{F}(c_1, \hat{\mu}_1(\hat{U}_2(c^{-T}d))))\big) \\
&\le \hat{F}\big(c_0, E_0(\hat{F}(c_1, E_1(\hat{U}_2(c^{-T}d))))\big) \le \hat{F}\big(c_0, \hat{F}(E_0(c_1), E_0(\hat{U}_2(c^{-T}d)))\big) \\
&\cdots \\
&\le \hat{F}\big(c_0, \hat{F}(E_0(c_1), \hat{F}(E_0(c_2), \ldots, \hat{F}(E_0(c_{T-1}), U(d)))))\big) = \hat{U}_0(E_0(c^{-T}d))
\end{aligned}$$

where we interchangeably used risk aversion of $\hat{\mu}$ and concavity of $\hat{F}$.

Passing to the limit as T goes to infinity in $\hat{U}_0(c^{-T}d) \le \hat{U}_0(E_0(c^{-T}d))$ and using tail-insensitivity of $\hat{U}$, we obtain (31) for $t = 0$ for arbitrary c. The proof for arbitrary $t \ge 1$ is the same. □

Corollary 5 provides sufficient conditions atemporal risk aversion of Epstein-Zin, risk-sensitive and Koopmans recursive utilities.

Corollary 5:

(i) Epstein-Zin recursive utility is atemporaly risk averse for every $0 < \alpha \le 1$ *and* $\rho \ge 0, \rho \ne 1$.

(ii) Risk-sensitive recursive utility is atemporaly risk averse for every $\sigma < 0$.

(iii) Koopmans recursive utility is atemporaly risk averse if period-utility function u *and discount function* f *are concave.*

Proof: For the representation of Epstein-Zin utility with CES aggregator (9) and CRRA utility index (12), the conditions of Proposition 6 hold if $\alpha \le 1$ and $\rho \ge 0$. For the representation with quasi-linear aggregator (11) and the transformed CRRA index (12), the conditions hold if $\alpha \le 1$ and $\frac{1-\rho}{\alpha} \le 1$. This is subsumed be the previous case. The representations with CES-like aggregator (13) does not provide any additional conditions.

For risk-sensitive utility, the representation with quasi-linear aggregator (3) and CARA certainty equivalent satisfies the conditions of Proposition 6 if the CARA utility index is risk averse. This holds if $\sigma < 0$. The representation with aggregator (16) does not provide any additional conditions. Summing up, we obtain (ii).

The case of the Koopmans utility is straightforward. □

[12]To simplify notation, we write $U_2(c^{-T}d)$ instead of $U_2(\{c^{-T}d\}^2)$ for date-2 continuation utility in this proof.

If the period-utility function in the risk-sensitive recursive utility is logarithmic, then the condition in Corollary 5 (ii) can strengthened to $\sigma \leq 1$. This is so because the Cobb-Douglas-like aggregator (17) if concave if $0 < \sigma < 1$.

Discounted expected utility with concave period-utility function has a stronger risk-aversion property not shared by any other utility function in Corollary 5. It is the *multi-period risk aversion* defined by

$$U_t(c^t) \leq U_t(c_t, \ldots, c_{\tau-1}, E_t[c_\tau], c^{\tau+1}), \tag{32}$$

for every $c \in C$ and every t and $\tau \geq t+1$, or more precisely,

$$U(s_t, c^t(s^t)) \leq U(s_t, (c_t(s^t), \ldots, c_{\tau-1}(s^t), E_{s_t}[c_\tau], c^{\tau+1}(s^t))),$$

for every s^t. Multi-period risk aversion implies temporal and atemporal risk aversions - the latter provided that U is tail-insensitive. It holds for Koopmans utility if the discount function is linear but may not hold otherwise.

7. Preference for Early Resolution of Uncertainty

Discounted time-separable expected utility exhibits indifference to timing of resolution of uncertainty. Recursive utilities offer flexibility in modeling different attitudes to timing of resolution.

The definition of preference for early resolution of uncertainty is adopted from Strzalecki (2013). We assume in this section that beliefs are not changing, that is, $Q(s, \cdot)$ is independent of s for iid beliefs, for otherwise beliefs would confound preference for timing of resolution. Accordingly, we assume that certainty equivalent μ_s is independent of s. This holds for quasi-arithmetic certainty equivalents.

Definition 5: *Recursive utility function U exhibits preference for early resolution of uncertainty if for every risk-free d_t, every $\tilde{z} \in R^S_+$ and s^t*

$$U(s_t, \hat{c}^t(s^t)) \geq U(s_t, \check{c}^t(s^t)) \tag{33}$$

where $\hat{c}_\tau = \check{c}_\tau = d_\tau$ for every $\tau \neq t+2$, $\hat{c}_{t+2}(\{-s^t\}) = \check{c}_{t+2}(\{-s^t\}) = d_{t+2}$ and $\hat{c}_{t+2}(s^{t+2}) = \tilde{z}(s_{t+1}), \check{c}_{t+2}(s^{t+2}) = \tilde{z}(s_{t+2})$, for every $s^{t+2} \subset s^{t+1}$ and every $s^{t+1} \subset s^t$.

Consumption plans $\hat{c}$ and $\check{c}$ in Definition 5 offer the same risky claim $\tilde{z}$ at date $t+2$, but differ by the timing of resolution of uncertainty of $\tilde{z}$. In $\hat{c}$, the uncertainty is resolved at date $t+1$ (early), while in $\check{c}$ it is resolved at date $t+2$ (late). The two consumption plans are risk-free at all dates other than date $t+2$. Thus the comparison of early and late resolutions of uncertainty is made in isolation of any other uncertainty. Note that because consumption plans $\hat{c}$ and $\check{c}$ are the same for all dates before t, the preference of early-resolution plan $\hat{c}$ over late-resolution $\check{c}$ in (33) holds for all dates from 0 to t.

We have the following proposition of which the first part has been demonstrated by Kreps and Porteus (1978) in the setting of consumption lotteries and the second part is closely related to Strzalecki (2013) in the setting of finite-time consumption.

Proposition 7: *Recursive utility with iid beliefs displays preference for early (late) resolution of uncertainty if there exists an ordinal representation $(\hat{F}, \hat{\mu})$ satisfying one of the following conditions:*

(i) $\hat{F}$ is convex (concave, respectively) in the second argument and $\hat{\mu}$ is the expectation operator.

(ii) $\hat{F}$ is quasi-linear and $\hat{\mu}$ is normalized, constant superadditive and subhomogeneous[13] (subadditive and superhomogeneous, respectively).

Proof: For consumption plan c equal to $\hat{c}$ or $\check{c}$ and any representation $(\hat{F}, \hat{\mu})$, we have

$$\hat{U}(s_t, c^t(s^t)) = \hat{F}\big(d_t, \hat{\mu}_{s_t}[\hat{F}(d_{t+1}, \hat{\mu}_{s_{t+1}}[\hat{F}(c_{t+2}, W))]\big) \tag{34}$$

where $W = \hat{U}_{t+3}(d^{t+3})$ and is risk-free. Clearly, the comparison of utilities $\hat{U}(s_t, \hat{c}^t(s^t))$ and $\hat{U}(s_t, \check{c}^t(s^t))$ amounts to comparing the second argument under outer $\hat{F}$ in (34), that is, $\hat{\mu}_{s_t}[\hat{F}(d_{t+1}, \hat{\mu}_{s_{t+1}}[\hat{F}(c_{t+2}, W))]$. Since $\hat{c}_{t+2}$ depends only on states s_{t+1} and is equal to $\tilde{z}$, and $\hat{\mu}_s$ is independent of s, this term for $c = \hat{c}$ is

$$\hat{\mu}[\hat{F}(d_{t+1}, \hat{F}(\tilde{z}, W))]. \tag{35}$$

[13] μ is subhomogeneous if $\mu(\lambda\tilde{z}) \geq \lambda\mu(\tilde{z})$ for every $\tilde{z} \in R^S_+$ and every $0 \leq \lambda \leq 1$. μ is superhomogeneous if the opposite inequality holds.

Since $\check{c}_{t+2}$ depends on states s_{t+2} and is equal to $\tilde{z}$, the respective expression for $c = \check{c}$ is

$$\hat{F}(d_{t+1}, \hat{\mu}[\hat{F}(\tilde{z}, W)]). \tag{36}$$

Under the assumptions of part (i), expression (35) equals $E[\hat{F}(d_{t+1}, \hat{F}(\tilde{z}, W))]$ while (36) equals $\hat{F}(d_{t+1}, E[\hat{F}(\tilde{z}, W)])$. If $\hat{F}$ is convex in the second argument, then the former exceeds the latter, that is, (35) exceeds (36) and (33) follows. The opposite holds if $\hat{F}$ is concave in the second argument.

In part (ii), expression (35) equals $\hat{\mu}(u(d_{t+1}) + \beta\hat{F}(\tilde{z}, W))$ while (36) equals $u(d_{t+1}) + \beta\hat{\mu}(\hat{F}(\tilde{z}, W))$. If $\hat{\mu}$ is normalized, constant superadditive and subhomogeneous, then

$$\hat{\mu}\big(u(d_{t+1}) + \beta\hat{F}(\tilde{z}, W)\big) \geq u(d_{t+1}) + \hat{\mu}(\beta\hat{F}(\tilde{z}, W)) \geq u(d_{t+1}) + \beta\hat{\mu}(\hat{F}(\tilde{z}, W)).$$

Thus (35) exceeds (36), and (33) follows. The opposite holds if $\hat{\mu}$ is normalized, constant subadditive and superhomogeneous. □

It can be easily shown that if certainty equivalent μ is normalized and concave, then it is constant superadditive and subhomogeneous.[14] Similarly, if μ is normalized and convex, then it is constant subadditive and superhomogeneous. Hardy et al (1934, Theorem 3.16) show that quasi-arithmetic certainty equivalent with CRRA utility index (10) is concave if $\rho \geq 0$ and convex if $\rho \leq 0$.

Corollary 6: *With iid beliefs, the following hold*

- *(i) Epstein-Zin recursive utility displays preference for early (late) resolution of uncertainty for $\alpha > 0$ and $\rho \neq 1$ if $\frac{\alpha}{1-\rho} \geq 1$ ($\frac{\alpha}{1-\rho} \leq 1$, respectively).*
- *(ii) Risk-sensitive recursive utility displays preference for early (late) resolution of uncertainty if $\sigma < 0$ ($\sigma > 0$, respectively).*
- *(iii) Koopmans recursive utility displays preference for early (late) resolution of uncertainty if discount function f is convex (concave, respectively).*

[14] Normalized and concave are the sufficient conditions on $\hat{\mu}$ in Strzalecki (2013, Theorem 2).

Proof: For the Epstein-Zin recursive utility, we consider the representation with quasi-linear aggregator (11) and the certainty equivalent of transformed CRRA index (12). Condition (ii) of Proposition 7 holds if the certainty equivalent is concave which is the case if $\frac{1-\rho}{\alpha} \leq 1$. Otherwise, if $\frac{1-\rho}{\alpha} \geq 1$, the certainty equivalent is convex and the preference for late resolution holds. The same conditions obtain from applying Proposition 7 (i) to the representations with CES-like aggregator and the expectation operator.

For the risk-sensitive utility, we consider the representation with aggregator (16) and the expectation operator. The aggregator is concave in the second argument for $\sigma > 0$ and convex for $\sigma < 0$. The result follows from Proposition 7 (i).

The case of the Koopmans utility is straightforward. □

Corollary 6 (i) can be found in Epstein and Zin (1989).

A restrictive feature of Definition 5 is that it applies to consumption plans that are risk-free at all dates except a single date $t+2$. Clearly, risk-free plan $(d_0, \ldots, d_{t+1})$ can be replaced by an arbitrary uncertain plan $(c_0, \ldots, c_{t+2})$ without affecting the results of Proposition 7. A further inspection of the proof reveals that the result of part (i) continues to hold if the continuation consumption from date $t+3$ on is Markov, so that its continuation utility is Markov.

8. Other Recursive Utilities

The methods and results of Sections 2-7 can be used to analyze other recursive utilities. We shall briefly discuss recursive utilities with stochastic discount factors and ambiguity.

Discounted time-separable expected utility is recursive if discount factor β is state dependent. More precisely, if the discount factor is β_s in state s with $0 < \beta_s < 1$, then the discounted time-separable expected utility

$$U_0(c) = E_0[\sum_{t=0}^{\infty} \beta_t u(c_t)],$$

where β_t takes value $\beta_{s_1} \cdot \beta_{s_2} \ldots \cdot \beta_{s_t}$ is event s^t, is recursive as it satisfies eq. (2)

with the aggregator $F(y,z) = u(y) + z$ and the certainty equivalent

$$\mu_s(\tilde{z}) = \beta_s E_s[\tilde{z}]. \tag{37}$$

In this case, the aggregator is a weak contraction and the certainty equivalent is a strict contraction. This certainty equivalent is not normalized.

Epstein-Zin, risk-sensitive, and Koopmans recursive utilities can be generalized to include stochastic discount factors. All results of Sections 4-6 remain unchanged. It suffices to adjust the certainty equivalents by discount factors as in (37). Of course, the analysis of early resolution of uncertainty of Section 7 does not apply as it necessitates state-independent certainty equivalent and hence state-independent discounting.

There is a vast class of recursive utilities under ambiguity, see Strzalecki (2013). We present an important example of *recursive multiple-prior utility* of Epstein and Schneider (2003). The recursive multiple-prior utility is a solution to

$$U_t(c^t) = u(c_t) + \beta \min_{P \in \mathcal{P}_t} E_{t,P}[U_{t+1}(c^{t+1})] \tag{38}$$

for $0 < \beta < 1$, with quasi-linear aggregator (3) and certainty equivalent given by

$$\mu_s(\tilde{z}) = \min_{P \in \mathcal{P}_s} E_P[\tilde{z}], \tag{39}$$

where $\mathcal{P}_s$ is a state-dependent set of probabilities on S. The certainty equivalent is constant additive and hence a contraction, and the aggregator satisfies the Blackwell condition. The resulting recursive utility function is

$$U_0(c) = \min_{\pi \in \Pi} E_\pi[\sum_{t=0}^{\infty} \beta^t u(c_t)],$$

where Π is a set of probabilities on S^∞ such that one-step-ahead conditional probabilities derived from Π are exactly the set $\mathcal{P}_{s_t}$ for every s_t.

The transformation rule of Proposition 1 does not apply to multiple-prior utility functions. Nevertheless, the results of Sections 4-7 can be applied to the representation with aggregator (3) and certainty equivalent (39). Propositions 3 and 4 imply that the recursive multiple-prior utility function displays preference for early consumption and is tail insensitive for arbitrary period-utility function. If the period-utility function is concave, then, by Proposition 6, the recursive multiple-prior utility is atemporaly risk averse with respect to any selection of probability measures from the set of priors. However, it may not be temporaly risk averse. Proposition 7 implies that recursive multiple-prior utility (with state-independent sets of priors) displays indifference to timing of resolution of uncertainty (see Strzalecki (2013, Theorem 1). This is so because the certainty equivalent (39) is normalized, constant additive, and homogeneous. Strzalecki (2013) provides a comprehensive analysis of attitudes to timing of resolution of uncertainty for a broad class of recursive utilities under ambiguity.

Koopmans recursive utilities can be generalized to include state-dependent sets of "priors" $\mathcal{P}_s$ instead of single-valued transition kernel Q, and the multiple-prior certainty equivalent.

9. Concluding Remarks

We studied properties of three classes of recursive utility functions: Epstein-Zin, risk-sensitive, and the novel Koopmans utilities. The properties under consideration were well-definiteness, time preference, and attitudes toward risk and timing of resolution of uncertainty. All three classes of utility functions can accommodate different attitudes toward risk and timing of resolution. While risk-sensitive and Koopmans utilities are well-defined, display preference for early consumption, and are tail insensitive for all parameters of their specifications, there is a range of parameters of Epstein-Zin utilities for which these desirable properties cannot be assured. Some parameters which are often used in applications lie in this range. Koopmans utilities offer the most clear-cut separation between timing and risk attitudes. Period-utility function expresses (temporal) risk attitudes through its concavity/convexity properties while discount function expresses attitudes to timing of resolution of uncertainty through its concavity/convexity properties and preference for early consumption through its contraction properties.

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