Speculative Trade under Ambiguity *

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Abstract: Ambiguous beliefs may lead to speculative trade and speculative bubbles. We demonstrate this by showing that the classical Harrison and Kreps (1978) example of speculative trade among agents with heterogeneous beliefs can be replicated with agents having common but ambiguous beliefs. More precisely, we show that the same asset prices and pattern of trade can be obtained in equilibrium with agents' having recursive multiple-prior expected utilities with common set of priors.

While learning about the true probabilities of dividends makes speculative bubbles vanish in the long run under heterogeneous beliefs, it may not do so under common ambiguous beliefs. Ambiguity need not disappear with learning over time, and speculative bubbles may persist.

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1. Introduction.

Ambiguous beliefs may lead to speculative trade and speculative bubbles. We demonstrate this by showing that the classical Harrison and Kreps (1978) example of speculative trade among agents with heterogeneous beliefs can be replicated with agents having common ambiguous beliefs. More precisely, we show that the same asset prices and speculative pattern of trade can be obtained in equilibrium with agents' having recursive multiple-prior expected utilities with common set of probabilities.

The key question of Harrison and Kreps (1978) was whether equilibrium prices in asset markets can persistently exceed all agents' valuations of the asset where valuation is defined by what an agent would be willing to pay if obliged to hold the asset forever. If price exceeds all valuations, then agents who buy the asset must intend to sell it in the future. Agents trade for short-term gain and hence engage in speculation. Harrison and Kreps considered a model of infinite-time asset markets where risk-neutral agents have heterogeneous beliefs about asset dividends and short selling is prohibited. Agents' beliefs exhibit perpetual switching: there is no single agent who is more optimistic at all future dates and states than other agents about next period dividends of the asset. In equilibrium, the agent who has the most optimistic belief buys the asset and agents with less optimistic beliefs sell the asset if they have some holdings from previous date. As beliefs keep switching, persistent speculative trade emerges.

Equilibrium asset price p_t at date t in the Harrison and Kreps (1978) model satisfies the relationship

$$p_t = \max_i \beta E_t^i [p_{t+1} + x_{t+1}], \tag{1}$$

where x_{t+1} denotes dividend at date $t+1$, E_t^i stand for date-t conditional expectation under agent's i belief, and β is a discount factor. The maximizing belief in (1) is the belief of the agent who is most optimistic about next period price plus dividend, and she holds the asset. Less optimistic agents have zero holdings and their beliefs give expected value of next period price plus dividend lower than price p_t. It turns out that the agent who is most optimistic about next period dividend is also most optimistic about price plus dividend in equilibrium. As the ranking of beliefs switches, holdings on the asset switch as well. Owning to risk-neutrality, agents' valuations of the asset are simply sums of discounted expected future dividends under their beliefs, that is $\sum_{\tau>t} \beta^{\tau-t} E_t^i[x_\tau]$. With perpetual switching of optimism, it follows from (1) that asset price p_t strictly exceeds every agent's valuation at every date and state. The difference between the asset price and the highest valuation is the speculative bubble.

We consider the same model of asset markets as Harrison and Kreps (1978) except for the specification of agents' beliefs. Instead of heterogeneous but exact beliefs, agents in our model have common ambiguous beliefs described by sets of one-period-ahead probabilities. Their decision criterion is the recursive multiple-prior expected utility - an extension of Gilboa and Schmeidler (1989) maxmin criterion to dynamic setting due to Epstein and Schneider (2003). Our key observation is that equilibrium pricing relationship (1) continues to hold with expectation E_t^i of agent i being now the one-period-ahead belief that minimizes expected value of that agent's date-$(t+1)$ continuation utility over the set of multiple probabilities. We call those probabilities effective one-period-ahead beliefs, and they feature in the valuation of agents' willingness to pay for the asset if obliged to hold it forever. If there is sufficient heterogeneity of agents' equilibrium consumption plans, then effective beliefs have the switching property and equilibrium prices strictly exceed all agents' asset valuations as in Harrison and Kreps (1978).[1] Further, there is speculative trade. Heterogeneity of equilibrium consumption is generated by heterogeneous endowments. Initial endowments play a critical role under common ambiguous beliefs, in contrast to the case of heterogeneous beliefs where they do not matter.

Two objections have been frequently raised against Harrison and Kreps' model of speculative trade. The first is that it departs from the common-prior assumption. This objection does not apply to our model. Agents in our model have common priors, or more precisely, common set of priors. The second is that agents have dogmatic beliefs and do not learn from observations of realized dividends over time. Learning and updating of beliefs is significantly different under ambiguity than with no ambiguity. Depending on the interaction between sources of uncertainty of dividends over time, ambiguity about dividends may or may not fade away in

[1]See Werner (2019) for a general condition on agents' heterogeneous beliefs which gives rise to speculative bubbles.

the long run. [2] The dividend process in our version of Harrison and Kreps' model can be thought of as resulting from sequences of indistinguishable but unrelated experiments, see Epstein and Schneider (2003b). Such experiments give rise to persistent ambiguity that is unaffected by learning.

In their comprehensive study of the dot.com bubble of 2000-2001 Ofek and Richardson (2003) concluded that short-sales restrictions and heterogeneity of investors' beliefs were the main reasons for the dramatic rise and fall of prices of internet stocks during that period. Short sales restrictions on internet stocks were particularly stringent because of the so-called lockups. Hong, Scheinkman, and Xiong (2006) offer a formal analysis of the dot.com bubble using a asset market model with heterogeneous beliefs and short-sales restrictions. The main argument in support of belief heterogeneity in Ofek and Richardson (2003) was relatively low level of institutional holdings of internet stocks. Individual investors tend to have more diverse beliefs. Yet, it is hard to believe that investors could have so diverse beliefs over a relatively long period of time. The argument of merging of opinions implies that diverse beliefs should quickly disappear. Our findings offer a different interpretation of Ofek and Richardson's analysis. Instead of being diverse, investors beliefs could have been ambiguous and imprecise, but common. Majority of dot.com stocks were new to the market justifying potential ambiguity of investors beliefs. Ambiguity of beliefs could persist over a long period of time.

Xiong and Yu (2011) argue that the Chinese warrant bubble of 2005-2008 can be attributed to heterogeneous beliefs and short-sales restrictions. Heterogeneous beliefs in that episode pertain to tail risk of deeply out-of-the-money options (or warrants). Those options were new products on the market, and ambiguous or imprecise beliefs appear more plausible than extremely diverse heterogeneous beliefs.

Ambiguity of beliefs has been primarily associated in the existing literature with reduction of trade in asset markets. This stands in stark contrast to this paper's finding that ambiguous beliefs may lead to speculative and excessive trade. We review the existing literature at the end of this section.

The paper is organized as follows. In Section 2 we review the example of Harrison and Kreps (1978) of speculative trade under heterogeneous beliefs. Section 3

[2] For a model of learning with multiple priors that may leave some ambiguity remaining in the long run, see Epstein and Schneider (2007)

introduces dynamic asset markets with no short-sales and recursive multiple-prior utilities under ambiguity. We provide characterizations of optimal portfolio choices and equilibrium asset prices. In Section 4 we show how the same asset prices and asset holdings as in the Harrison Kreps's example of Section 2 can be obtained in equilibrium with recursive multiple-prior utilities and common sets of beliefs. Section 5 contains concluding remarks concerning robustness of the example and learning with ambiguous beliefs.

Related literature: This paper addresses primarily two distinct strands of the literature in financial economics. The first is about speculative trade and speculative bubbles in asset markets with heterogeneous beliefs and short-sales restrictions. It has its origin in Harrison and Kreps (1978). In Harrison and Kreps (1978), traders beliefs are heterogeneous and dogmatic. They remain unchanged regardless of any observations of dividend histories. In Hong, Scheinkman and Xiong(2006) (see also Scheinkman and Xiong (2003)), heterogeneous beliefs result from agents' being overconfident about the precision of their observations of public signals. They are too optimistic after good signals and too pessimistic after bad signals. Morris (1996) introduced learning in the model of speculative trade. He showed that if agents have heterogeneous prior beliefs in a parametric model of Bayesian learning, then their posterior beliefs exhibit a switching property that leads to speculative trade. Werner (2019) develops a general theory of speculative trade and speculative bubbles with heterogeneous beliefs and short sales restrictions. He identifies a sufficient condition on agents' beliefs for speculative bubble and studies asymptotic properties of bubbles under Bayesian learning.

Implications of ambiguous beliefs (or ambiguity aversion) on equilibrium in asset markets has been studied in a number of papers. The most prevailing conclusion has been that ambiguity impedes trade. It is particularly transparent in the literature on static equilibrium models. Inspired by the portfolio-inertia result of Dow and Werlang (1992), the literature has strived to demonstrate that non-participation in trade by some agents with ambiguous beliefs may arise in equilibrium. In Cao, Wang and Zhang (2005) agents have heterogeneous ambiguity and those with the highest degree of ambiguity opt out of trading risky assets in equilibrium. [3] Illeditsch (2011) demonstrates portfolio inertia and excess price volatility in equilibrium when investors receive ambiguous information. Mukerji and Tallon (2001) show that ambiguous beliefs concerning idiosyncratic risk may lead to break down of trade of some assets.

In dynamic models, Uppal and Wang (2003) show (in a continuous-time model with multiplier utilities) that differential ambiguity across assets leads to reduction of investment in most ambiguous assets and under-diversification of portfolios. There have been numerous studies of dynamic asset markets with recursive multiple-prior utility functions with different objectives. Epstein and Wang (1994) consider representative agent's setting and argue that equilibrium asset prices may be indeterminate and therefore more volatile. Condie (2008) considers complete markets for Arrow securities and studies survival of agents with recursive multiple-prior utilities. Hara and Kajii (2006) consider two-period asset markets with CARA utilities and ambiguous beliefs, and show that equilibrium risk-free return in incomplete markets is higher than if markets are complete.

2. Speculation under Heterogeneous Beliefs.

The following example is due to Harrison and Kreps (1978). There is a single infinitely-lived asset with uncertain dividend equal to either 0 (low dividend) or 1 (high dividend) at every date $t \geq 1$. Date-0 dividend is equal to zero. Let $S = \{0, 1\}$ be the set of states for every date. There are two agents who perceive the dividend process $\{x_t\}$ as Markov chain with different transition probabilities, that is, they have *heterogeneous beliefs*. Agents' matrices of transition probabilities are

$$Q_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \tag{2}$$

where the first column and the first row correspond to the state of zero dividend. The key feature of transition probabilities (2) is the property of switching beliefs: Agent 1 is more optimistic than agent 2 about next period high dividend when current dividend is zero, while it is the other way round when current dividend is

[3] A related CARA-normal model has been considered by Easley and O'Hara (2009).

one.

Agents are risk-neutral with utility functions over infinite-time consumption plans given by the discounted expected value

$$u^i(c) = E^i[\sum_{t=0}^{\infty} \beta^t c_t], \tag{3}$$

where E^i denotes expectation under the unique probability measure on S^∞ derived from transition probabilities Q_i. The common discount factor is $\beta = 0.75$. Consumption endowments don't matter and are left unspecified. The asset supply is normalized to one share which is initially held by agent 1. Short selling of the asset is prohibited in that there is zero short-sales constraint.

In equilibrium, the agent who is more optimistic at any date and state holds the asset and the price reflects his one-period valuation of the payoff. The less optimistic agent wants to sell the asset short and ends up with zero holding because of the short-sales constraint. There exists a stationary equilibrium with asset prices that depend only on the current dividend. Equilibrium prices, denoted by $p(0)$ and $p(1)$, obtain from the first-order conditions for the respective optimistic agent,

$$p(0) = \beta[\frac{1}{2}p(0) + \frac{1}{2}(p(1) + 1)] \tag{4}$$

$$p(1) = \beta[\frac{1}{4}p(0) + \frac{3}{4}(p(1) + 1)] \tag{5}$$

They are

$$p(0) = \frac{24}{13}, \qquad p(1) = \frac{27}{13}. \tag{6}$$

Security holdings are

$$h^1(0) = 1, \quad h^1(1) = 0, \quad h^2(0) = 0, \quad h^2(1) = 1. \tag{7}$$

The first-order conditions for agent 2 when the dividend is zero and for agent 1 when the dividend is one hold as strict inequalities. Transversality conditions hold, too.

The discounted expected value of the asset's future dividends at date t under agent's i beliefs is

$$F_t^i(s) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E^i[x_\tau | x_t = s] \tag{8}$$

for $s = 0, 1$. The value $F_t^i(s)$ does not depend on t and we drop subscript t from the notation. Because of linearity of utility functions, discounted expected value of dividends is the agent's willingness to pay for the asset if obliged to hold forever. Therefore $F^i(s)$ may be called the *fundamental value.*[4] Elementary algebra shows that

$$F^1(0) = \frac{4}{3}, \qquad F^1(1) = \frac{11}{9}, \tag{9}$$

$$F^2(0) = \frac{16}{11}, \qquad F^2(1) = \frac{21}{11}. \tag{10}$$

It holds

$$p(0) > F^i(0) \text{ and } \ p(1) > F^i(1)$$

for $i = 1, 2$. Thus, the asset price strictly exceeds every agent's fundamental value. The difference between the price and the maximum of fundamental values is termed *speculative bubble.*

3. Dynamic Asset Market with Ambiguous Beliefs and No Short-Sales.

The set of possible states at each date is an arbitrary finite set S. The product set S^∞ represents all sequences of states. We use s^t the denote the partial history $(s_0, \ldots, s_t)$ through date t. Partial histories are date-t events. The set S^∞ together with the σ-field Σ of products of subsets of S is the measurable space describing the uncertainty. Consumption plans are positive and adapted processes on the state-space S^∞. There is a single asset with date-t dividend x_t that depends only on the current state s_t.

Agents have common ambiguous beliefs about states that are described by sets of transition (or one-period-ahead) probabilities on S. The recursive multiple-prior expected utility, with linear period utility, is defined by

$$u_t(c, s^t) = c_t(s^t) + \beta \min_{P \in \mathcal{P}(s_t)} E_P[u_{t+1}(c)|s^t], \tag{11}$$

where $\mathcal{P}(s_t)$ is the common set of transition probabilities. $\mathcal{P}(s_t)$ is assumed to depend only on the current state s_t. Date-0 utility function implied by the recursive

[4]Needless to say, this is a different notion of fundamental value than the one used in the literature on rational price bubbles, see Section 3.

relation (11) is

$$u_0(c) = \min_{\pi \in \Pi} E_\pi[\sum_{t=0}^{\infty} \beta^t c_t],$$

where Π is a set of probabilities on S^∞ such that conditional one-period-ahead probabilities at any date t are $\mathcal{P}(s_t)$, see Epstein and Schneider (2003). There are I agents. Endowments e_t^i are positive, and depend only on the current state s_t. Initial holdings of the asset are $\hat{h}_0^i \geq 0$. The supply of the asset $\hat{h}_0 = \sum_i \hat{h}_0^i$ is strictly positive.

Agent i faces the following budget and portfolio constraints

$$c(0) + p(0)h(0) \leq e^i(0) + p(0)\hat{h}_0^i, \tag{12}$$

$$c(s^t) + p(s^t)h(s^t) \leq e^i(s^t) + [p(s^t) + x(s^t)]h(s_-^t) \quad \forall s^t, \tag{13}$$

$$h(s^t) \geq 0, \quad \forall s^t, \tag{14}$$

where s_-^t is the predecessor event of s^t at $t-1$. Condition (14) is the short-sales constraint.

An equilibrium consists of prices p and consumption-portfolio allocation $\{c^i, h^i\}$ such that plans (c^i, h^i) are optimal and markets clear. Market clearing is

$$\sum_i c_t^i = \bar{e}_t^i + \hat{h}_0 x_t, \text{ and } \sum_i h_t^i = \hat{h}_0,$$

for every t.

The portfolio choice problem can be characterized by the following Bellman equation:

$$V^i(h, s) = \max_{c, h'} \{c + \beta \min_{P \in \mathcal{P}(s)} E_P[V^i(h', \cdot)|s]\} \tag{15}$$

$$\text{s.t } c + p(s)h' \leq e^i(s) + [p(s) + x(s)]h, \quad h' \geq 0. \tag{16}$$

If $c^i = \{c_t^i\}$ and $h^i = \{h_t^i\}$ are sequences of consumption and asset holdings generated from the Bellman equation (15), then $V^i(h_{t-1}^i, s^t) = u_t(c^i, s^t)$.

Value function V^i is concave in h. Further, it is differentiable as long as the solution $(h^i(s), c^i(s))$ is interior, i.e., $h^i(s) > 0$ and $c^i(s) > 0$. It is so because the Lagrange multiplier of the budget constraint (16) is unique. This follows from Marimon and Werner (2019, Corollary 2). The unique multiplier is equal to 1 because of the linearity with respect to c. Therefore $\partial V^i(h,s) = [p(s) + x(s)]$, so that V^i is in fact linear in h whenever solutions are interior.

The first-order condition for optimal asset holding $h^i(s^t) > 0$ generated by (15) is

$$p(s^t) = \beta E_{P^i(s^t)}[(p_{t+1} + x_{t+1})|s^t], \tag{17}$$

for some probability measure $P^i(s^t) \in \mathcal{P}(s_t)$ such that

$$P(s^t) \in \operatorname{argmin}_{P \in \mathcal{P}(s_t)} E_P[V^i(h^i, \cdot)|s^t]. \tag{18}$$

Since $V^i(h^i_t, s_{t+1}) = u_t(c^i, s_{t+1})$, we can equivalently write

$$P^i(s^t) \in \operatorname{argmin}_{P \in \mathcal{P}(s_t)} E_P[u_{t+1}(c^i)|s^t]. \tag{19}$$

Probability measure $P^i(s^t)$ satisfying (19) is called the agent's *effective belief* at c^i. Condition (17) implies that

$$\min_{P \in \mathcal{P}(s_t)} \beta E_P[(p_{t+1} + x_{t+1})|s^t] \le p(s^t) \le \max_{P \in \mathcal{P}(s_t)} \beta E_P[(p_{t+1} + x_{t+1})|s^t] \tag{20}$$

see Epstein and Wang (1994). The respective condition for a solution to (15) with binding short-sales constraint, i.e. $h^i(s) = 0$, is

$$p(s^t) \ge \beta E_{P^i(s^t)}[(p_{t+1} + x_{t+1})|s^t] \tag{21}$$

for some effective belief $P^i(s^t) \in \mathcal{P}(s_t)$ such that (19) holds.

Equilibrium asset price $p(s^t)$ in the market with no short-sales satisfies the relation

$$p(s^t) = \beta \max_i E_{P^i(s^t)}[p_{t+1} + x_{t+1}|s^t] \tag{22}$$

for every s^t, where probability vector $P^i(s^t)$ is the agent's i effective belief at c^i. Agents whose effective beliefs are the maximizing one on the right-hand side of (22) hold the asset in s^t while the other agents whose belief gives lower expectation have zero holding.

Let $\hat{P}(\cdot|s^t)$ be the maximizing probability in (22). We call $\hat{P}(\cdot|s^t)$ the *market belief* at s^t. The probability measure $\hat{P}$ on S^∞ derived from one-period-ahead probabilities $\hat{P}(\cdot|s^t)$ is a risk-neutral pricing measure (or state-price process) for p. Since the asset is in strictly positive supply, the standard no-bubble theorem implies that equilibrium price of the asset is equal to the infinite sum of discounted expected dividends under the market belief (see Werner (2019) for details). That is,

$$p(s^t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\hat{P}}[x_\tau | s^t], \tag{23}$$

for every s^t.

The fundamental value of the asset is the agent's willingness to pay if obliged to hold it forever. Since effective beliefs are the willingness to pay for holding the asset over one period, it follows that the discounted sum of expected dividends under agent's i effective beliefs is the fundamental value. That is,

$$F^i(s^t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\hat{P}^i}[x_\tau | s^t], \tag{24}$$

where $\hat{P}^i$ is the probability measure on S^∞ derived from one-period-ahead effective beliefs $P^i(s^t)$. It follows from (22) that

$$p(s^t) \geq F^i(s^t), \tag{25}$$

for every i, every s^t. Thus the asset price exceeds every agent's fundamental valuation.

We say that there is *speculative bubble* in event s^t, if

$$p(s^t) > max_i F^i(s^t). \tag{26}$$

If (26) holds, then the agent who buys the asset at s^t pays the price exceeding her willingness to pay if she were to hold the asset forever. This means, of course, that she intends to sell the asset at a later date. Thus, speculative bubble implies speculative trade.

A sufficient condition for the existence of speculative bubble is that there is no valuation dominant agent, see Theorem 1 in Werner (2019). Agent i is *valuation dominant* in s^t if

$$F^i(s^\tau) \geq max_j F^j(s^\tau), \tag{27}$$

for every event s^τ, $\tau \geq t$, which is a successor of s^t. For short, we say that there is *valuation switching* at s^t, if there is no valuation dominant agent. Thus valuation switching is a sufficient condition for speculative bubbles in equilibrium. It is not a necessary condition though. In the example of Section 2 of a market with heterogeneous beliefs, agent 2 is valuation dominant and yet there is speculative bubble at every date. [5]

Note that valuation switching with ambiguous beliefs is a condition on endogenous effective beliefs in equilibrium. This is in contrast to heterogeneous unambiguous beliefs where the condition pertains to exogenous beliefs.

4. Speculation under Ambiguous Beliefs.

Consider the asset with dividends equal to 0 or 1 at every date as in Section 2. Suppose now that agents have common ambiguous beliefs about the dividend process that are described by sets of transition (or one-period-ahead) probabilities. If the current dividend is zero, the set of transition probabilities is the convex hull of two probability vectors $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{2}{3}, \frac{1}{3})$ from eq. (2). If the current dividend is one, the set of transition probabilities is the convex hull of $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{4}, \frac{3}{4})$, again from eq. (2).

Agents' endowments are uncertain. The endowment e_t^1 of agent 1 can take two possible values 5 or 10 at every date $t \geq 1$. Agent's 2 endowment is $e_t^2 = 15 - e_t^1$. Thus endowments are negatively comonotone - one is high when the other is low.[6] Agents have ambiguous beliefs about the endowment process, too. The time- and state-independent set of one-period-ahead probabilities of endowment e_t^1 is taken to be the whole simplex of probabilities, that is, the convex hull of two extreme probability vectors $(0, 1)$ and $(1, 0)$, where the first coordinate is the probability of $e_t^1 = 5$.

The joint process (x_t, e_t^1) takes four possible values $(0, 5), (0, 10), (1, 5), (1, 10)$ for every date $t \geq 1$. Those four values are the set of states S. Date-0 state is $(0, 10)$. Agents' common ambiguous beliefs about the dividend-endowment process are constructed from above specified marginal ambiguous beliefs in the following

[5]Further sufficient condition in the special case of heterogeneous beliefs with two-state Markov process of dividends can be found in Slawski (2008).

[6]The symmetry of endowments is not important. See Section 6 for discussion.

way: If the current dividend is zero, the set of transition probabilities on S is

$$\mathcal{P}(0) = co\ \{(\frac{1}{2}, 0, \frac{1}{2}, 0),\ (0, \frac{2}{3}, 0, \frac{1}{3})\}. \tag{28}$$

That is, $\mathcal{P}(0)$ is the convex hull of two probability vectors each of which is independent products of extreme probabilities in the sets of priors on dividends and on endowments. The first is product of $(\frac{1}{2}, \frac{1}{2})$ and $(1, 0)$, that is low probability of zero dividend and high probability of low endowment $e^1 = 5$. The second is product of $(\frac{2}{3}, \frac{1}{3})$ and $(0, 1)$ - high probability of zero dividend and low probability of low endowment $e^1 = 5$.

A simple Ellsberg urn experiment which generates the set of probabilities $\mathcal{P}(0)$ is as follows (see Couso et al (1999), Example 5): There are two urns, each with red and blue balls of unknown fractions of the total number of balls. Dividends are generated by drawing a ball from the first urn with either $\frac{1}{2}$ or $\frac{2}{3}$ fraction of red balls. Red ball indicates zero dividend. Endowments are generated by drawing from the second urn with either all or none red balls. Red ball indicates endowment $e^1 = 5$ in the second urn. Balls are drawn independently from the two urns, but it is known that having more red balls in one urn implies having less in the other. This implies that only the two joint probabilities seen in (28) are possible.

This experiment and the resulting set of joint probabilities reflect what is called independence in the selection, see Couso et al (1999). The experiments are independent, but there is an interaction between them that leads to the specific selection from among ambiguous marginal priors. Independence in the selection implies epistemic irrelevance of the experiment generating dividends to the experiment generating endowments. This in turn means that the set of conditional probabilities (obtained by Bayes' rule) of endowments conditional on dividends equals the given set of marginal probabilities of endowments.[7]

If the current dividend is one, the set of transition probabilities is

$$\mathcal{P}(1) = co\ \{(\frac{2}{3}, 0, \frac{1}{3}, 0),\ (0, \frac{1}{4}, 0, \frac{3}{4})\}. \tag{29}$$

The first probability vector is the product of $(\frac{2}{3}, \frac{1}{3})$ and $(1, 0)$ while the second is the product of $(\frac{1}{4}, \frac{3}{4})$ and $(1, 0)$. Here, high (low) probability of zero dividend is

[7]Because of the possibility of zero probabilities for endowments, epistemic irrelevance of experiments does not hold in the other direction, see Section 6.

associated with high (low) probability of low endowment e^1. Justification for the set $\mathcal{P}(1)$ is analogous to the one for $\mathcal{P}(0)$.

We claim that asset prices (6) and asset holdings (7) derived in Section 2 together with the implied consumption plans are an equilibrium. Figure 1 in the Appendix provides graphical intuition (albeit with two states) for the type of equilibrium we obtained. It makes clear that heterogeneity of agents endowments, which appears here in the strong form of negative comonotonicity between e_t^1 and e_t^2, is important for speculative trade under ambiguity. We proceed now with formal arguments.

Consider a date-t event with current dividend equal to zero. Transition probabilities in $\mathcal{P}(0)$ that minimize the expected value of agent's 1 next period endowment are $(\frac{1}{2}, 0, \frac{1}{2}, 0)$. We demonstrate at the end of this section that the same probability vector minimizes the expected value of agent's 1 next period continuation utility of consumption plan c^1. Thus, $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ is the effective belief (19) of agent 1 if current dividend is zero. Transition probabilities in $\mathcal{P}(0)$ that minimize the expected value of agent's 2 next period endowment are the opposite extreme vector $(0, \frac{2}{3}, 0, \frac{1}{3})$ because of negative comonotonicity between agents' endowments. As for agent 1, transition probabilities that minimize the expected value of agent's 2 next period continuation utility of c^2 are the same as for her endowment. Thus, $(0, \frac{2}{3}, 0, \frac{1}{3})$ is the effective belief of agent 2. It follows that asset prices (6) satisfy relation (22) which in turn implies that asset holdings (7) are optimal if current dividend is zero.

Consider next a date-t state with current dividend equal to one. We show below that agent's 1 effective belief in that state at c^1 is $(\frac{2}{3}, 0, \frac{1}{3}, 0)$ while agent's 2 effective belief at c^2 is $(0, \frac{1}{4}, 0, \frac{3}{4})$. Again, those effective belief are the same the effective beliefs at their respective endowments. Asset prices (6) satisfy relation (22) if current dividend is one. Consequently, asset holdings (7) are optimal as well. The equilibrium prices $p(0)$ and $p(1)$ are equal to the maximum discounted expected one-period payoff over all probabilities in the respective set of transition probabilities $\mathcal{P}(0)$ or $\mathcal{P}(1)$.

An agent's willingness to pay for the asset if obliged to hold forever (24) is the discounted expected value of future dividends under probabilities that minimize the expected value of the agent's next period continuation utility. Therefore, fundamental values of the asset remain the same $F^i(0)$ and $F^i(1)$ for $i = 1, 2$ as in eq. (10). Equilibrium prices $p(0)$ and $p(1)$ strictly exceed both agents' fundamental valuations and there is speculative bubble.

Effective beliefs:
We prove that agents' effective beliefs at consumption plans c^i implied by asset holdings (7) and prices (6) are as claimed above. We have $c^i(s_t) = e^i(s_t) + x(s_t)h^i(s_{t-1}) + p(s_t)[h^i(s_{t-1}) - h^i(s_t)]$. According to eq. (15), continuation utility (or value) equals current consumption plus discounted conditional expectation of next period continuation utility. If the dividend in s_t is zero, agent's 1 asset holding is 1 and next period continuation utilities in the four successor events are

$$(5 + \Delta^1(0), 10 + \Delta^1(0), 6 + p(1) + \Delta^1(1), 11 + p(1) + \Delta^1(1)) \tag{30}$$

where $\Delta^1(0)$ and $\Delta^1(1)$ denote discounted expected values of next period continuation utility which depend only on current dividend. If the dividend is one and asset holding is zero, next period continuation utilities are

$$(5 - p(0) + \Delta^1(0), 10 - p(0) + \Delta^1(0), 5 + \Delta^1(1), 10 + \Delta^1(1)). \tag{31}$$

The values of $\Delta^1(0)$ and $\Delta^1(1)$ can be calculated from two recursive equations

$$\Delta^1(0) = \beta[\frac{1}{2}(5 + \Delta^1(0)) + \frac{1}{2}(6 + p(1) + \Delta^1(1))] \tag{32}$$

$$\Delta^1(1) = \beta[\frac{2}{3}(5 - p(0) + \Delta^1(0)) + \frac{1}{3}(\Delta^1(1) + 5)], \tag{33}$$

where we assumed (to be verified later) that effective beliefs at c^1 are $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ if dividend is zero and $(\frac{2}{3}, 0, \frac{1}{3}, 0)$ if dividend is one. For $\beta = 0.75$, the solutions are $\Delta^1(0) = 16\frac{11}{13}$ and $\Delta^1(1) = 15$.

Because there is relatively small variability in Δ^1 and in net gains from asset trade across states, the probability in $\mathcal{P}(0)$ that minimizes the expected value of (30) is the same as for endowment $(5, 10, 5, 10)$, as can be easily verified. Thus $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ is indeed the effective belief if dividend is zero. Similarly, the probability in $\mathcal{P}(1)$ that minimizes the expected value of (31) is $(\frac{2}{3}, 0, \frac{1}{3}, 0)$. Calculations of agent's 2 effective beliefs are omitted as they are very similar.

5. Robustness of the Example and Concluding Remarks.

We have demonstrated that speculative bubbles and speculative trade can arise in asset market equilibrium with short-sales constraints when agents have common but ambiguous beliefs. We presented an example of an asset market with two agents with recursive multiple-prior expected utilities whose effective beliefs in equilibrium replicate the heterogeneous beliefs in the example of Harrison and Kreps (1978). A critical condition for generating disagreement of effective beliefs with common set of priors is heterogeneity of initial endowments. It gives rise to heterogeneous consumption and disagreement of beliefs. Heterogeneity of endowments takes in our example the strong form of negative comonotonicity. It gives rise to nearly negatively comonotone equilibrium consumption which in turn leads to disagreement of effective beliefs and speculative bubbles. It should be noted though that the strong disagreement of beliefs as in the Harrison and Kreps' example is not necessary for speculative bubbles (see Section 3), nor is the negative comonotonicity of endowments. The example is robust in the choice of all numerical parameters. In particular, the specification of the set of ambiguous beliefs over endowments being the entire simplex $co\ \{(0,1),(1,0)\}$ is inessential. At the cost of slightly more complicated calculations, that set of beliefs could be replaced by $co\ \{(\gamma, 1-\gamma),(1-\gamma,\gamma)\}$ for small $\gamma > 0$. In fact, with this latter set of priors, the resulting sets of joint probabilities $\mathcal{P}(0)$ and $\mathcal{P}(1)$ of (28) and (29) would exhibit epistemic irrelevance of experiments in both directions, that is, epistemic independence, see Couso et al (1999).

Unlike heterogeneous beliefs as in the Harrison and Kreps' example of Section 2, ambiguous beliefs need not be dogmatic in order to persist in the long run with learning from past observations. Learning and updating of beliefs is significantly different under ambiguity than with no ambiguity. With no ambiguity, the classical Blackwell and Dubins (1962) merging-of-opinions result states that if agents prior beliefs are absolutely continuous with respect to each other, then conditional beliefs for the future given the past converge over time. Werner (2019) (see also Morris (1996)) shows that if the true probability measure on dividends is absolutely continuous with respect to agents' beliefs, then their valuations converge to the true valuation and, moreover, asset price converges to the true valuation.

This makes speculative bubble vanish in the limit. The same holds under weaker condition of consistency of priors with the true parameter. Slawski (2008) provides an example of speculative trade with Bayesian learning and misspecified priors in which speculative bubbles persist over time.

Things can be quite different under ambiguity. Whether or not ambiguity fades away in the long run in repeated experiments depends on the interaction between those experiments. The dividend-endowment process in the model of Section 4 has sets of transition probabilities that do not change over time. Time-invariant sets may arise in repeated experiments that are indistinguishable but have unknown relationship as in the I.I.D. process of Epstein and Schneider (2003b). IID processes have persistent ambiguity. Epstein and Schneider (2007) propose a model of learning with multiple priors that may leave some ambiguity remaining in the long run (see also Zimper and Ma (2017)). On the other hand, Marinacci (2002) shows that ambiguity vanishes in experiments of sampling with replacement from an ambiguous urn.

Appendix.

Static portfolio choice under ambiguity with risky endowments.

We explain the features of optimal portfolio choice under ambiguity that the results of Section 4 rely on in a two-period model.

Consider an agent whose preferences over date-1 state-dependent consumption plans are described by multiple-prior expected utility with the set of probabilities $\mathcal{P}$ and linear utility function. Date-1 endowment $\tilde{e}$ is risky. There is a single asset with date-1 payoff $\tilde{x}$ and date-0 price p. Initial holdings of the asset are zero. At first, we assume that short-sales are unrestricted.

The investment problem is

$$\max_{c_0,h} \left[c_0 + \beta \min_{P\in\mathcal{P}} E_P(\tilde{e} + \tilde{x}h)\right], \tag{34}$$

$$\text{subject to} \;\; c_0 + ph = w_0,$$

where w_0 is date-0 wealth.

For any date-1 consumption plan $\tilde{c}$ we denote the set of minimizing probabilities at $\tilde{c}$ by $\mathcal{P}(\tilde{c}) = \text{ argmin}_{P\in\mathcal{P}} E_P[\tilde{c}]$. A necessary and sufficient condition for h^* to be a solution to (34) is that

$$\min_{P\in\mathcal{P}(c^*)} \beta E_P[\tilde{x}] \leq p \leq \max_{P\in\mathcal{P}(c^*)} \beta E_P[\tilde{x}] \tag{35}$$

where $c^* = \tilde{e} + \tilde{x}h^*$. Note that (35) can be equivalently written as $p = \beta E_P[\tilde{x}]$ for some $P \in \mathcal{P}(c^*)$. The left-hand and the right-hand sides of (35) are equal if $\mathcal{P}(c^*)$ is singleton which is exactly when the multiple-prior utility is differentiable at c^*. The proof of (35) is a simple application of superdifferential calculus.

If follows from (35) that zero holding, $h^* = 0$, is a solution to (34) if and only if

$$\min_{P\in\mathcal{P}(\tilde{e})} \beta E_P[\tilde{x}] \leq p \leq \max_{P\in\mathcal{P}(\tilde{e})} \beta E_P[\tilde{x}] \tag{36}$$

If (36) holds with strict inequalities, then $h^* = 0$ is the unique solution.

If there is zero short-sales constraint in the investment problem (34), then the optimal investment is $h^* = 0$ if and only if $p \geq \min_{P\in\mathcal{P}(\tilde{e})} \beta E_P[\tilde{x}]$. If the inequality is strict, then $h^* = 0$ is the unique optimal investment.

These observations extend the portfolio-inertia result of Dow and Werlang (1992). If date-1 endowment $\tilde{e}$ is risk-free, then $\mathcal{P}(\tilde{e}) = \mathcal{P}$ and it follows from (36) that the optimal investment is zero for all asset prices in the interval between the minimum and the maximum discounted expected payoff over all beliefs in $\mathcal{P}$.

Figure 1 illustrates the optimal investment under ambiguity with background risk and zero short-sales constraint. There are two states and consumption takes place only at date 1. The set of probabilities is $\mathcal{P} = \{(\pi, 1-\pi) : \; 0.4 \leq \pi \leq 0.6\}$. The unique probability measure in $\mathcal{P}$ that minimizes the expected value of endowment $e_1 = (10, 5)$ is $\pi_1 = (0.4, 0.6)$. For asset price $p = E_{\pi_1}[\tilde{x}]$, holding one share of the asset - which results in consumption equal to $e_1 + (\tilde{x} - p)$ - is an optimal investments (as is zero holding). At this price p, the optimal investment for initial endowment $e_2 = (5, 10)$ and subject to the short-sales constraint is zero. This is so because $p > E_{\pi_2}[\tilde{x}]$ where $\pi_2 = (0.6, 0.4)$ is the probability minimizing the expected value of e_2 over $\mathcal{P}$.

In a two-agent economy where both agents have the set of probabilities $\mathcal{P}$ but one has endowment e_1 while the other has e_2 and there is unitary supply of the asset traded under zero short-sales constraint, the equilibrium price of the asset is p. The first agent holds the asset. Note that $p = \max_{P \in \mathcal{P}} E_P[\tilde{x}]$.

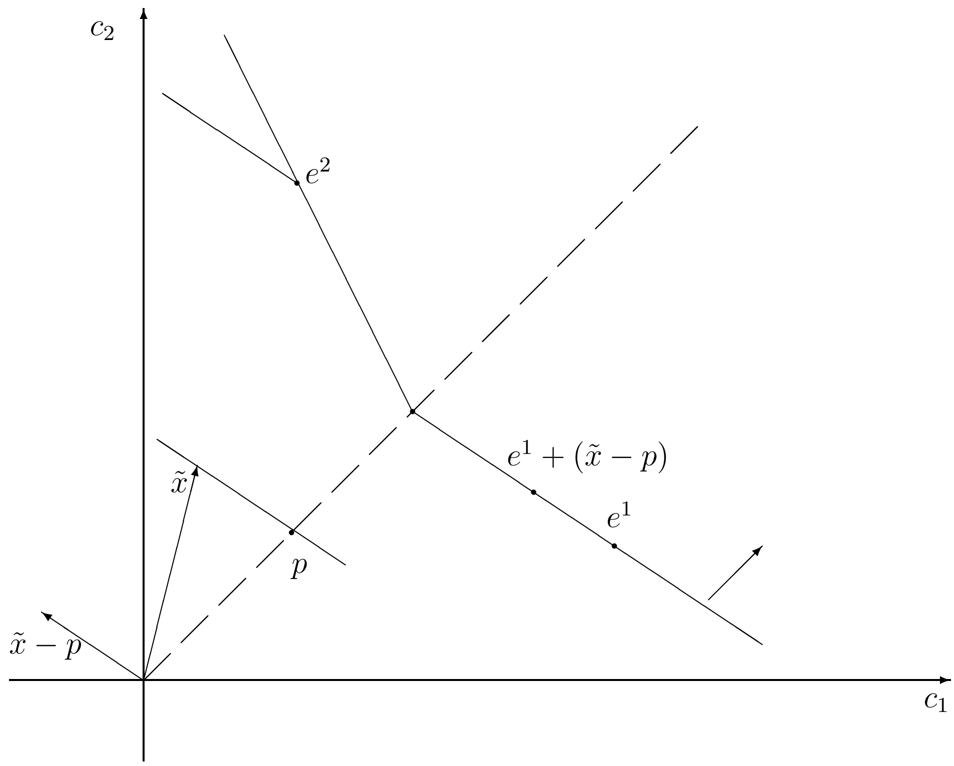

Figure 1: Equilibrium under ambiguity

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