Speculative Bubbles, Heterogeneous Beliefs, and Learning *

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January 2023.

Abstract: This paper develops a general theory of speculative bubbles and speculative trade in dynamic asset markets with short sales restrictions when agents have heterogeneous beliefs and are risk neutral. Speculative bubble arises when the price of an asset exceeds every trader's valuation measured by her willingness to pay if obliged to hold the asset forever. Speculative bubble indicates speculative trade - whoever holds the asset intends to sell it at a later date. We identify a sufficient condition on agents' heterogeneous beliefs for speculative bubbles in equilibrium. Our main focus is on heterogeneous beliefs arising from updating different prior beliefs in the Bayesian model of learning. A sufficient condition on prior beliefs in the Bayesian model is that no single prior dominates other agents' priors in the sense of monotone likelihood ratio order. We study long-run properties of speculative bubbles, in particular, their vanishing and persistence, in light of merging of conditional beliefs, consistency and misspecification of priors.

*We thank participants in seminars and conferences where this work has been presented, in particular, Luis Araujo, Gadi Barlevy, John Conlon and Tony Doblas-Madrid of the Greater Fool Bubble (on-line) seminar series. Parts of the work have been done during visits to the Center for Interdisciplinary Research (ZiF) in Bielefeld within the research program "Robust Finance" and the Australian National University in Canberra.

1. Introduction

If traders in asset markets have diverse (or heterogeneous) beliefs and short sales are restricted, asset prices will reflect the most optimistic beliefs. Pessimists who would want to short sell the asset, will be excluded from the market by the restriction on short selling. Harrison and Kreps (1978) pointed out that if heterogeneous beliefs are randomly changing over time so that different traders become the most optimistic at different times, asset prices may strictly exceed the most optimistic valuations because those traders anticipate to sell at a future date to new optimists. Trade becomes speculative as every trader who buys the asset intends to sell it at a future date, and hence she trades for short-term gain. Harrison and Kreps (1978) presented an example of a dynamic infinite-time market where agents are risk neutral, have heterogeneous beliefs about asset dividends, and short selling is prohibited. Because of risk neutrality, agents' valuation of the asset which in general stands for the willingness to pay if obliged to hold the asset forever, is simply the discounted expected value of dividends under individual beliefs. Agents' beliefs exhibit perpetual switching: there is no single agent who is more optimistic at all future dates and states than other agents about next period dividends of the asset. In equilibrium, the agent who has the most optimistic belief buys the asset and agents with less optimistic beliefs want to short-sell the asset but are restricted by the constraint. Asset prices persistently exceed all agents' discounted expected values of future dividends.

Heterogeneity of beliefs and short sales restrictions are generally believed to be the primary reasons for the rapid rise and fall of stock prices during the dot.com bubble of 2000-2001. Ofek and Richardson (2003) provided compelling empirical evidence that traders beliefs about newly issued internet stocks were vastly diverse and that there were stringent short sales restrictions because of lockups. Hong, Scheinkman and Xiong (2006) developed a formal analysis in a model of asset markets with heterogeneous beliefs and short sales restrictions, and demonstrated that the model can account for the behavior of stock prices of dot.com companies. Heterogeneity of beliefs in Hong, Scheinkman and Xiong's (2006) model results from traders being overconfident about precision of their information, that is, thinking that information signals are more accurate than they actually are. Xiong and Yu (2011) argue that the Chinese warrants bubble of 2005-2008, when seemingly worthless warrants (i.e., put options) on stocks of several Chinese companies traded at surprisingly high prices, can be attributed to heterogeneous beliefs and short-sales restrictions. Spectacular rise and fall of Tesla stock in 2020-2022 (1100% gain in 2020-2021 followed by 70% loss in 2022) bear the marks of a speculative bubble. There have been strong differences of opinions about Tesla throughout this period as well as significant short interest, in particular in 2019 and 2020.

Asset prices in Harrison and Kreps (1978) and Hong, Scheinkman and Xiong (2006) strictly exceed the valuation of the most optimistic agent. The difference between the price and the highest valuation is termed *speculative bubble.* Speculative bubbles should not be confused with rational bubbles as the respective definitions are based on different notions of fundamental valuation.[1] For speculative bubble, fundamental valuation is the willingness to pay for the asset if obliged to hold it forever. For rational bubble, fundamental valuation is the discounted expected value of future dividend under the risk-neutral pricing measure (or stochastic discount factor). While rational bubbles can arise in equilibrium under rather special conditions[2], it is not so for speculative bubbles. Dynamic properties of rational and speculative bubbles are different, too. Speculative bubbles may "burst," while rational bubbles have to persist indefinitely, with positive probability.

Heterogeneous beliefs in a dynamic environment can arise in many different ways. In Harrison and Kreps 1978), traders initial beliefs are heterogeneous and dogmatic. They remain unchanged regardless of patterns of realized dividends. Heterogeneous beliefs in Hong, Scheinkman and Xiong(2006) (see also Scheinkman and Xiong (2003)) result from agents' overestimating the precision of publicly observed signals.[3] Overconfident agents have different posterior beliefs from "rational" agents and they overreact to signals. They are too optimistic after good signals and too pessimistic after bad signals, but they never correct their behavior. Morris (1996) introduced Bayesian learning in the model of speculative trade. He considered an i.i.d. dividend process parametrized by a single parameter of its dis-

[1]See Barlevy (2015) for a survey of various theories of asset price bubbles. Related theories of price bubbles in markets with heterogeneous beliefs and short sales restrictions are Allen et al (1993), Conlon (2004), Doblas-Madrid (2012), and Fostel and Geanakoplos (2012).

[2]By the no-bubble theorem of Santos and Woodford (1997), see also LeRoy and Werner (2014), rational bubbles arise only with low interest rates, i.e., infinite present value of total resources.

[3]See also Harris and Raviv (1993).

tribution (probability of high dividend) that is unknown to the agents. Agents have heterogeneous prior beliefs about that parameter. Morris (1996) showed that, as the agents update their beliefs over time, their posterior beliefs will exhibit switching of optimism that leads to speculative trade as long as the prior beliefs are not ranked in the maximum likelihood ratio order. Werner (2022) showed that speculative bubbles may arise with ambiguous beliefs that are common to all traders.

This paper develops a general theory of speculative bubbles and speculative trade in dynamic asset markets with short sales restrictions when agents have heterogeneous beliefs and are risk neutral. There is a single asset with arbitrary dividend process over (discrete) infinite time-horizon. Dividends may be paid infrequently which makes the model better suited to applications to the aforementioned real-world episodes in which no dividends were paid in their duration. Heterogeneous beliefs may arise in the model because of overconfidence in estimation of the precision of public information, Bayesian learning with heterogeneous priors, or simply be dogmatic beliefs. We show that a condition of *valuation switching* is sufficient for speculative bubble and speculative trade. Valuation switching holds if for every event at every date there does not exist an agent whose discounted expected value of future dividends exceeds all other agents' discounted expected values from that date on forever. The condition of valuation switching is sufficient but not necessary for speculative bubbles. Interestingly, the example of Harrison and Kreps (1978) provides an illustration. One of the traders in that example is valuation dominant at every date, in every event.

Our main analysis is focused on beliefs arising from updating heterogeneous prior beliefs in the Bayesian model of learning. We consider a general model of priors on a parametric set of probability measures on states of nature. Valuation dominance in the setting of Bayesian learning and priors with common support is closely related to the maximum likelihood ratio (MLR) order of priors. We show that dominance in the MLR order implies valuation dominance. For i.i.d. binomial uncertainty, valuation dominance is equivalent to MLR dominance (see Morris (1996)), so that valuation switching and speculative bubbles obtain whenever there is no MLR dominant prior. Speculative bubbles can easily arise when agents' priors have different supports. We show that if there is no single agent for whom the maximum and the minimum valuations of dividends over parameters in the support of her prior exceed the respective values for other agents, then there is valuation switching and speculative bubbles. For example, if there is an agent with ignorant (uniform) prior with full support and another agent who knows the true distribution of states, then there is valuation switching and speculative bubbles.

An important issue arising in settings with heterogeneous beliefs is whether or not difference in beliefs can persist in the long run when beliefs are updated on public information. During dot.com and Chinese warrants bubbles and Tesla stock rise and fall differences of opinions persisted for quite a long time. The classical Blackwell and Dubins (1962) merging-of-opinions result states that if agents prior beliefs are absolutely continuous with respect to each other, then conditional beliefs for the future given the past converge over time. Slawski (2008) was the first to point out the relevance of merging of beliefs for the asymptotic behavior of speculative bubbles, see also Morris (1996). We show that if the true probability measure on dividends is absolutely continuous with respect to agents' beliefs, then their valuations converge to the true valuation and, moreover, asset price converges to the true valuation. This makes speculative bubble vanish in the limit. The condition of absolute continuity in infinite time is a restrictive condition. In the setting of Bayesian learning with heterogeneous priors, a weaker condition of consistency of priors with the true parameter combined with absolute continuity of priors with respect to each other is shown to be sufficient for aforementioned asymptotic properties of prices and valuations. Yet again, consistency of priors with the true parameter is not an innocuous condition and may be easily violated, for example, in infinite-dimensional parameter sets or misspecified priors. We conclude that persistent (or non-vanishing) speculative bubbles are not at all unlikely.

The paper is organized as follows. In Section 2 we present the model of dynamic asset markets with heterogeneous beliefs and short sales restrictions. We prove the main result about sufficiency of valuation switching for the existence of speculative bubbles. In Sections 3 we discuss speculative bubbles in settings with heterogeneous priors and Bayesian learning in general and with i.i.d. dividends. We derive our results about valuation switching with common and heterogeneous supports of prior beliefs. Section 5 is about long-run properties of speculative bubbles in light of merging of conditional beliefs, consistency and misspecification of priors. Some concluding remarks are provided in Section 6.

2. Heterogeneous Beliefs and Speculative Trade.

Time is discrete with infinite horizon and begins at date 0. The set of possible states at each date is a finite set S. The product set S^∞ represents all sequences of states. For a sequence (or path) of states $(s_0, \ldots, s_t, \ldots)$, we use s^t the denote the partial history $(s_0, \ldots, s_t)$ through date t. Partial histories are date-t events. The set S^∞ together with the σ-field Σ of products of subsets of S is the measurable space describing the uncertainty. There is a single asset with date-t dividend x_t. Dividend x_t is a random variable on (S^∞, Σ) assumed bounded and measurable with respect to $\mathcal{F}_t$, the σ-field of date-t events. Ex-dividend price of the asset at date t is a random variable p_t, measurable with respect to $\mathcal{F}_t$.

There are I agents. Each agent i is risk-neutral and discounts future consumption by discount factor β, common to all agents. Agent's i beliefs are represented by a probability measure P^i on (S^∞, Σ) such that $P^i(s^t) > 0$ for every s^t. Agent's i utility function of consumption plan $c = \{c_t\}_{t=0}^\infty$ adapted to $\mathcal{F}_t$ is

$$\sum_{t=0}^{\infty} \beta^t E^i[c_t], \tag{1}$$

where E^i denotes the expectation under probability measure P^i. Endowments e_t^i are measurable w.r. to $\mathcal{F}_t$, positive, and bounded. Initial holdings of the asset are $\hat{h}_0^i \geq 0$. The supply of the asset $\hat{h}_0 = \sum_i \hat{h}_0^i$ is strictly positive.

The agent faces the following budget and portfolio constraints for the choice of an optimal consumption plan c and portfolio holding h,

$$c_0 + p_0 h_0 \leq e_0^i + p_0 \hat{h}_0^i, \tag{2}$$

$$c_t(s^t) + p_t(s^t) h_t(s^t) \leq e_t^i(s^t) + [p_t(s^t) + x_t(s^t)] h_{t-1}(s_-^t) \quad \forall s^t, \tag{3}$$

$$h_t(s^t) \geq 0, \quad \forall s^t \tag{4}$$

where s_-^t denotes the predecessor event of s^t at date $t-1$. Condition (4) is the short-sales constraint.

An equilibrium consists of prices p and consumption-portfolio allocation $\{c^i, h^i\}$

such that plans (c^i, h^i) are optimal and markets clear. Market clearing is

$$\sum_i c_t^i = \bar{e}_t^i + \hat{h}_0 x_t, \text{ and } \sum_i h_t^i = \hat{h}_0,$$

for every t.

Because of the short-sales constraint, equilibrium asset price p_t at date t satisfies the relationship

$$p_t(s^t) = \max_i \beta E^i[p_{t+1} + x_{t+1}|s^t]. \tag{5}$$

The agent (or agents) whose one-period-ahead conditional belief $P^i(\cdot|s^t)$ is the maximizing one on the right-hand side of (5) holds the asset in s^t while the other agents whose conditional beliefs give lower expectation have zero holding. We call the agent whose beliefs is the maximizing one the optimist (about next-period price plus dividend) at s^t.

Market belief at s^t is the maximizing probability in (5), i.e., the optimist's belief, and is denoted by $\hat{P}(\cdot|s^t)$. Let $\hat{P}$ be the probability measure on S^∞ derived from one-period-ahead probabilities $\hat{P}(\cdot|s^t)$.[4] It follows that $\hat{P}$ is a risk-neutral pricing measure (or state-price process) for p. Since the asset is in strictly positive supply and the discounted present value of the aggregate endowment $\sum_{t=0}^\infty \beta^t E_{\hat{P}}[\bar{e}_t]$ is finite, the no-bubble theorem (see Theorem 3.3 in Santos and Woodford (1997), or Theorem 30.6.1 in LeRoy and Werner (2014)) implies that equilibrium price of the asset is equal to the infinite sum of discounted expected dividends under the market belief. That is,

$$p_t(s^t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\hat{P}}[x_\tau|s^t], \tag{6}$$

for every s^t. The fundamental value of the asset under agent's i belief is the discounted sum of expected dividends conditional on event s^t, that is,

$$V^i(s^t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E^i[x_\tau|s^t]. \tag{7}$$

Because of risk-neutral utilities, agents' fundamental values represent their willingness to pay for the asset if obliged to hold it forever. It follows from (5) that

$$p_t(s^t) \geq V_t^i(s_t), \tag{8}$$

[4]The existence of probability measure $\hat{P}$ on S^∞ follows from the Kolmogorov Extension Theorem, see Halmos (1974), Sec. 38.

for every i, every s^t. The following lemma will be used in the analysis to follow.

Lemma 1: *If $p_t(s^t) > V_t^i(s^t)$ for agent i in some event s^t, then $p_\tau(s^\tau) > V_\tau^i(s^\tau)$ for every predecessor event s^τ of s^t, where $\tau < t$.*

Proof: We first prove that $p_{t-1}(s^{t-1}) > V_{t-1}^i(s^{t-1})$ for the immediate predecessor of s^t. From (5) we have

$$p_{t-1}(s^{t-1}) \geq \beta E_{t-1}^i[p_t + x_t|s^{t-1}] > \beta E^i[V_t^i + x_t|s^{t-1}] = V_{t-1}^i(s^{t-1}), \tag{9}$$

where we used (8) and, for the strict inequality, the assumption that $p_t(s^t) > V_t^i(s_t)$. The proof for non-immediate predecessor events is an iteration of the argument in (9). $\square$.

We say that there is *speculative bubble* in event s^t, if

$$p_t(s^t) > max_i V_t^i(s^t). \tag{10}$$

If (10) holds, then the optimist who buys the asset at s^t pays the price exceeding her valuation of the asset if she were to hold the asset forever. This means, of course, that she intends to sell the asset at a later date. Thus, speculative bubble indicates speculative trade. It follows from Lemma 1 that if there is speculative bubble in event s^t at date t, then there is speculative bubble at every date $\tau < t$, in each predecessor event. Thus speculative bubble has to originate at date 0, or more generally at the time of initial offering, but it can either cease to exist ("burst") at a later date or be permanent.

Agent i is (weakly) *valuation dominant* in event s^t if

$$V^i(s^\tau) \geq max_j V^j(s^\tau), \tag{11}$$

for every event s^τ which is a successor of s^t. If there is no valuation dominant agent in event s^t, then we say that agents' beliefs exhibit *valuation switching* at s^t. There is *perpetual valuation switching* from s^t on if beliefs exhibit valuation switching in every successor of s^t.

The main result of this section shows that valuation switching is sufficient for the existence of speculative bubble.

Theorem 1: *If agents' beliefs exhibit valuation switching in event s^t, then in equilibrium there is speculative bubble in s^t.*

PROOF: Suppose by contradiction that $p_t(s^t) = V_t^i(s^t)$ for some agent i. It follows from Lemma 1 that $p_\tau(s^\tau) = V_\tau^i(s^\tau)$ for every successor event s^τ. Since agent i is not valuation dominant, there exists j and a successor event s^τ such that $V_\tau^j(s^\tau) > V_\tau^i(s^\tau) = p_\tau(s^\tau)$. But this contradicts (8). $\square$.

If there is perpetual valuation switching from s^t on, then, by Theorem 1, there is permanent speculative bubble in every successor event of s^t. The following example illustrates Theorem 1 and also shows that valuation switching is not a necessary condition for speculative bubble.

Example 1 (Harrison and Kreps (1978)): We first present a variation of the original example in which there is valuation switching. The dividend process x_t is a Markov chain taking two values 0 and 1 for every $t \geq 1$. States are identified with dividends. There are two agents whose beliefs are described by transition matrices Q_1 and Q_2 given by

$$Q_1 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \tag{12}$$

Note that agent 1 is more optimistic than agent 2 about next-period high dividend when current dividend is 0 and vice versa when the current dividend is 1. Discount factor is $\beta = 0.9$.

Fundamental values of the asset depend only on the current dividend and can be calculated from the respective recursive relations. They are $V^1(0) = 4.66$, $V^1(1) = 4.34$, $V^2(0) = 4.09$, and $V^2(1) = 4.91$. Since $V^1(0) > V^2(0)$ and $V^2(1) > V^1(1)$, there is perpetual valuation switching.

In equilibrium, the agent who is more optimistic about next period dividend is also the optimist about price plus dividend, and holds the asset. Equilibrium prices can be found from equation (5). We have

$$p(0) = \beta[\frac{1}{4}p(0) + \frac{3}{4}(p(1) + 1)], \text{ and } p(1) = \beta[\frac{1}{4}p(0) + \frac{3}{4}(p(1) + 1)]. \tag{13}$$

Because of the symmetry of agents' beliefs, equilibrium price is state independent and equal to $p(0) = p(1) = 7\frac{1}{2}$. One can easily verify that the right-hand sides of equations (13) are the respective maximal values among the two agents. We have

$$p(0) > \max_i V^i(0) \text{ and } \ p(1) > \max_i V^i(1).$$

There is speculative bubble in accordance with Theorem 1.

In the original Harrison and Kreps (1978) example the transition matrices are

$$Q_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \tag{14}$$

and the discount factor is $\beta = 0.75$. As before, agent 1 is the optimist about next-period high dividend when the current dividend is 0, while agent 2 is the optimist when it is 1. Fundamental values are $V^1(0) = \frac{4}{3}, V^1(1) = \frac{11}{9}$, and $V^2(0) = \frac{16}{11}, V^2(1) = \frac{21}{11}$. Here, agent 2 is valuation dominant. Equilibrium prices can be found in the same way as before. They are $p(0) = \frac{24}{13}$ and $p(1) = \frac{27}{13}$, so that there is speculative bubble. $\square$

3. Speculative Trade and Bayesian Learning.

Bayesian learning in the setting of Section 2 is described as follows: There is a family of probability measures P_θ on (S^∞, Σ) parametrized by θ in the set of parameters Θ. The set Θ can be finite or infinite. There is σ-field $\mathcal{G}$ of subsets of Θ, and the mapping $\theta \to P_\theta(A)$ is measurable for every $A \in \Sigma$. We assume that $P_\theta(s^t) > 0$ for every θ and every finite-time event s^t. An agent, who does not know the true probability measure on (S^∞, Σ), has a prior belief μ on $(\Theta, \mathcal{G})$. The support of prior μ is the smallest closed subset of Θ of μ-measure 1, or equivalently a closed set $C \subset \Theta$ with $\mu(C) = 1$ and such that if $\theta \in C$, then $\mu(U) > 0$ for every neighborhood U of θ in Θ.

Prior μ induces a joint distribution Π_μ of states and parameters defined by

$$\Pi_\mu(A \times B) = \int_A P_\theta(B)\mu(d\theta),$$

for $A \in \mathcal{G}$ and $B \in \Sigma$. Conditional probability on $\mathcal{G} \times \Sigma$ upon observing date-t history of states s^t is $\Pi_\mu(\cdot|s^t)$ and it induces the posterior belief on Θ denoted by $\mu_t(\cdot|s^t)$ and conditional probability of the future given the past on Σ denoted by

$P_\mu(\cdot|s^t)$. For example, if μ is a Dirac point-mass measure at some θ, then $\mu_t = \mu$ for every t and $P_\mu(\cdot|s^t) = P_\theta(\cdot|s^t)$. This is "dogmatic" belief, as in Example 1, that is unaffected by learning.

Returning to the model of asset trading of Section 2, let agent's i prior belief be μ^i on $(\Theta, \mathcal{G})$. We use E^i to denote the expectation under probability measure P_{μ^i} and $E^i[\cdot|s^t]$ (or simply E^i_t) for conditional expectation under conditional probability $P_{\mu^i}(\cdot|s^t)$. The asset's dividends are a stochastic process x_t as in Section 2, and $V^i(s^t)$ is the fundamental value under agent's i updated belief $P_{\mu^i}(\cdot|s^t)$. Note that agents learn the true probability of states, and in contrast to Morris (1996), states are not identified with dividends.

The condition of valuation dominance is closely related to the monotone likelihood ratio order of priors (see Morris (1996)). Suppose that $\Theta \subset R$ and that each prior μ^i has strictly positive density function f^i on Θ. Thus, all priors have full support Θ. We say that prior μ^i dominates μ^j in the monotone likelihood ratio (MLR) order if

$$\frac{f^i(\theta')}{f^i(\theta)} \geq \frac{f^j(\theta')}{f^j(\theta)} \quad \text{whenever} \quad \theta' \geq \theta, \tag{15}$$

for every $\theta, \theta' \in \Theta$.[5] Let $V_\theta(s^t)$ denote the fundamental value of the asset in event s^t under conditional probability $P_\theta(\cdot|s^t)$.

Proposition 1: *Suppose that $V_{\theta'}(s^t) \geq V_\theta(s^t)$ for every $\theta' \geq \theta$ and every $s^t \in \mathcal{S}^\infty$. If agent's i prior μ^i MLR-dominates every other agents' prior, then agent i is valuation dominant.*

Proof: It is well-known that if μ^i MLR-dominates μ^j, then μ^i dominates μ^j in the sense of first-order stochastic dominance. Since $V_\theta(s^0)$ is increasing in θ, it follows that

$$V^i(s^0) = \int V_\theta(s^0) f^i(\theta) d\theta \geq \int V_\theta(s^0) f^j(\theta) d\theta = V^j(s^0) \tag{16}$$

for every j. Further, if μ^i MLR-dominates μ^j, then the posterior $\mu^i(\cdot|s^t)$ MLR-dominates the posterior $\mu^j(\cdot|s^t)$ for every s^t. As in (16), this implies $V^i(s^t) \geq V^j(s^t)$ for every j. □

[5] With condition (15) written as $f^i(\theta')f^j(\theta) \geq f^j(\theta')f^i(\theta)$ for $\theta' \geq \theta$, definition of MLR-dominance extends to priors with different supports.

3.1 IID Model of Learning and Regular Dividends

An important special case of Bayesian learning is the i.i.d. model. There is a family of probability measures π_θ on the state space $\mathcal{S}$. Measure P_θ is the product measure π_θ^∞ making states independently and identically distributed with π_θ on $\mathcal{S}$. the mean of the posterior distribution $\mu_t^i(\cdot|s^t)$ on Θ is the Bayes estimate of the unknown true parameter.

We assume in this subsection that dividends are a time-invariant function of states and are paid with constant frequency $\kappa \geq 1$. That is, dividend x_t is given by $x_t(s^t) = d(s_t)$ for every $t = n\kappa$ where $n = 0, 1, \ldots,$ for some function $d : R \to R$, and $x_t(s^t) = 0$ for every $t \neq n\kappa$. We refer to such specification as *regular dividends*.[6] Let $E_\theta[d]$ denote the expected value of the dividend under π_θ and $E^i[d|s^t]$ be the expected value of one-period dividend under the conditional probability $P_{\mu^i}(\cdot|s^t)$. We have

$$V^i(s^t) = \beta^{\kappa(t)} \frac{\beta^\kappa}{1-\beta^\kappa} E^i[d|s^t], \tag{17}$$

for every s^t, where $\kappa(t)$ denotes the number of dates from last dividend payment to date t, so that $0 \leq \kappa(t) < \kappa$. It follows that agent i is valuation dominant in event s^t if and only if her conditional expectation of one-period dividend (weakly) exceeds every other agent's conditional expectation of the dividend in every successor event s^τ of s^t. Otherwise, there is valuation switching in s^t.

If each prior μ^i has density function f^i on Θ and $\Theta \subset R$, then we have the following corollary to Proposition 1:

Corollary 1: *In the i.i.d. model with regular dividends, if agent's i prior μ^i MLR-dominates every other agents' prior and $E_\theta[d]$ is a non-decreasing function of θ, then agent i is valuation dominant.*

The following example of speculative bubbles with Bayesian learning and heterogeneous prior beliefs is variation of Morris(1996) with infrequent but regular dividends.

Example 2: Suppose that the state is a binary variable taking one of two values, 0 or 1. Let θ be the probability of state 1 (high) where $\theta \in [0, 1] = \Theta$. Consider a prior μ on $[0, 1]$ with density function f. Expected value of the dividend $E_\mu[d|s^t]$

[6]Dividends on firms' stocks are typically paid quarterly or annually.

conditional on s^t is equal to the conditional probability of next-period high state. That probability depends only on the number of high states from date 0 through t, and is denoted by $\nu(t,k)$ for $k \leq t$. We have, by Bayesian updating,

$$\nu(t,k) = \frac{\int_0^1 \theta^{k+1}(1-\theta)^{t-k} f(\theta) d\theta}{\int_0^1 \theta^{k}(1-\theta)^{t-k} f(\theta) d\theta} \tag{18}$$

An important class of priors on the interval $[0,1]$ are beta priors with density functions of the form $f(\theta) \sim \theta^{\alpha-1}(1-\theta)^{\beta-1}$ for some $\alpha > 0$ and $\beta > 0$. The posterior probability of high state under beta prior is

$$\nu(t,k) = \frac{(k+\alpha)}{(t+\alpha+\beta)}, \tag{19}$$

see Ghosh and Ramamoorthi (2003).

Examples of beta priors are the uniform prior with density $f^u(\theta) \equiv 1$ and the Jeffrey's prior with density $f^J(\theta) \sim \frac{1}{\sqrt{\theta(1-\theta)}}$. Their respective posterior probabilities are

$$\nu^u(t,k) = \frac{(k+1)}{(t+2)}, \tag{20}$$

and

$$\nu^J(t,k) = \frac{(k+1/2)}{(t+1)}. \tag{21}$$

It can be easily seen that for every t and k there exist $\tau' > t$ and $k' > k$, such that $\nu^J(\tau',k') > \nu^u(\tau',k')$. Similarly, there exists (τ'',k'') with $\tau'' > t$ and $k'' > k$ such that $\nu^J(\tau'',k'') < \nu^u(\tau'',k'')$. It follows that these two popular priors under ignorance give rise to perpetual valuation switching. In a market where some agents have uniform prior and others have Jeffrey's prior, there is, by Theorem 1, permanent speculative bubble.

More generally, if μ^i and μ^j have beta distribution, with (α_i, β_i) and (α_j, β_j) respectively, then μ^j valuation-dominates μ^i at every (t,k) if and only if $\alpha_j \geq \alpha_i$ and $\beta_j \leq \beta_i$. Otherwise, there is perpetual valuation switching between μ^i and μ^j. One can show that prior μ^j dominates μ^i in the MLR-order if and only if $\alpha_j \geq \alpha_i$ and $\beta_j \leq \beta_i$. Thus, MLR-order dominance and valuation dominance are equivalent within the class of beta priors on $[0,1]$ and i.i.d. binomial states (see Morris (1996)). In a market where all agents have beta priors and there is no single agent whose prior dominates all other agents' priors in the MLR-order, there is, by Theorem 1, permanent speculative bubble. □

Speculative bubbles can arise when agents' priors have different supports. Let $C_i \subset \Theta$ be the support of agent's i prior belief μ^i. The condition of heterogeneity of prior supports that leads to valuation switching in the i.i.d. model with regular dividends concerns the maximal and the minimal expected values of the dividend over priors in respective supports. Let

$$M_i = \max_{\theta \in C_i} E_\theta[d], \quad \text{and } m_i = \min_{\theta \in C_i} E_\theta[d]. \tag{22}$$

We have

Proposition 2: *In the i.i.d. model with regular dividends, suppose that the mapping $\theta \to \pi_\theta$ is 1-to-1 and continuous. If there is no single agent i such that $M_i \geq M_j$ and $m_i \geq m_j$ for all $j \neq i$, then there is perpetual valuation switching and speculative bubble in an equilibrium.*

PROOF: Consider an arbitrary event s^t and let i be the agent with the highest valuation $V^i(s^t)$ among all agents. Note that the value $E^i[d|s^t]$ is the highest for agent i as well, because of (17). We shall prove that there exists a successor event s^τ for some $\tau > t$ such that $V^j(s^\tau) > V^i(s^\tau)$ for some agent j. By assumption, there exists an agent j such that either $M_j > M_i$ or $m_j > m_i$. Consider the former case first. Then there exists $\theta \in C_j$ such that $E_\theta[d] > M_i$. Note that $M_i \geq E^i[d|s^\tau]$ for every s^τ since the support of $\mu^i(s^\tau)$ is C_i. Consider the event $\{s^\infty : \lim_{T\to\infty} E^j[d|s^T] = E_\theta[d]\}$. By the Theorem of Doob (1948), see Section 5, this event has P_θ-probability 1. It follows that for τ large enough, date-τ event $\{s^\tau : E^j[s^\tau] > M_i\}$ has strictly positive P_θ-probability, and hence is non-empty. Since $E^j[s^\tau] > M_i$ implies that $V^j(s^\tau) > V^i(s^\tau)$, this concludes the proof of the first case. In the second case we have $m_j > m_i$. Then there exists $\theta \in C_i$ such that $m_j > E_\theta[d]$. The same argument as before shows that there is s^τ for some $\tau > t$ such that $V^j(s^\tau) > V^i(s^\tau)$. This concludes the proof. □

The hypothesis of Proposition 2 holds if there is an agent whose prior has full support Θ and another agent whose prior is Dirac point-mass on some parameter in the interior of Θ. Those agents could be a Bayesian learner with ignorant prior and another agent who knows the true distribution. We have

Corollary 2: *In the i.i.d. model with regular dividends and continuous 1-to-1 mapping $\theta \to \pi_\theta$, if there is an agent whose prior has full support Θ and another agent with Dirac point-mass prior on some parameter $\theta \in \text{int}\ \Theta$, then there is perpetual valuation switching and speculative bubble in an equilibrium.*

Example 3 illustrates Proposition 2 and Corollary 2.

Example 3: Consider again the setting of i.i.d. binomial states with parameter space $\Theta = [0, 1]$ as in Example 2. Suppose that agents have uniform priors on different intervals. More specific, there are two type of agents, i and j, with i having the uniform prior on $[0, 1]$ and j having uniform prior on an interval of parameters $[a, b]$ where $0 < a < b < 1$. Thus, type-j agents have more concentrated prior. Since the posterior belief at t conditional on k high states has strictly positive density on the interval $[a, b]$ and $\nu^j(t, k)$ is its mean, it follows that $a < \nu^j(t, k) < b$. Using (20), we see that for every (t, k), there exist (τ', k') with $\tau' \geq t$ and $k' \geq k$ such that $\nu^i(\tau', k') < a$. Similarly, there exists (τ'', k'') with $\tau'' \geq t$ and $k'' \geq k$ such that $b < \nu^i(\tau'', k'')$. This implies permanent valuation switching and speculative bubble in an equilibrium, as does Proposition 2. If $a = b$, then agent's j prior is point-mass measure on a. The argument for valuation switching continues to hold as in Corollary 2. $\square$

4. Speculative Bubbles in the Long Run.

In this section we discuss asymptotic properties of speculative bubbles. Slawski (2009) pointed out the relevance of the Blackwell and Dubins (1962) merging-of-opinions result for the asymptotic properties of bubbles. If conditional beliefs merge in the sense of becoming close to each other in variational norm, then fundamental values merge as well. Further, as we show, prices merge with fundamental values. Blackwell and Dubins theorem says that conditional beliefs merge if initial beliefs are absolutely continuous with respect to each other.

As in Section 2, suppose that the beliefs of agent i are represented by a probability measure P^i on (S^∞, Σ). Further, let P^0 be the true probability measure on (S^∞, Σ). We assume that such that $P^i(s^t) > 0$ and $P^0(s^t) > 0$ for every s^t. Blackwell and Dubins theorem says that if P^0 is absolutely continuous with respect to

P^i, then

$$\lim_{t\to\infty}\{\sup_{A\in\Sigma}|P^i(A|s^t)-P^0(A|s^t)|\}=0,\quad P^0-a.e. \tag{23}$$

Condition (23) is called (strong) merging of conditional beliefs.[7] If the merging condition holds, then $\lim_t[V_t^i(s^t)-V_t^0(s^t)]=0$, P^0-a.e, where V^0 is the fundamental value of the asset under the true measure P^0. Thus, the agent's fundamental value merges with the true value of the asset.

Absolute continuity of P^0 with respect to P^i says that $P^0(A)=0$ for every event $A\in\Sigma$ such that $P^i(A)=0$. It is a strong condition. It does not follow from the assumed innocuous condition that date-t marginal P_t^0 is absolutely continuous with respect to P_t^i for all t. For example, if P^0 and P^i are infinite products of measures on S as in the case of i.i.d. true distribution and i.i.d. beliefs, then P^0 is absolutely continuous with respect to P^i only if they are identical. The same holds for stationary Markov beliefs. The beliefs in Example 1 are, of course, not absolutely continuous with respect to each other.

By the same argument of the Blackwell and Dubins Theorem, if the true measure P^0 is absolutely continuous with respect to the market belief $\hat{P}$, then then equilibrium asset price p_t merges with the true fundamental value, P^0-a.e. We shall prove that P^0 is absolutely continuous with respect to $\hat{P}$ if P^0 is absolutely continuous with respect to every agent's belief P^i. We apply a criterion for absolutely continuity of measures on the product space (S^∞,Σ) due to Darwich (2009), which is a simplified version of the main result of a seminal paper by Kabanov, Liptser and Shiryaev (1985). It says that probability measure P^0 is absolutely continuous with respect to another measure Q on (S^∞,Σ) if

$$\sum_{t=0}^{\infty}E_Q[(1-\frac{Q(s^{t+1}|s^t)}{P^0(s^{t+1}|s^t)})^2|s^t]<\infty,\quad P^0-a.e., \tag{24}$$

where the ratio of conditional probabilities is set to zero if the denominator is zero.

Recall from Section 2 that the market belief $\hat{P}$ is formed by selecting at each s^t the one-period-ahead probability $P^i(\cdot|s^t)$ which maximizes (5). If the sum in (24) is finite for $Q=P^i$ for each i, then the sum for $Q=\hat{P}$ must be finite, as well. It

[7] Absolute continuity of probability measures on the product space (S^∞,Σ) is not only sufficient but also necessary for merging of conditional beliefs for any pair of measures whose date-t marginals are absolutely continuous for every t.

follows that P^0 is absolutely continuous with respect to the market belief $\hat{P}$. We summarize our discussion in the following theorem

Theorem 2: *Suppose that P^0 is absolutely continuous with respect to P^i for every i. Then*

$$\lim_t [V_t^i(s^t) - V_t^0(s^t)] = 0, \quad P^0 - a.e. \tag{25}$$

Moreover P^0 is absolutely continuous with respect to the market belief $\hat{P}$ and

$$\lim_t [p_t(s^t) - V_t^0(s^t)] = 0, \quad P^0 - a.e. \tag{26}$$

Consequently, if there is speculative bubble, it vanishes in the limit P^0-almost surely.

4.1 Speculative Bubbles in the Long Run under Bayesian Learning

The analysis of asymptotic properties of speculative bubbles is different when beliefs arise from Bayesian learning with heterogeneous priors. As in Section 3, let Θ be the set of parameters with a σ-field of subsets $\mathcal{G}$. Prior belief of agent i is measure μ^i on $(\Theta, \mathcal{G})$. Let θ_0 be the true parameter so that the true probability distribution on states is $P^0 = P_\theta$ for $\theta = \theta_0$. If Θ is a finite set, then the condition $\mu^i(\theta_0) > 0$ guarantees that the Dirac point-mass measure at θ_0 is absolutely continuous with respect to μ^i. This in turn implies that P^0 is absolutely continuous with respect to P_μ, and Theorem 2 can be applied. If Θ is an infinite set, then the condition $\mu^i(\theta_0) > 0$ may be unnatural. In Example 3, there is no θ in the support of any of the priors that has strictly positive measure. Consistency of prior belief μ^i at θ_0 becomes an important issue.

Recall that prior μ^i is consistent at θ_0 if posterior beliefs μ_t^i converge to the Dirac measure at θ_0 in the weak topology, that is

$$\lim_{t\to\infty} \int_\Theta g d\mu_t^i(\cdot|s^t) = g(\theta_0), \qquad P^0 - a.e. \tag{27}$$

for every continuous and bounded function g on Θ. We have

Theorem 3: *Suppose that $\theta \to P_\theta$ is continuous. If every prior μ^i is consistent at θ_0 and μ^i are absolutely continuous with respect to each other, then the conclusions (25) and (26) of Theorem 2 hold. If there is speculative bubble, it vanishes in the limit P^0-almost surely.*

Proof: If μ^i is consistent at θ_0, then μ_t^i converges weakly to the Dirac measure at θ_0. This implies that $\lim_t[V_t^i(s^t) - V_t^0(s^t)] = 0$, P^0-a.e. Furthermore, if μ^i are absolutely continuous with respect to each other, then P_{μ^i} are absolutely continuous with respect to each other and, by the same argument as in Theorem 2, P_{μ^i} is absolutely continuous with respect to the market belief $\hat{P}$. This implies that $\lim_t[p_t(s^t) - V_t^0(s^t)] = 0$. P^0-a.e. $\square$.

We illustrate Theorem 3 with the following example.

Example 4: Consider again the setting of Example 2. The uniform prior and the Jeffrey's prior are absolutely continuous with respect to each other. Further, they are consistent at the true parameter θ_0 for every $\theta_0 \in [0, 1]$. This follows from Freedman (1963), but it can also be seen from the results in Example 2. By the strong Law of Large Numbers, the frequency k/t of high states converges to θ_0 with π_0–probability 1. Means of the posteriors of μ^i and μ^j are $\nu^i(t, k)$ in (20) and $\nu^j(t, k)$ in (21), respectively, and they converge to θ_0, as well. Variances of the posteriors converge to zero (see Ghosh and Ramamoorthi (2003)) implying consistency.

Fundamental values of the asset merge with the true value $\beta^{\kappa(t)} \frac{\beta^\kappa}{1-\beta^\kappa}\theta_0$ in the sense of (25), with π_0^∞-probability 1. By Theorem 3, the price of the asset merges with the true value as well. There is permanent speculative trade, but the speculative bubble vanishes in the limit. $\square$.

Conditions for consistency of the prior belief at true parameter have been extensively studied in Bayesian statistics (see, for example, Ghosh and Ramamoorthi (2003)). The classical Theorem of Doob (1948) for the i.i.d. model states that if the mapping $\theta \to \pi_\theta$ is 1-to-1, then prior μ on Θ is consistent at μ-almost every parameter θ. The almost-every nature of the theorem is unsatisfactory, and there are stronger results in the literature. With our maintained assumption of the finite set of states S, consistency holds for every parameter in the support of prior μ. This follows from Freedman (1963), and also from a general result of Schwartz (1965). If S were infinite (countable or not), then the parameter set Θ could naturally be infinite dimensional, and consistency - beyond the Theorem of Doob - may not hold (see Diaconis and Freedman (1986)). A recent account of conditions for consistency for non-i.i.d. processes can be found in Shalizi (2009).

4.2 Persistent Bubbles when Beliefs are Misspecified

Prior beliefs may be misspecified. Prior μ is called misspecified if the true parameter θ_0 lies outside of its support, that is, if $\theta_0 \notin C$. An important theorem of Berk (1966) says that, under some regularity conditions, posterior beliefs concentrate, with true P^0-probability 1, on the subset $C_0^* \subset C$ of parameters θ that minimize the Kullback-Liebler divergence[8] from π_θ to the true measure π^0 over all θ in the support of μ. The set of minimizers C_0^* need not in general be a singleton, but in many cases of interest it does contain only one parameter (see Berk (1966) and Bunke and Mihaud(1998)). We have

Proposition 3: *In the i.i.d. model with regular dividends, suppose that $\theta \to \pi_\theta$ is 1-to-1 and continuous. If prior μ^i is misspecified, that is, $\theta_0 \notin C^i$, and there is a single parameter θ_0^* in the set C_0^*, then*

$$\lim_t [V_t^i(s^t) - V_{\theta_0^*}(s^t)] = 0, \quad P^0 - a.e. \tag{28}$$

If some agents have misspecified beliefs while others have well specified beliefs or if misspecified beliefs have different minimizers of the divergence from the true probability measure, then fundamental values will not merge over time and speculative bubbles may not vanish in the limit. Slawski (2009) provides an example persistent speculative bubble with misspecified prior beliefs.

6. Concluding Remarks.

This paper is a contribution to theory of speculative bubbles and speculative trade in dynamic asset markets with short sales restrictions when agents have heterogeneous beliefs and are risk neutral. We demonstrated that the condition of valuation switching is sufficient for there being speculative bubbles in equilibrium. Our main focus has been on heterogeneous beliefs arising from updating different prior beliefs in Bayesian model of learning. The condition of valuation switching is closely related to agents' priors not being ordered in the monotone likelihood ratio order. We showed that valuations switching and speculative bubbles can easily arise when agents' priors have different supports. The take-away from the analysis

[8]For discrete state space $(S, \mathcal{S})$, the Kullback-Leibler divergence of probability measure π_θ from π_0 is $K(\pi_0, \pi_\theta) = \sum_{s \in S}[\pi_0(s) \ln(\frac{\pi_0(s)}{\pi_\theta(s)})]$.

of speculative bubbles for heterogeneous beliefs in Bayesian model of learning is that speculative bubbles are not at all unlikely. We studied asymptotic properties of speculative bubbles, in particular, their vanishing and persistence. These properties are closely related to merging of conditional beliefs and consistency of priors in the long run. Here again, persistence of speculative bubbles over long period of time is not unlikely. Further, misspecified priors may easily lead to non-vanishing bubbles.

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