Liquidity and Asset Prices in Rational Expectations Equilibrium with Ambiguous Information*

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Abstract: The quality of information in financial asset markets is often hard to estimate. This paper analyzes information transmission in asset markets when agents treat information of unknown quality as ambiguous. We consider a market with risk-averse informed investors, risk-neutral competitive arbitrageurs, and noisy supply of the risky asset, first studied in Vives (1995a,b) with unambiguous information. Ambiguous information gives rise to the possibility of illiquid market where arbitrageurs choose not to trade in a rational expectations equilibrium. When market is illiquid, small informational or supply shocks have relatively large effects on asset prices. We show that trading volume decreases and liquidity risk increases with ambiguity about probability distribution of asset payoffs. High ambiguity may lead to excess volatility of asset prices.

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1. Introduction

Information in financial markets is plentiful. There are earnings reports, announcements of macroeconomic indices, political news, expert opinions, and many others. Investors use various pieces of information to update their expectations about asset returns. In standard models of asset markets, agents update their prior probabilistic beliefs about asset returns in Bayesian fashion upon observing an information signal drawn from a precisely known distribution. The quality of some information signals in the markets may be difficult to judge. Investors may not have a single probability belief about the information signal.

The situation of insufficient knowledge of probability distribution is, of course, reminiscent of the famous Ellsberg Paradox where agents have to choose between bets based on draws from an urn with a specified mix of balls of different colors and an urn with an unspecified mix. Many agents choose bets with known odds over the bets with the same stakes but unknown odds. A decision criterion which - unlike the standard expected utility - is compatible with this pattern of preferences is the maxmin (or multiple-prior) expected utility. Under the maxmin expected utility, an agent has a set of probability beliefs (priors) instead of a single one, and evaluates an action, such as taking a bet, according to the minimum expected utility over the set of priors. Such behavior is often referred to as ambiguity aversion, for it indicates the dislike of uncertainty with unknown or ambiguous odds. Axiomatic foundations of maxmin expected utility are due to Gilboa and Schmeidler (1987).

The effects of ambiguous uncertainty and ambiguity aversion in financial markets have been studied over the past two decades. Dow and Werlang (1992) showed that an agent facing ambiguous uncertainty about payoff of an asset will choose not to trade the asset for a range of prices. Building on this result, Cao, Wang and Zhang (2005) showed that ambiguity aversion may lead to limited participation in trading in asset market equilibrium. They consider a model of asset markets with heterogeneous ambiguity and study the effects of limited market participation on asset prices. Easley and O'Hara (2009) use a similar model to study the role of regulation in mitigating the effects of nonparticipation induced by ambiguity aversion. Mukerji and Tallon (2003) showed that ambiguity aversion may have adverse effects on risk sharing in asset markets. The recent paper by Epstein and Schneider (2008) studies asset prices in dynamic markets with ambiguous information signals, but without information transmission.[1]

[1]Other papers in the growing literature on asset pricing implications of ambiguous uncertainty and ambiguity

This paper analyzes information transmission in asset markets when the quality of some information signals is unknown and agents treat information of unknown quality as ambiguous. The questions we ask are: how does ambiguity of information affect the process of information transmission in markets, and how does ambiguous information affect asset prices and trading in equilibrium.

Information transmission in financial markets has been extensively studied when information is of precisely known quality. Models of competitive markets with asymmetric information that is partially revealed by asset prices have been developed by Grossman and Stiglitz (1980), Hellwig (1980), Diamond and Verecchia (1981), and Admati (1985). Models of strategic trading under asymmetric information take their origin in the work of Kyle (1985) and Glosten and Milgrom (1985). We consider a market with risk-averse informed investors, risk-neutral uninformed arbitrageurs and random supply of a single risky asset, first studied in Vives (1995a,b) with unambiguous information. Prior to the arrival of information, all investors have ambiguous beliefs about probability distribution of the asset payoff. Ambiguous beliefs are described by a set of probability distributions. Informed agents receive a private information signal about the payoff. The signal reveals the payoff only partially so that the payoff remains uncertain, but it removes the ambiguity of informed investors' beliefs. They know precisely the conditional distribution of the payoff. Uninformed arbitrageurs do not observe the signal and extract information from prices. Their beliefs about payoff distribution remain ambiguous. Our main focus is on the process of information transmission through prices in the presence of ambiguity.

Our description of ambiguous information is different from the one seen in Epstein and Schneider (2008). In Epstein and Schneider (2008) investors have unambiguous beliefs prior to the arrival of information. Their posterior beliefs after receiving information signals become ambiguous. Investors in our model have ambiguous prior beliefs. The ambiguity disappears for investors who observe the information signal but persists for uninformed investors.

The model has CARA-normal specification. Informed investors' utility of wealth has CARA form and all random variables are normally distributed. The set of multiple prior beliefs about the distribution of the asset payoff consists of normal distributions with means and variances from some bounded intervals. Informed investors and arbitrageurs behave as competitive price-takers.

aversion are Routledge and Zin (2004), Garlappi, Uppal and Wang (2007), Condie and Ganguli (2007), Caskey (2008), Ju and Miao (2009), and Ui (2009).

Arbitrageurs, who have linear utility of wealth, take arbitrary positive or negative positions in the asset if the price equals the minimum expected payoff over their set of beliefs or, respectively, the maximum expected payoff. If the price is between the maximum and the minimum expected values, they do not trade. Arbitrageurs are uniformed and extract information from market prices. We derive a rational expectations equilibrium in closed-form. The equilibrium price function is piecewise linear. If asset supply is deterministic, equilibrium is fully revealing. Otherwise it is partially revealing.

As long as there is ambiguity about the mean of the information signal, there exists a range of values of the signal and random asset supply such that the arbitrageurs choose not to trade in equilibrium. Since arbitrageurs provide liquidity to the market whenever they trade, we identify this range as the regime of market illiquidity. When market is illiquid sensitivity of prices to information signal and to asset supply is lower than when it is liquid. Using reciprocals of price sensitivities as measures of market depth, the depth is low in state of illiqudity and high otherwise. This is a typical feature of market liquidity, see Kyle (1985).

We study the effects of ambiguity on market depth, liquidity risk and trading volume. We compare rational expectations equilibrium under ambiguity with equilibrium under no ambiguity. We find that market depth is lower when there is ambiguity. Trading volume also decreases with ambiguity. Another interesting property concerns volatility of prices. While volatility of equilibrium prices is lower than volatility of payoff when there is no ambiguity, the relation can be reversed if there is sufficiently large ambiguity about the variance of the signal. Thus there can be *excess volatility* in the sense of LeRoy and Porter (1981) and Shiller (1981).

The plan of the paper is as follows: The model of asset markets with ambiguous and asymmetric information is introduced in Section 2. In Section 3 we derive the rational expectations equilibria under ambiguity and with no ambiguity. Market liquidity is introduced and discussed in Section 4. Section 5 contains comparative statics results on the effects of ambiguity, and Section 6 is about the possibility of excess volatility under ambiguity. Section 7 concludes the paper.

2. Asset Markets with Ambiguous Information

There are two assets: a single risky asset, and a risk-free bond. The payoff of the risky asset is described by random variable $\tilde{v}$. The bond has deterministic payoff equal to one. Asset are traded in a market with risk-averse informed agents, risk-neutral uninformed arbitrageurs, and noise traders. The price of the risky asset is denoted by p. The price of the bond is normalized to 1.

There is a single informed agent, called the *speculator,* whose utility of end-of period wealth w has the CARA form $-e^{\rho w}$, where $\rho > 0$ is the Arrow-Pratt measure of absolute risk aversion. The *arbitrageur* has linear utility of end-of-period wealth equal to w. These two agents should be thought of as representative agents for large numbers of two types of traders. Initial wealth is w_s for the speculator and w_a for the arbitrageur.

The payoff of the risky asset is the sum of two random variables,

$$\tilde{v} = \tilde{\theta} + \tilde{\omega}. \tag{1}$$

Random variables $\tilde{\theta}$ and $\tilde{\omega}$ are independent and have normal distributions with respective means μ and m and variances σ^2 and τ^2.

The informed speculator observes realization θ of random variable $\tilde{\theta}$. Observation θ resolves partially, but not fully, the riskiness of payoff of the risky asset. The arbitrageur does not observe θ. Further, she has ambiguous beliefs about the distribution of $\tilde{\theta}$. The ambiguity of beliefs about $\tilde{\theta}$ is described by a set $\mathcal{P}$ of probability distributions. Each probability distribution in $\mathcal{P}$ is assumed normal and independent of $\tilde{\omega}$. More specific, we take $\mathcal{P}$ to be the set of normal distributions with mean lying in an interval $[\underline{\mu}, \bar{\mu}]$ and variance in an interval $[\underline{\sigma}^2, \bar{\sigma}^2]$. We assume that the true distribution with mean μ and variance σ^2 lies in the set $\mathcal{P}$, hence $\mu \in [\underline{\mu}, \bar{\mu}]$ and $\sigma^2 \in [\underline{\sigma}^2, \bar{\sigma}^2]$. These intervals of means and variances could be confidence intervals resulting from statistical estimation of moments (see Cao, Wang, and Zhang (2005) and Garlappi, Uppal and Wang (2007)), or they could reflect investor's subjective aversion to ambiguity.

We assume that the supply of the risky asset is random. Random asset supply serves as an additional source of uncertainty, other than information signal, and prevents asset prices from fully revealing agents' information. Randomness in $\tilde{L}$ can be thought as resulting from trade by noise traders. We assume that $\tilde{L}$ is normally distributed, independent of $\tilde{\theta}$ and $\tilde{\omega}$, with mean zero and variance σ_L^2. Assuming zero mean of asset supply $\tilde{L}$ is inessential (see a remark at the end of Section 3.)

The speculator's random wealth resulting from purchasing x shares of the risky security at price p is $\tilde{w} = w_s + (\tilde{v} - p)x$. Upon observing realization θ of $\tilde{\theta}$ her information is $I_s = \{\tilde{\theta} = \theta\}$.

Her portfolio choice is described by

$$\max_x \mathrm{E}[-e^{-\rho(w_s+(\tilde{v}-p)x)}|I_s]. \tag{2}$$

Since the conditional distribution of $\tilde{v}$ on I_s is normal, maximization problem (2) simplifies to

$$\max_x \left(\rho \left(w_s + x \, \mathrm{E} \left[(\tilde{v} - p) | \, I_s \right] \right) - \frac{1}{2} \rho^2 x^2 \mathrm{var} \left[\tilde{v} | \, I_s \right] \right). \tag{3}$$

The solution to (3) is the speculator's risky asset demand, and it takes the form

$$x_s(I_s, p) \quad = \quad \frac{\mathrm{E} \left[\tilde{v} | \, I_s \right] - p}{\rho \, \mathrm{var} \left[\tilde{v} | \, I_s \right]}. \tag{4}$$

The ambiguity about the distribution of $\tilde{\theta}$ is reflected in the arbitrageur's choice of portfolio. Arbitrageur's preferences are represented by maxmin expected utility with linear utility function and the set of priors $\mathcal{P}$. These preferences are motivated by the famous Ellsberg Paradox which most clearly exemplifies the impact of ambiguous information on agent's decision. Maxmin expected utility prescribes that the agent considers the worst-case distribution when making a decision. It has been extensively studied in decision theory, and an axiomatization has been given by Gilboa and Schmeidler (1989).

The arbitrageur's maxmin expected utility of random wealth resulting from purchasing x shares of risky security is

$$\min_{\pi \in \mathcal{P}} \mathrm{E}_\pi [w_a + (\tilde{v} - p) x | I_a], \tag{5}$$

where E_π denotes expectation under belief $\pi \in \mathcal{P}$, and I_a is the arbitrageur's information. Her portfolio choice given information I_a is thus

$$\max_x \min_{\pi \in \mathcal{P}} \mathrm{E}_\pi \left[w_a + (\tilde{v} - p) x | I_a \right]. \tag{6}$$

The set of solutions $x_a(I_a, p)$ to this maximization problem is

$$x_a(I_a, p) = \begin{cases} 0 & \text{if } \ \min_{\pi \in \mathcal{P}} \mathrm{E}_\pi \left[\tilde{v} | I_a \right] < p < \max_{\pi \in \mathcal{P}} \mathrm{E}_\pi \left[\tilde{v} | I_a \right] \\ [0, +\infty) & \text{if } \ p = \min_{\pi \in \mathcal{P}} \mathrm{E}_\pi \left[\tilde{v} | I_a \right] \\ (-\infty, 0] & \text{if } \ p = \max_{\pi \in \mathcal{P}} \mathrm{E}_\pi \left[\tilde{v} | I_a \right] \end{cases} \tag{7}$$

The arbitrageur's portfolio choice problem has no solution for other values of price p. Arbitrageur's demand shows an "inertia" that is typical to maxmin expected utilities, as first pointed out by Dow and Werlang (1992). For a range of prices, the agent chooses not to trade the asset at all.

3. Rational Expectations Equilibrium

The definition of rational expectations equilibrium in our model is standard (see Grossman and Stiglitz (1980), Hellwig (1980), and Diamond and Verecchia (1981) among many others.) Rational expectations equilibrium consists of an *equilibrium price function* $P(\theta, L)$ and *equilibrium demand functions* $X_s(\theta, L)$ and $X_a(\theta, L)$ such that, for $p = P(\theta, L)$, it holds

$$X_s(\theta, L) = x_s(I_s^*, p), \qquad X_a(\theta, L) \in x_a(I_a^*, p) \tag{8}$$

$$X_a(\theta, L) \;=\; L - X_s(\theta, L), \tag{9}$$

for almost every realizations θ and L of $\tilde{\theta}$ and $\tilde{L}$. Condition (8) expresses optimality of agents' portfolio demands. Condition (9) is the market clearing condition. Information set I_s^* of the speculator and I_a^* of the arbitrageur reflect rational expectations, that is, they result from observations of private signals and equilibrium prices. We describe these information sets next.

The speculator observes realization θ of $\tilde{\theta}$. Equilibrium price could reveal extra information about realization of asset supply $\tilde{L}$, but such information would be irrelevant for the speculator. This is so because the probability distribution of risky part $\tilde{\omega}$ of the asset payoff is independent of the supply $\tilde{L}$. Consequently, the information set I_s^* is equal to $\{\tilde{\theta} = \theta\}$. We write $I_s^* = \{\theta\}$ for short. The arbitrageur does not observe θ and extracts information about θ from equilibrium price. Her information set I_a^* is $\{P(\tilde{\theta}, \tilde{L}) = p\}$.

Let us consider information revealed to the arbitrageur by the order flow against her, instead of the equilibrium price. If the observed order flow is f and asset price is p, that information is described by the set $\{\tilde{L} - x_s(I_s^*, p) = f\}$. Conditions (8) and (9) imply that in equilibrium, that is, if $p = P(\theta, L)$ and $f = L - X_s(\theta, L)$, information revealed by order flow is the same as information revealed by price. We will use this observation when deriving an equilibrium and verify it again at the end of this section for the derived equilibrium.

The equivalence between information revealed in equilibrium by order flow and information revealed by price has been pointed out by Vives (1995b) (see also Romer (1993)). In the absence of ambiguity, this equivalence and the fact that the arbitrageur has linear utility permit a different interpretation of the model. Instead of competitive market with price-taking speculator and arbitrageur, one could imagine the arbitrageur acting as a market maker who sets asset price using zero-expected-profit rule and executes orders submitted by the speculator and noise traders. The arbitrageur sets price equal to the expected payoff conditional on information revealed by total order flow. This zero-expected-profit condition could be justified by Bertrand competition among many risk-neutral market makers. Speculator's orders are price-dependent "limit orders" determined by competitive demand. Such market structure resembles Kyle's (1985) auction, an important difference being that speculator's order is competitive instead of monopolistic.

We proceed now to derive a rational expectations equilibrium. The demand function of the speculator at information $I_s^* = \{\theta\}$ can be obtained from (4),

$$x_s(I_s^*, p) = \frac{m + \theta - p}{\rho\tau^2}. \tag{10}$$

The information revealed by the order flow $\tilde{L} - x_s(I_s^*, p)$ when its observed value is f is

$$\{\tilde{L} - \frac{m + \tilde{\theta} - p}{\rho\tau^2} = f\}.$$

This information set can be written as $\{\rho\tau^2\tilde{L} - \tilde{\theta} = a\}$, where $a = m - p + \rho\tau^2 f$ is a parameter known to the arbitrageur. Thus the content of information revealed by the order flow against the arbitrageur is the same as the content of observing random variable $\rho\tau^2\tilde{L} - \tilde{\theta}$. We have

$$I_a^* = \{\rho\tau^2 L - \theta\}. \tag{11}$$

The arbitrageur's information under rational expectations is ambiguous.

Conditional expectation of asset payoff $\tilde{v}$ on I_a^* under a probability distribution π from the set of multiple priors $\mathcal{P}$ can be obtained using the Projection Theorem. We have

$$\begin{aligned} \mathrm{E}_\pi[\tilde{v}|\, I_a^*] &= \mathrm{E}_\pi[\tilde{v}|\rho\tau^2 L - \theta] \\ &= m + \mathrm{E}_\pi[\tilde{\theta}] + \frac{\mathrm{cov}_\pi\left(\tilde{\theta}, \rho\tau^2\tilde{L} - \tilde{\theta}\right)}{\mathrm{var}_\pi\left(\rho\tau^2\tilde{L} - \tilde{\theta}\right)}\left(\rho\tau^2 L - \theta - \mathrm{E}_\pi\left[\tilde{\rho}\tau^2 L - \tilde{\theta}\right]\right) \\ &= m + \mu_\pi - \frac{\sigma_\pi^2}{\sigma_\pi^2 + \sigma_L^2\rho^2\tau^4}\left(\rho\tau^2 L + \mu_\pi - \theta\right), \end{aligned} \tag{12}$$

where μ_π and σ_π denote the mean and the variance of distribution π of $\tilde{\theta}$.

Equation (12) allows us to find the maximum and the minimum expected payoffs over the set of priors $\mathcal{P}$ needed for the arbitrageur's asset demand (7). Market clearing condition (9) is then used to determine equilibrium price function and equilibrium asset demands. These calculations have been relegated to the Appendix.

Our main result is

Theorem 1 *There exists a unique rational expectations equilibrium with price function given by*

$$P(\theta, L) = \begin{cases} m + \underline{\mu} + \frac{\bar{\sigma}^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \underline{\mu}\right), & \text{if} \quad \theta - \rho\tau^2 L \leq \underline{\mu} \\ m + \theta - \rho\tau^2 L, & \text{if} \quad \underline{\mu} < \theta - \rho\tau^2 L < \bar{\mu} \\ m + \bar{\mu} + \frac{\bar{\sigma}^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \bar{\mu}\right), & \text{if} \quad \theta - \rho\tau^2 L \geq \bar{\mu}, \end{cases} \tag{13}$$

speculator's demand given by

$$X_s(\theta, L) = \begin{cases} \frac{\rho\tau^2\sigma_L^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \underline{\mu}\right) + \frac{\bar{\sigma}^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2} L, & \text{if} \quad \theta - \rho\tau^2 L \leq \underline{\mu} \\ L, & \text{if} \quad \underline{\mu} < \theta - \rho\tau^2 L < \bar{\mu} \\ \frac{\rho\tau^2\sigma_L^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \bar{\mu}\right) + \frac{\bar{\sigma}^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2} L, & \text{if} \quad \theta - \rho\tau^2 L \geq \bar{\mu}, \end{cases} \tag{14}$$

and arbitrageur's demand given by

$$X_a(\theta, L) = \begin{cases} -\frac{\rho\tau^2\sigma_L^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \underline{\mu}\right), & \text{if} \quad \theta - \rho\tau^2 L \leq \underline{\mu} \\ 0, & \text{if} \quad \underline{\mu} < \theta - \rho\tau^2 L < \bar{\mu} \\ -\frac{\rho\tau^2\sigma_L^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \bar{\mu}\right), & \text{if} \quad \theta - \rho\tau^2 L \geq \bar{\mu}. \end{cases} \tag{15}$$

Equilibrium price function P of (13) is piecewise linear. It depends on θ and L only through the value of $\rho\tau^2 L - \theta$ and is strictly increasing in it. This implies that information revealed by equilibrium price function P is the same as observing realization of $\rho\tau^2\tilde{L} - \tilde{\theta}$ which in turn is the same as observing order flow $\tilde{L} - x_s(I_s^*, p)$ against the arbitrageur. This verifies again that (11) holds and shows that the price and the demand functions of Theorem 1 are indeed a rational expectations equilibrium.

Price function P is partially revealing, that is, it reveals signal θ to the uninformed arbitrageur only partially. If asset supply is deterministic, that is, if $\sigma_L^2 = 0$, then the equilibrium fully reveals signal θ.

We can use Theorem 1 to derive rational expectations equilibrium with no ambiguity. Setting $\underline{\mu}, \bar{\mu}, \bar{\sigma}^2$ and $\underline{\sigma}^2$ equal to the respective true values μ and σ^2, we obtain

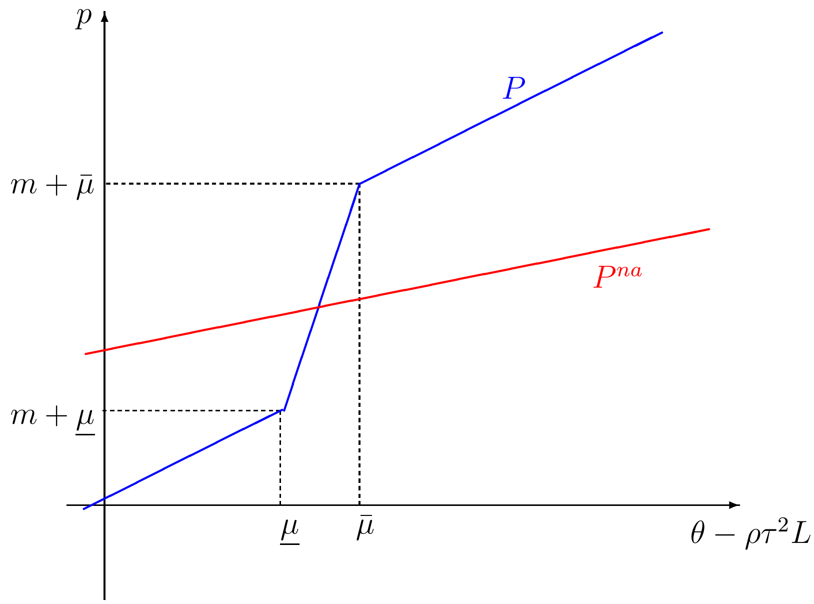

Figure 1: Equilibrium price functions P^{na} and P.

Corollary 2 *If there is no ambiguity, then the unique rational expectations equilibrium has linear price function given by*

$$P^{na}(\theta, L) = m + \mu + \frac{\sigma^2}{\sigma^2 + \rho^2\sigma_L^2\tau^4}\left(\theta - \rho\tau^2 L - \mu\right), \tag{16}$$

and demand functions given by

$$\begin{aligned} X_s^{na}(\theta, L) &= \frac{\rho\tau^2\sigma_L^2}{\sigma^2 + \rho^2\tau^4\sigma_L^2}(\theta - \mu) + \frac{\sigma^2}{\sigma^2 + \rho^2\tau^4\sigma_L^2} L \\ X_a^{na}(\theta, L) &= -\frac{\rho\tau^2\sigma_L^2}{\sigma^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \mu\right). \end{aligned}$$

The equilibrium of Corollary 2 is a modification of the one obtained by Vives (1995, Proposition 1). Equilibrium price function P^{na} is linear and partially revealing as long as the asset supply is non-deterministic. Otherwise, if $\sigma_L^2 = 0$, P^{na} is fully revealing. Conditions for partial and full revelation are therefore the same without ambiguity as they are with ambiguity. Figure 1 shows equilibrium price functions with and without ambiguity.

It is interesting to note that if there is ambiguity about the variance but not about the mean of the information signal, that is if $\underline{\mu} = \bar{\mu}$, then the equilibrium takes exactly the same form as the equilibrium with no ambiguity, but with variance of the signal set at the upper bound $\bar{\sigma}^2$. Loosely speaking, market behaves as if the variance were unambiguously known as the upper bound.

Equilibrium price function P^{na} equals the expected payoff conditional on arbitrageur's information I_a^*. Expression (16) can be decomposed in two parts: the first part, $m + \mu$, is the unconditional expected payoff and the remaining part is the *information premium* for arbitrageur's information. The premium is proportional to the order flow against the arbitrageur. It is positive when the order flow is positive, and negative (a discount) when the order flow is negative. The information premium increases with the variance of signal $\tilde{\theta}$, or, in other words, decreases with the signal's precision.

Equilibrium price function P can take one of three values as indicated in (13). If $\theta - \rho\tau^2 L \leq \underline{\mu}$, then the equilibrium price equals the minimum over the set of priors of expected payoff conditional on information I_a^*. In this case equilibrium order flow against the arbitrageur is positive, see (15). Similarly, the equilibrium price equals maximum expected payoff conditional on I_a^* when the order flow is negative. The price lies between the maximum and the minimum values when the order flow is zero. Therefore equilibrium prices satisfy a multiple-prior version of the zero-expected-profit condition. If the order flow against the arbitrageur is positive, equilibrium price is the sum of minimum expected payoff and information premium. The information premium is positive and proportional to the order flow. If the order flow is negative, the price is the sum of maximum expected payoff and negative information premium (i.e., discount). The discount is proportional to the order flow. Information premium per unit of order flow is greater when there is ambiguity than when there is no ambiguity.

4. Market Liquidity and Price Sensitivities

The rational expectations equilibrium (13) has the following interesting feature: If the arbitrageur participates in trading, then she is indifferent among all long or all short positions in the asset depending on whether the price equals her minimum or maximum expected payoff. Her trade is determined by matching the order flow of the speculator and noise traders. Thus the arbitrageur provides liquidity to the market whenever she participates in trading. She does not always participate in trading though. If θ and L are such that $\underline{\mu} \leq \theta - \rho\tau^2 L \leq \bar{\mu}$, then her asset demand is zero. The range of values of θ and L such that the arbitrageur does not trade is called the *regime of market illiquidity.* Market illiquidity arises only if there is ambiguity about the mean. A strong reason for there being ambiguity about the mean is that it is usually quite difficult to estimate expected payoff of an asset.

The way market illiquidity arises in our model is quite similar to how limited market participation occurs in equilibrium in Cao, Wang and Zhang's (2005) model of asset markets with symmetric information and deterministic supply. In Cao, Wang and Zhang (2005) there is heterogeneous ambiguity about the mean of the payoff and agents with the highest degree of ambiguity may choose not to trade in equilibrium. In our model, there is ambiguity of beliefs about the mean for uninformed arbitrageurs and no ambiguity for informed speculators after information transmission takes place. It is arbitrageurs who may choose not to trade.

Figure 1 indicates that equilibrium price function is more sensitive to changes in information and asset supply when market is illiquid than when it is liquid. We define price sensitivities as $\frac{\partial}{\partial\theta}P$ and $|\frac{\partial}{\partial L}P|$, where we take the absolute value of the latter to have positive number. Since information signal θ directly affects the asset payoff, the sensitivity of price to informational shocks (i.e., changes in information signal) can be considered as fundamental. In contrast, the sensitivity of price to supply shocks (i.e., changes in asset supply) is non-fundamental since asset supply changes are uncorrelated with the payoff.

Sensitivities of equilibrium price function P to changes in information and asset supply depend on whether the asset market is liquid or illiquid. They are

$$\frac{\partial}{\partial\theta}P(\theta, L) = \begin{cases} 1, & \text{if} \quad \underline{\mu} < \theta - \rho\tau^2 L < \bar{\mu}, \\ \frac{\bar{\sigma}^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}, & \text{otherwise.} \end{cases} \tag{17}$$

and

$$\left|\frac{\partial}{\partial L}P(\theta, L)\right| = \begin{cases} \rho\tau^2, & \text{if} \quad \underline{\mu} < \theta - \rho\tau^2 L < \bar{\mu} \\ \frac{\rho\tau^2\bar{\sigma}^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}, & \text{otherwise.} \end{cases} \tag{18}$$

Price sensitivities change in a discontinuous way as the market switches from one regime to the other.

It follows from (17) and (18) that

Proposition 3 *Assume that $\bar{\mu} > \underline{\mu}$. Sensitivities of equilibrium asset price to informational and supply shocks are higher when market is illiquid than when it is liquid.*

Reciprocals of price sensitivities are measures of market depth. Kyle (1985) identifies market depth as one of the main characteristics of liquidity. In Kyle's model with price-setting market maker, market depth is the reciprocal of sensitivity of price to changes in order flow. When market depth is high, or in other words, price sensitivity is low, market can absorb a shock to the order flow without large effect on the price. When all agents are price takers, as in our model, sensitivities of price to informational and supply shocks can be used to define market depth since these shocks induce changes in equilibrium order flow (see Vives (1985a)). Proposition 3 says that market depth is low when market is illiquid and high when it is liquid.

Price sensitivities or related measures of price fluctuations are often used in empirical research as measures of market liquidity. Pástor and Stambaugh (2003) construct a measure of liquidity for US stocks, both individual and aggregate. They show that their measure is a significant factor in explaining expected stock returns. They find several episodes of extremely low aggregate liquidity, including the October-1987 crash and the LTCM crisis of September 1998.

5. Effects of Ambiguity on Liquidity Risk, Price Sensitivity, and Trading Volume

In this section we study the effects of ambiguity on liquidity risk, price sensitivity, and trading volume. For each of these market characteristics we first compare equilibrium when there is no ambiguity with equilibrium under ambiguity. There is no ambiguity if arbitrageurs know the true distribution of $\tilde{\theta}$. Recall that the true distribution has mean μ and variance σ^2 such that $\mu \in [\underline{\mu}, \bar{\mu}]$ and $\sigma^2 \in [\underline{\sigma}^2, \bar{\sigma}^2]$. Second, we conduct comparative statics of changes in ambiguity. To fix ideas, we consider changes in the upper bounds of the intervals of ambiguous mean and variance of signal $\tilde{\theta}$. The higher is $\bar{\mu}$, the greater is ambiguity about the mean, keeping all other parameters fixed. The higher is $\bar{\sigma}^2$, the greater is ambiguity about variance.

The analysis of the effects of ambiguity has potential policy implications. Ambiguity of beliefs can be influenced in a variety of ways ranging from investors' education to payoff guarantees, transparency of disclosure, and microstructure of markets. Easley and O'Hara (2009) (see also Easley and O'Hara (2006)) provide an extensive discussion of these factors and how they affect ambiguity.

5.1. Liquidity Risk

Liquidity risk is measured by the probability of market illiquidity, that is, the probability that $\underline{\mu} \leq \tilde{\theta} - \rho\tau^2\tilde{L} \leq \bar{\mu}$ under true probability distribution of $\tilde{\theta}$ and $\tilde{L}$. Of course, liquidity risk is strictly positive only if there is ambiguity about the mean of $\tilde{\theta}$.

Proposition 4 *Liquidity risk increases with ambiguity about the mean but does not depend on ambiguity about the variance.*

Furthermore, liquidity risk decreases with variance τ^2, variance of asset supply σ_L^2, and speculator's risk aversion ρ.

5.2. Price Sensitivity

If there is no ambiguity, sensitivities of equilibrium price are

$$\frac{\partial}{\partial\theta}P^{na} = \frac{\sigma^2}{\sigma^2 + \rho^2\tau^4\sigma_L^2} \quad \text{and} \quad \left|\frac{\partial}{\partial L}P^{na}\right| = \frac{\rho\tau^2\sigma^2}{\sigma^2 + \rho^2\tau^4\sigma_L^2} \tag{19}$$

Comparing equations (17, 18) with (19) reveals that

Proposition 5 *Sensitivities of asset price to informational and supply shocks are greater in equilibrium under ambiguity than with no ambiguity.*

Sensitivities of equilibrium price depend on ambiguity about the mean only in so far as the regime of illiquidity expands with ambiguity about the mean. The values of price sensitivities when market is liquid and when it is illiquid remain unchanged. In other words, market depth remain unchanged in the two regimes. Price sensitivities increase (and market depth decreases) with ambiguity about the variance when market is liquid.

5.3. Trading Volume

The total trading volume is defined as

$$V(\theta, L) = |X_a(\theta, L)| + |X_s(\theta, L)| + |L| \tag{20}$$

and is a function of information signal and asset supply. It consists of trading volume of arbitrageur $|X_a(\theta, L)|$, trading volume of speculator $|X_a(\theta, L)|$ and the volume of supply $|L|$. The expected total trading volume $E[V(\tilde{\theta}, \tilde{L})]$ – with the expectation taken under the true distribution of $\tilde{\theta}$ and $\tilde{L}$ – is an ex-ante measure of trading volume.

First, we consider the effects of ambiguity about the mean. We have

Proposition 6 *Total trading volume $V(\theta, L)$ weakly decreases with ambiguity about the mean, for every θ and L. The expected total trading volume decreases with ambiguity about the mean.*

It follows that total trading volume is higher in equilibrium under ambiguity about the mean than with no ambiguity. The proof of Proposition 6 (see Appendix) shows that trading volume of the arbitrageur weakly decreases while trading volume of speculator may increase or decrease with ambiguity about the mean, depending on realization of information signal and supply. The former effect overweights that latter and the total trading volume decreases.

Figure 2 shows results of numerical simulations of expected trading volume. All graphs have $\bar{\mu}$ on the horizontal axes. The dotted red curve shows the expected trading volume of arbitrageur, the dashed blue curve shows the expected trading volume of speculator, and the orange curve shows the expected total volume. The true mean and variance of $\tilde{\theta}$ are $\mu = 1$ and $\sigma = 1$. Other parameters are $\underline{\mu} = 1$, $\bar{\sigma} = 1$, $\tau = 1$, $\sigma_L = 1$, and $\rho = 1$.

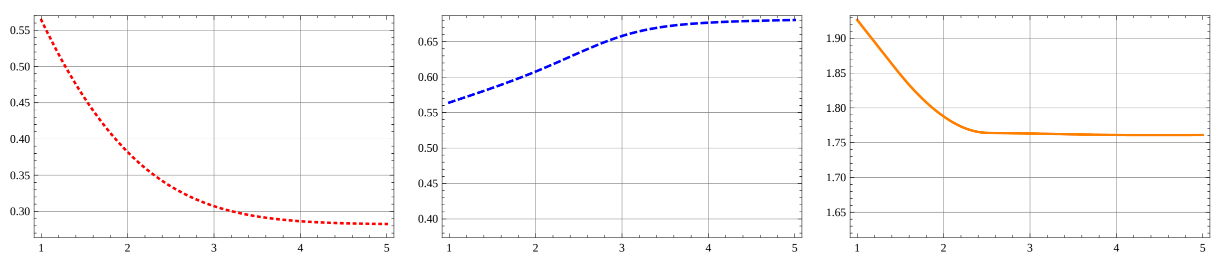

Figure 2: Expected trading volumes of arbitrageur, speculator, and total as functions of $\bar{\mu}$.

We turn our attention now to the effects of ambiguity about the variance. We have

Proposition 7 *Total trading volume $V(\theta, L)$ weakly decreases with ambiguity about the variance for every θ and L. The expected total trading volume decreases with ambiguity about the variance.*

It follows that total trading volume is higher in equilibrium under ambiguity about the variance than with no ambiguity.

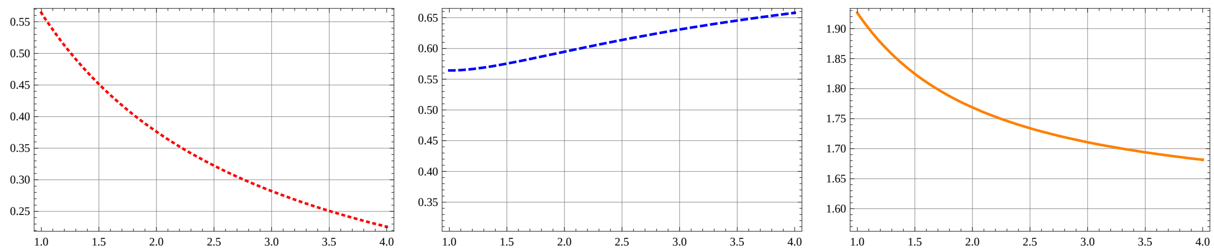

Figure 3: Expected trading volumes of arbitrageur, speculator, and total as functions of $\bar{\sigma}^2$.

Figure 3 shows results of numerical simulations of expected trading volumes of arbitrageur and speculator, and expected total trading volume as functions of $\bar{\sigma}^2$ (from left to right). It is assumed that there is no ambiguity about the mean. More precisely, we set $\bar{\mu} = \underline{\mu} = \mu = 1$. Other parameters are as in Figure 2.

Summarizing the results of the section we can say that high ambiguity about the mean leads to high liquidity risk and low trading volume, but it does not affect market depth. High ambiguity about the variance leads to low trading volume and low market depth, but it does not affect liquidity risk. Johnson (2008) argues that market depth and trading volume are positively related in cross-section of stocks although no such relation is observed in time-series data. Johnson's main finding is a positive relation between liquidity risk and trading volume, but his dynamic measure of liquidity risk is quite different from our measure which is the probability of market illiquidity.

6. Excess Volatility

The conventional efficient markets hypothesis asserts that changes in asset prices are driven by arrival of new information about future payoffs. If prices equal discounted expected value of future payoffs, then volatility of prices should be related to volatility of payoffs. Shiller (1981) and LeRoy and Porter (1981) developed a variance-bound inequality between the two volatilities and used it in empirical tests. They found systematic violations of the inequality in the U.S. stock markets. This *excess volatility* of stock prices has been confirmed by other studies such as

Campbell and Shiller (1988a, b), LeRoy and Parke (1992), Mankiw, Romer and Shapiro (1985, 1991).

Dow and Werlang (1992b) showed that the presence of ambiguous information may account for violations of the variance-bound inequality. They gave an example of an equilibrium in asset markets with investors having multiple-prior preferences such that the variance of asset prices exceeds the variance of dividends where variances are taken under some probability measure from the set of priors. If that measure represents the true distribution of payoffs and prices, then excess volatility would be observed. In the Dow and Werlang (1992b) example investors have symmetric information and asset supply is deterministic. We show that excess volatility may arise in a rational expectations equilibrium under ambiguity with asymmetric information and noisy asset supply.

First, we demonstrate that there is no excess volatility in rational expectations equilibrium with no ambiguity. Volatilities of payoff and price are measured by the respective variances. The variance of payoff $\tilde{v}$ is $\sigma^2 + \tau^2$. The variance of equilibrium price P^{na} given by (16) is

$$\mathrm{var}[P^{na}] = \frac{\sigma^4}{(\sigma^2 + \rho^2 \sigma_L^2 \tau^4)} \tag{21}$$

One can easily check that $\mathrm{var}[P^{na}] < \sigma^2 + \tau^2$, so that there is no excess volatility.

Excess volatility may arise in equilibrium if there is sufficiently high ambiguity about the variance of signal $\tilde{\theta}$. To illustrate this, suppose that there is ambiguity about the variance but not about the mean of $\tilde{\theta}$. Volatilities of payoff and price are measured by variances taken under the true distribution of $\tilde{\theta}$, with mean μ and variance σ^2. As discussed in Section 3, equilibrium price P is given by (16) with variance of signal set at $\bar{\sigma}^2$. The variance of P under the true distribution of signal with variance σ^2 is

$$\mathrm{var}_{\mu,\sigma}[P] = \frac{\bar{\sigma}^4}{(\bar{\sigma}^2 + \rho^2 \sigma_L^2 \tau^4)^2} (\sigma^2 + \rho^2 \sigma_L^2 \tau^4). \tag{22}$$

For large values of $\bar{\sigma}$, expression (22) approaches $\sigma^2 + \rho^2 \sigma_L^2 \tau^4$. If $\rho \tau \sigma_L > 1$, then this limiting value exceeds $\sigma^2 + \tau^2$ which is the variance of payoff under the true distribution. Thus, if $\rho \tau \sigma_L > 1$, then there is excess volatility for large enough ambiguity about the variance. This argument applies if there is small ambiguity about the mean of $\tilde{\theta}$.

Proposition 8 provides an explicit bound on variance $\bar{\sigma}$ that is is sufficient for excess volatility.

Proposition 8 *If* $\rho\tau\sigma_L > 1$ *and* $\bar{\sigma}^2 > A(\sigma)$, *then*

$$\text{var}_{\mu,\sigma}[P] \quad > \quad \sigma^2 + \tau^2 \tag{23}$$

for every $\mu \in [\underline{\mu}, \bar{\mu}]$.

Bound $A(\sigma)$ is defined by equation (39) in the Appendix. It depends on $\bar{\mu}$ and is a decreasing function of $\bar{\mu}$. This implies that high ambiguity about the mean makes excess volatility more likely.

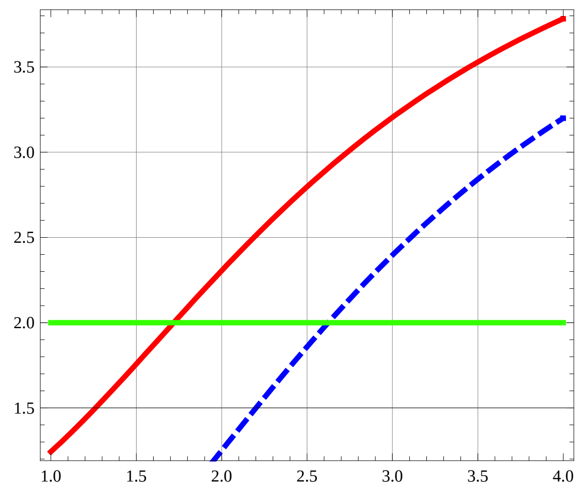

Figure 4: Excess volatility.

The possibility of excess volatility is illustrated by Figure 4 which has the upper bound $\bar{\sigma}$ of the interval of ambiguous variance on horizontal axis and variance of price on vertical axis. Solid red curve shows the variance $\text{var}_{\mu,\sigma}[P]$ as function of $\bar{\sigma}^2$ for $\mu = 1$ and $\sigma = 1$. Other parameters are $\bar{\mu} = 2$, $\underline{\mu} = 0$, $m = 0$, $\tau = 1$, $\sigma_L = 1$, and $\rho = 2$. The variance of payoff equals 2 and is marked by the green solid line. For values of $\bar{\sigma}^2$ greater than 1.72 the variance of price exceeds the variance of payoff and there is excess volatility. Dashed blue curve shows the variance $\text{var}_{\mu,\sigma}[P]$ for $\mu = 1$ and $\sigma = 1$ when there is no ambiguity about the mean, that is, when $\bar{\mu} = \underline{\mu} = 1$.

7. Concluding Remarks

Investors in our model have ambiguous prior beliefs about the asset payoff. Ambiguity is resolved for informed investors upon their observing information signal, but it remains present for uninformed investors. One could, of course, imagine different scenarios. Making $\tilde{\omega}$ ambiguous would make both informed and uninformed agents have ambiguous beliefs in rational expectations equilibrium. We have not been able to find a closed-form solution for such variation of the model.

The rational expectations equilibrium derived in this paper has a special feature that information revealed by equilibrium prices is the same with and without ambiguity. Condie and Ganguli (2007) give a robust example of an economy in which a rational expectations equilibrium is partially revealing under ambiguity and fully revealing with no ambiguity. Their example does not have CARA-normal specification, but a similar example can be constructed with such specification.

Appendix

Proof of Theorem 1. We derive $\min_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right]$ and $\max_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right]$ using (12). We have:

- If $\theta - \rho\tau^2 L < \underline{\mu}$, then

$$\begin{aligned}\min_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right] &= m + \underline{\mu} + \frac{\bar{\sigma}^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \underline{\mu}\right), \quad (24)\\ \max_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right] &= m + \bar{\mu} + \frac{\underline{\sigma}^2}{\underline{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \bar{\mu}\right). \quad (25)\end{aligned}$$

- If $\underline{\mu} \le \theta - \rho\tau^2 L \le \bar{\mu}$, then

$$\begin{aligned}\min_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right] &= m + \underline{\mu} + \frac{\underline{\sigma}^2}{\underline{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \underline{\mu}\right), \quad (26)\\ \max_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right] &= m + \bar{\mu} + \frac{\underline{\sigma}^2}{\underline{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \bar{\mu}\right). \quad (27)\end{aligned}$$

- If $\theta - \rho\tau^2 L > \bar{\mu}$, then

$$\min_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right] = m + \underline{\mu} + \frac{\underline{\sigma}^2}{\underline{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \underline{\mu}\right), \quad (28)$$

$$\max_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right] = m + \bar{\mu} + \frac{\bar{\sigma}^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \bar{\mu}\right). \quad (29)$$

Next, we calculate speculator's demand $x_s(I_s^*, p)$ using (10) for $p = \min_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right]$ and $p = \max_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right]$. Since $\min_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right]$ can take either one of two values (24) or (26), we obtain two possible values of speculator's demand:

- If $p = m + \underline{\mu} + \frac{\underline{\sigma}^2}{\underline{\sigma}^2+\rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \underline{\mu}\right)$, then

$$x_s(I_s^*, p) = \frac{\rho\tau^2\sigma_L^2}{\underline{\sigma}^2 + \rho^2\tau^4\sigma_L^2}(\theta - \underline{\mu}) + \frac{\underline{\sigma}^2}{\underline{\sigma}^2 + \rho^2\tau^4\sigma_L^2} L, \quad (30)$$

- If $p = m + \underline{\mu} + \frac{\bar{\sigma}^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \underline{\mu}\right)$, then

$$x_s(I_s^*, p) = \frac{\rho\tau^2\sigma_L^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2}(\theta - \underline{\mu}) + \frac{\bar{\sigma}^2}{\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2} L, \quad (31)$$

Similarly, $\max_{P\in\mathcal{P}} \mathrm{E}_P\left[\tilde{v}|I_a^*\right]$ can take one of two values (25) or (29). Speculator's demand is

- If $p = m + \bar{\mu} + \frac{\bar{\sigma}^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \bar{\mu}\right)$, then

$$x_s(I_s^*, p) = \frac{\rho\tau^2\sigma_L^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}\left(\theta - \bar{\mu}\right) + \frac{\bar{\sigma}^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}\, L, \tag{32}$$

- If $p = m + \bar{\mu} + \frac{\underline{\sigma}^2}{\underline{\sigma}^2+\rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \bar{\mu}\right)$, then

$$x_s(I_s^*, p) = \frac{\rho\tau^2\sigma_L^2}{\underline{\sigma}^2+\rho^2\tau^4\sigma_L^2}\left(\theta - \bar{\mu}\right) + \frac{\underline{\sigma}^2}{\underline{\sigma}^2+\rho^2\tau^4\sigma_L^2}\, L, \tag{33}$$

If $p = m + \underline{\mu} + \frac{\bar{\sigma}^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \underline{\mu}\right)$, then the net order flow of noise traders and the speculator $L - x_s(I_a^*, p)$ is positive. Since $p = \min_{\pi\in\mathcal{P}} \mathrm{E}_\pi\left[\tilde{v}|I_a^*\right]$ holds in this case, the arbitrageur's demand is arbitrarily positive, see (7). Thus $p = m + \underline{\mu} + \frac{\bar{\sigma}^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \underline{\mu}\right)$ is a rational equilibrium price for $\theta - \rho\tau^2 L \leq \underline{\mu}$. Similarly, $p = m + \bar{\mu} + \frac{\bar{\sigma}^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}\left(\theta - \rho\tau^2 L - \bar{\mu}\right)$ is the equilibrium price for $\theta - \rho\tau^2 L \geq \bar{\mu}$ with negative equilibrium demand of the arbitrageur. Finally, if $\underline{\mu} < \theta - \rho\tau^2 L < \bar{\mu}$, then $p = m + \theta - \rho\tau^2 L$ satisfies $\min_{\pi\in\mathcal{P}} \mathrm{E}_\pi\left[\tilde{v}|I_a^*\right] < p < \max_{\pi\in\mathcal{P}} \mathrm{E}_\pi\left[\tilde{v}|I_a^*\right]$ and is an equilibrium price with the corresponding arbitrageur's demand equal to zero. Summing up, the rational expectations equilibrium is as specified in (13), (14) and (15). $\square$

Proof of Proposition 4. The probability of market illiquidity (i.e., liquidity risk) under probability distribution of $\tilde{\theta}$ with mean μ and variance σ^2 is

$$\begin{aligned} Pr\left(\underline{\mu} < \tilde{\theta} - \rho\tau^2\tilde{L} < \bar{\mu}\right) &= \frac{1}{2}\left(\mathrm{erf}\left(\frac{\bar{\mu} - \mathrm{E}[\tilde{\theta} - \rho\tau^2\tilde{L}]}{\sqrt{2\mathrm{var}[\tilde{\theta} - \rho\tau^2\tilde{L}]}}\right) - \mathrm{erf}\left(\frac{\underline{\mu} - \mathrm{E}[\tilde{\theta} - \rho\tau^2\tilde{L}]}{\sqrt{2\mathrm{var}[\tilde{\theta} - \rho\tau^2\tilde{L}]}}\right)\right) \\ &= \frac{1}{2}\left(\mathrm{erf}\left(\frac{\bar{\mu} - \mu}{\sqrt{2(\sigma^2 + \rho^2\tau^4\sigma_L^2)}}\right) - \mathrm{erf}\left(\frac{\underline{\mu} - \mu}{\sqrt{2(\sigma^2 + \rho^2\tau^4\sigma_L^2)}}\right)\right) \end{aligned} \tag{34}$$

where erf is the Gauss error function defined by

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)du \tag{35}$$

for $x \geq 0$, and erf$(x) = -\mathrm{erf}(-x)$ for $x \leq 0$.

Clearly, $Pr\left(\underline{\mu} < \tilde{\theta} - \rho\tau^2\tilde{L} < \bar{\mu}\right)$ increases with $\bar{\mu}$. By differentiating (34) with respect to τ^2, σ_L^2 and ρ, it can be seen that $Pr\left(\underline{\mu} < \tilde{\theta} - \rho\tau^2\tilde{L} < \bar{\mu}\right)$ decreases with these parameters. □

Proof of Proposition 6. Let $V(\bar{\mu})$ denote the total trading volume (20) as function of the upper bound $\bar{\mu}$ of the interval of ambiguous mean of the signal. We need to show that V is a non-increasing function of $\bar{\mu}$ for $\bar{\mu} \geq \underline{\mu}$, for every (θ, L) (which we suppressed from the notation). We will repeatedly use (15) and (14) in our calculations.

We consider two cases:

- First, if $\theta - \rho\tau^2 L \leq \underline{\mu}$, then V does not depend on $\bar{\mu}$ for $\bar{\mu} \geq \underline{\mu}$. Hence, it is a constant function of $\bar{\mu}$.

- Second, if $\theta - \rho\tau^2 L > \underline{\mu}$, then

$$V(\bar{\mu}) = 2|L| \tag{36}$$

  for $\bar{\mu} \geq \theta - \rho\tau^2 L$, and

$$V(\bar{\mu}) = \begin{cases} 2\,|L|, & \text{if } \rho\tau^2\sigma_L^2(\theta - \bar{\mu}) + \bar{\sigma}^2 L < 0, \\ \frac{2\rho\tau^2\sigma_L^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}\,(\theta - \bar{\mu}) + \frac{\bar{\sigma}^2-\rho^2\tau^4\sigma_L^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}\,L + |L|, & \text{if } \rho\tau^2\sigma_L^2(\theta - \bar{\mu}) + \bar{\sigma}^2 L \geq 0, \end{cases} \tag{37}$$

  for $\bar{\mu} < \theta - \rho\tau^2 L$. In this case, V is a piece-wise linear continuous function of $\bar{\mu}$. The linear pieces are either constant or decreasing in $\bar{\mu}$.

This shows that $V(\bar{\mu})$ is a non-increasing function of $\bar{\mu}$ for $\bar{\mu} \geq \underline{\mu}$, for every (θ, L). Since V is decreasing in $\bar{\mu}$ for a set of (θ, L) that has strictly positive probability, the expected value $E[V(\bar{\mu})]$ is decreasing. □

To provide additional insight, we show that trading volume of the arbitrageur $\left|X_a(\theta, L; \bar{\mu})\right|$ is a non-increasing function of $\bar{\mu}$. If follows from (15) that $\left|X_a\right|$ does not depend on $\bar{\mu}$ if $\theta - \rho\tau^2 L \leq \underline{\mu}$. If $\theta - \rho\tau^2 L > \underline{\mu}$, then $\left|X_a\right|$ is a decreasing function of $\bar{\mu}$ for $\bar{\mu} \geq \theta - \rho\tau^2 L$ and is constant equal to zero for $\bar{\mu} < \theta - \rho\tau^2 L$. Trading volume of the speculator $\left|X_s\right|$ may increase or decrease with $\bar{\mu}$ depending on (θ, L).

Proof of Proposition 7. Let $V(\bar{\sigma})$ denote the total trading volume (20) as function of the upper bound $\bar{\sigma}$ of the interval of ambiguous variance of the signal. We need to show that $V(\bar{\sigma})$ is a non-increasing function of $\bar{\sigma}$ for $\bar{\sigma} \geq \underline{\sigma}$, for every (θ, L). We will repeatedly use (15) and (14) in our calculations.

We consider three cases:

- If $\theta - \rho\tau^2 L \leq \underline{\mu}$, then

$$V(\bar{\sigma}) = \begin{cases} 2\,|L|, & \text{if} \quad \rho\tau^2\sigma_L^2(\theta - \underline{\mu}) + \bar{\sigma}^2 L \geq 0, \\ -\frac{2\rho\tau^2\sigma_L^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}(\theta - \underline{\mu}) - \frac{\bar{\sigma}^2-\rho^2\tau^4\sigma_L^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2} L + |L|, & \text{if} \quad \rho\tau^2\sigma_L^2(\theta - \underline{\mu}) + \bar{\sigma}^2 L < 0, \end{cases}$$

  for $\bar{\sigma} \geq \underline{\sigma}$. If $\bar{\sigma}$ is such that $\rho\tau^2\sigma_L^2(\theta - \underline{\mu}) + \bar{\sigma}^2 L > 0$, then $\frac{\partial V}{\partial \bar{\sigma}^2} = 0$. If, on the other hand, $\rho\tau^2\sigma_L^2(\theta - \underline{\mu}) + \bar{\sigma}^2 L < 0$, then

$$\frac{\partial V}{\partial \bar{\sigma}^2} = \frac{2\rho\tau^2\sigma_L^2\left(\theta - \underline{\mu} - \rho\tau^2 L\right)}{\left(\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2\right)^2} \leq 0. \tag{38}$$

  V is a continuous function of $\bar{\sigma}$ and differentiable except for one point. Therefore it is non-increasing in $\bar{\sigma}$.

- If $\underline{\mu} < \theta - \rho\tau^2 L < \bar{\mu}$, then $V(\bar{\sigma}) = 2|L|$ and V does not depend on $\bar{\sigma}$.

- If $\theta - \rho\tau^2 L \geq \bar{\mu}$, then

$$V(\bar{\sigma}) = \begin{cases} 2\,|L|, & \text{if} \quad \rho\tau^2\sigma_L^2(\theta - \bar{\mu}) + \bar{\sigma}^2 L < 0, \\ \frac{2\rho\tau^2\sigma_L^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2}(\theta - \bar{\mu}) + \frac{\bar{\sigma}^2-\rho^2\tau^4\sigma_L^2}{\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2} L + |L|, & \text{if} \quad \rho\tau^2\sigma_L^2(\theta - \bar{\mu}) + \bar{\sigma}^2 L \geq 0. \end{cases}$$

  for $\bar{\sigma} \geq \underline{\sigma}$. If $\bar{\sigma}$ is such that $\rho\tau^2\sigma_L^2(\theta - \bar{\mu}) + \bar{\sigma}^2 L < 0$, then $\frac{\partial V}{\partial \bar{\sigma}^2} = 0$. If $\rho\tau^2\sigma_L^2(\theta - \bar{\mu}) + \bar{\sigma}^2 L \geq 0$, then

$$\frac{\partial V}{\partial \bar{\sigma}^2} = -\frac{2\rho\tau^2\sigma_L^2\left(\theta - \underline{\mu} - \rho\tau^2 L\right)}{\left(\bar{\sigma}^2 + \rho^2\tau^4\sigma_L^2\right)^2} \leq 0.$$

It follows that $V(\bar{\sigma})$ non-increasing in $\bar{\sigma}^2$ for every θ and L. Since V is decreasing in $\bar{\sigma}$ for a set of (θ, L) that has strictly positive probability, the expected value $E[V(\bar{\sigma})]$ is decreasing. $\square$

Proof of Proposition 8. We first give a definition of bound $A(\sigma)$:

$$A(\sigma)=\frac{\rho^2\tau^2\sigma_L^2\Big((2-b)\sqrt{(\sigma^2+\tau^2)\left(\sigma^2+\rho^2\tau^4\sigma_L^2\right)}+\Big((2-b)\sigma^2+(2-b\rho^2\tau^2\sigma_L^2)\tau^2\Big)\Big)}{2\Big(\rho^2\tau^2\sigma_L^2-1\Big)}, \tag{39}$$

where b is

$$b=\operatorname{erf}\left(\frac{\bar{\mu}-\underline{\mu}}{\sqrt{2(\sigma^2+\rho^2\tau^4\sigma_L^2)}}\right), \tag{40}$$

and erf is the Gauss error function (35). $A(\sigma)$ is a decreasing function of $\bar{\mu}$.

The proof of Proposition 8 relies on the following lemma:

Lemma I. *Let $\tilde{Z}$ be a normally distributed random variable and $g:\Re\to\Re$ a differentiable function. Suppose that $\operatorname{var}[g(\tilde{Z})]<\infty$ and $\operatorname{E}[|g'(\tilde{Z})|]<\infty$, where g' is the derivative of g. Then*

$$\operatorname{var}[g(\tilde{Z})]\ \geq\ \Big(\operatorname{E}[g'(\tilde{Z})]\Big)^2\ \operatorname{var}(\tilde{Z}). \tag{41}$$

Further, (41) holds with equality if and only if g is linear.

The proof can be found in Cacoullos (1982), Proposition 3.2. Lemma I continues to hold for functions g that are differentiable except for a finite set of points.

We apply Lemma I to $\tilde{Z}=\tilde{\theta}-\rho\tau^2\tilde{L}$ and $g(\tilde{Z})=P(\tilde{\theta},\tilde{L})$ given by (13). Using (41), we obtain

$$\begin{aligned}
&\operatorname{var}_{\mu,\sigma}\left[P(\tilde{\theta},\tilde{L})\right]\\
\geq\ &\frac{1}{2\pi}\left(\frac{\bar{\sigma}^2}{\bar{\sigma}^2+\rho^2\tau^4{\sigma_L}^2}\left(\int_{-\infty}^{\underline{\mu}}e^{-\frac{(Z-\mu)^2}{2(\sigma^2+\rho^2\tau^4{\sigma_L}^2)}}dZ+\int_{\bar{\mu}}^{\infty}e^{-\frac{(Z-\mu)^2}{2(\sigma^2+\rho^2\tau^4{\sigma_L}^2)}}dZ\right)+\int_{\underline{\mu}}^{\bar{\mu}}e^{-\frac{(Z-\mu)^2}{2(\sigma^2+\rho^2\tau^4{\sigma_L}^2)}}dZ\right)^2\\
=\ &\left(\frac{2\bar{\sigma}^2+B\rho^2\tau^4\sigma_L^2}{2(\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2)}\right)^2\left(\sigma^2+\rho^2\tau^4\sigma_L^2\right),
\end{aligned} \tag{42}$$

where

$$B\ =\ \operatorname{erf}\left(\frac{\bar{\mu}-\mu}{\sqrt{2(\sigma^2+\rho^2\tau^4\sigma_L^2)}}\right)\ +\ \operatorname{erf}\left(\frac{\mu-\underline{\mu}}{\sqrt{2(\sigma^2+\rho^2\tau^4\sigma_L^2)}}\right). \tag{43}$$

For every $\mu\in[\underline{\mu},\bar{\mu}]$, it holds $B\geq b$, where b is defined by (40). Thus (42) implies

$$\operatorname{var}_{\mu,\sigma}\left[P(\tilde{\theta},\tilde{L})\right]\geq\left(\frac{2\bar{\sigma}^2+b\rho^2\tau^4\sigma_L^2}{2(\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2)}\right)^2\left(\sigma^2+\rho^2\tau^4\sigma_L^2\right). \tag{44}$$

Calculation shows that inequality

$$\left(\frac{2\bar{\sigma}^2+b\rho^2\tau^4\sigma_L^2}{2(\bar{\sigma}^2+\rho^2\tau^4\sigma_L^2)}\right)^2\left(\sigma^2+\rho^2\tau^4\sigma_L^2\right)>\sigma^2+\tau^2 \tag{45}$$

holds if

$$\sigma^2 \geq A(\sigma) \quad \text{and} \quad \rho\tau\sigma_L \; > \; 1. \tag{46}$$

Using (44) and (45), we conclude that (46) implies that

$$\mathrm{var}_{\mu,\sigma}[P(\tilde{\theta}, \tilde{L})] \; > \; \sigma^2 + \tau^2$$

for every $\mu \in [\underline{\mu}, \bar{\mu}]$. $\square$

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