Efficient Allocations under Ambiguity*

Tomasz Strzalecki — Harvard University

Jan Werner — University of Minnesota

September 2009

Abstract

An important implication of the expected utility model under risk aversion is that if agents have the same probability belief, then the efficient allocations under uncertainty are comonotone with the aggregate endowment, and if their beliefs are concordant, then the efficient allocations are measurable with respect to the aggregate endowment. We study these two properties of efficient allocations for models of preferences that exhibit ambiguity aversion using the concept of subjective conditional probability belief which we introduce in this paper. We provide characterizations of such conditional beliefs for the standard models of preferences used in applications.

*We thank the audiences at the conference in honor of Truman Bewley at the University of Texas, Austin, at EWGET in Barcelona, 2009 SAET Conference on Ischia, and SWET 09 at Universite Paris I. A part of this research was done while Strzalecki was visiting the Economic Theory Center at Princeton University to which he is very grateful for its support and hospitality.

1 Introduction

The hypothesis of expected utility and risk aversion has strong implications for risk sharing among multiple agents. In the case of no aggregate risk, that is, when aggregate resources are state independent, the consumption plans in any Pareto optimal allocation are risk free, provided that all of agents' probability beliefs are the same. In this case, agents are unwilling to bet against each other. This is, of course, the well known result that no aggregate risk implies no individual risk in an efficient allocation.

Billot et al [2] extended the no-individual-risk result to multiple-prior (or MaxMin) expected utility of Gilboa and Schmeidler [10]. They show that if agents have at least one prior in common and their von Neumann-Morgenstern utility functions are concave, then the consumption plans in Pareto optimal allocations are risk free. Rigotti, Shannon and Strzalecki [17] provide further extensions of that result to models of ambiguity aversion such as variational preferences of Maccheroni, Marinacci and Rustichini [14] and the smooth ambiguity model of Klibanoff, Marinacci and Mukherji [12]. They introduce the concept of *subjective beliefs* revealed by agents' unwillingness to take fair bets and show that no aggregate risk implies no individual risk if agents have and at least one common subjective belief.

In this paper we study stronger properties of optimal risk sharing such as *measurability* and *comonotonicity* of individual consumption plans with respect to the aggregate endowment, which apply to economies where the aggregate uncertainty is present. The former property asserts that consumption plans are state independent in every event in which the aggregate endowment is state independent (no individual risk conditional on every event in which there is no aggregate risk). A sufficient condition for this property under risk averse expected utility is that agents' probability beliefs are *concordant* (Milgrom and Stokey [15])[1], that is, beliefs conditional on every event in which there is no aggregate risk are the same. The property of comonotonicity asserts that the consumption plans are non-decreasing functions of the aggregate endowment, i.e., the greater the aggregate resources, the greater each agent's consumption. A sufficient condition for this property under risk averse expected utility is that agents' probability beliefs are identical (see LeRoy and Werner [13])[2]. Of course,

[1] This result has been known in the literature on sunspot equilibria, see Cass and Shell [3].

[2] This result has been known much earlier. For references, see Chateauneuf, Dana and Tallon [4].

comonotonicity of consumption plans implies their measurability, which in turn implies no individual risk when there is no aggregate risk.

We extend the approach of Rigotti, Shannon and Strzalecki [17] by introducing the notion of *conditional subjective beliefs*. These are the conditional beliefs revealed by agents' unwillingness to take fair conditional bets. We show that a necessary and sufficient condition for measurability is that agents have at least one conditional subjective belief in common for every event in the partition induced by the aggregate endowment. This condition is a generalization of the concordancy of beliefs under expected utility. The comonotonicity of individual consumption plans with aggregate endowment requires a stronger condition: we show that if there is at least one common subjective conditional belief for every partition coarser than the one induced by the aggregate endowment, then agents' consumption plans in all Pareto optimal allocations are comonotone with aggregate endowment.

We provide characterizations of conditional subjective beliefs for the most important models of ambiguity aversion, such as the multiple-prior expected utility of Gilboa and Schmeidler [10], the variational preferences of Marinacci et al [14], and the smooth ambiguity model of Klibanoff, Marinacci and Mukherji [12]. For the multiple-prior expected utility the subjective beliefs conditional on an event are the conditional probabilities derived from the prior beliefs that minimize the expected utility at the consumption plans that are state independent conditional on that event. The existence of at least one subjective conditional belief common to all agents for every event in which aggregate endowment is state independent is in general much more restrictive than the Billot et al [2] condition of non-empty intersection of sets prior beliefs. We provide examples that illustrate this observation. We also relate our results to the conditions for comonotonicity of optimal consumption plans under the Choquet expected utility obtained by Chateauneuf, Dana and Tallon [4] and the recent work of de Castro and Chateauneuf [6] who study optimal allocations for multiple-prior and Choquet expected utility when the aggregate endowment is unambiguous.

The paper is organized as follows: We introduce the conditional subjective beliefs in Section 2. In Section 3 we prove our main results about optimal risk sharing. Characterizations of conditional beliefs for various models of ambiguity averse preferences are presented in Section 4. Section 5 concludes and discusses the relation of our results to the literature.

2 Conditional Subjective Beliefs

Uncertainty is described by a finite set of states S. The set of consequences is $\mathbb{R}_+$, which we interpret as monetary payoffs. Acts are functions from states to consequences and can be identified with vectors in $\mathbb{R}^S_+$. Acts are denoted by f, g or h. The set of all acts is $\mathcal{F} = \mathbb{R}^S_+$. The set of probability measures on S is denoted by Δ.

Let $\mathcal{G}$ be a partition of the set of states S consisting of K subsets G_j for $j = 1, \ldots, K$. An act f is $\mathcal{G}$-*measurable*, if $f(s) = f(s')$ for every $s, s' \in G_j$, for every j. Let $\mathring{\Delta}_{\mathcal{G}} = \{P \in \Delta : P(G_j) > 0 \text{ for every } j\}$ denote the set of probability measures that assign strictly positive probability to each cell in the partition $\mathcal{G}$. Two probability measures $P, Q \in \mathring{\Delta}_{\mathcal{G}}$ are $\mathcal{G}$-*concordant* if they induce the same conditional probabilities on $\mathcal{G}$, that is

$$\frac{P(s)}{P(G_j)} = \frac{Q(s)}{Q(G_j)}, \quad \forall s \in G_j,\ \forall j. \tag{1}$$

$\mathcal{G}$-concordancy is an equivalence relation on $\mathring{\Delta}_{\mathcal{G}}$ and it identifies classes of probability measures with the same $\mathcal{G}$-conditional probabilities. We will often use conditional expectation of an act on a partition of states. We write $\mathbb{E}_P[f|\mathcal{G}]$ to denote a $\mathcal{G}$-measurable act in $\mathcal{F}$ that is equal to the conditional expectation $\mathbb{E}_P[f|G_j]$ in each state $s \in G_j$. Note that if P and Q are $\mathcal{G}$-concordant, then $\mathbb{E}_P[f|\mathcal{G}] = \mathbb{E}_Q[f|\mathcal{G}]$ for every act f.

An agent's preferences on acts are described by a binary relation $\succsim$ on $\mathcal{F}$. We assume throughout that $\succsim$ is complete, transitive and continuous. Additional relevant properties that $\succsim$ may have are: *monotonicity* (for all $f, g \in \mathcal{F}$, if $f(s) > g(s)$ for every $s \in S$, then $f \succ g$), $\mathcal{G}$-*monotonicity* (for all $f, g \in \mathcal{F}$, if $f \geq g$ and $f(s) > g(s)$ for every $s \in G_j$ for some j, then $f \succ g$), *convexity* (for all $f \in \mathcal{F}$, the set $\{g \in \mathcal{F} : g \succsim f\}$ is convex), and *strict convexity* (for all $f \neq g$ and $\alpha \in (0, 1)$, if $f \succsim g$, then $\alpha f + (1-\alpha)g \succ g$).

Rigotti, Shannon and Strzalecki ([17]) define subjective beliefs at an act $f \in \mathcal{F}$ as follows

Definition 1. A probability measure $P \in \Delta$ is a *subjective belief* at an act $f \in \mathcal{F}$ if

$$\mathbb{E}_P(g) \geq \mathbb{E}_P(f) \ \text{ for every } g \in \mathcal{F} \text{ such that } g \succsim f. \tag{2}$$

Subjective beliefs at f correspond to hyperplanes supporting upper contour set of f.

The idea of relating subjective beliefs to supporting hyperplanes was proposed by Yaari [26]. If a preference relation $\succsim$ has a concave utility representation U, then it follows from a standard result in the theory of superdifferentials (see Rockafellar ([19]) and Aubin ([1])) that subjective beliefs at an interior act f are normalized supergradients of U at f. More precisely, a probability measure P is a subjective belief at a strictly positive act f if $P = \lambda\phi$ for some $\phi \in \partial U(f)$ and $\lambda > 0$, where $\partial U(f)$ denotes the superdifferential of U at f. Superdifferential $\partial U(f)$ is the set of all vectors $\phi \in \mathbb{R}^S$ (supergradients) such that

$$U(g) \leq U(f) + \phi(g - f) \quad \text{for every } g \in \mathcal{F}. \tag{3}$$

If the utility representation U is differentiable, then the superdifferential is the gradient vector $DU(f)$ in the usual sense.

RSS [17] provide characterizations of subjective beliefs for the most important models of preferences under uncertainty. Particularly important are beliefs at constant acts as they play a critical role in their study of optimal risk sharing with no aggregate risk. For the expected utility a with differentiable von Neumann-Morgenstern utility function, the subjective belief at a constant act is simply the probability measure of the expected utility representation. For multiple-prior expected utility of Gilboa and Schmeidler [10] (with differentiable utility), subjective beliefs at constant acts are the set of all probability priors. For variational preferences of [14], they are all probability measures with zero cost. For confidence preferences of [5], they are the measures with full confidence. For smooth preferences of [12], they are the average subjective probability measure.

Our focus in this paper is on conditional probabilities induced by subjective beliefs at acts that are measurable with respect to a partition of states. We identify conditional probabilities from subjective beliefs using the relation of concordancy.

Definition 2. Probability measure $Q \in \mathring{\Delta}_{\mathcal{G}}$ is a *conditional subjective belief* at an act $f \in \mathcal{F}$ for a partition $\mathcal{G}$ if Q is $\mathcal{G}$-concordant with some subjective belief P at f such that $P \in \mathring{\Delta}_{\mathcal{G}}$.

The set of all subjective $\mathcal{G}$-conditional beliefs at an act f is denoted by $\boldsymbol{\pi}_{\mathcal{G}}(f)$. Clearly, every subjective belief that lies in $\mathring{\Delta}_{\mathcal{G}}$ is a conditional subjective belief.

Particularly important are $\mathcal{G}$-conditional beliefs at $\mathcal{G}$-measurable acts. For concave expected utility with probability measure $P \in \mathring{\Delta}_{\mathcal{G}}$, every measure in $\mathring{\Delta}_{\mathcal{G}}$ that is $\mathcal{G}$-concordant with P is a $\mathcal{G}$-conditional belief at every $\mathcal{G}$-measurable act f. The following is an important characterization of $\mathcal{G}$-conditional beliefs at $\mathcal{G}$-measurable acts.

Proposition 1. *The following hold for every $\mathcal{G}$-measurable act f:*

(i) If $Q \in \mathring{\Delta}_{\mathcal{G}}$ is a $\mathcal{G}$-conditional belief at f, then

$$f \succsim g \quad \text{for every } g \in \mathcal{F} \text{ such that } \mathbb{E}_Q[g|\mathcal{G}] = f. \tag{4}$$

(ii) Conversely, if $\succsim$ is $\mathcal{G}$-monotone and convex, and (4) holds for a strictly positive act f and $Q \in \mathring{\Delta}_{\mathcal{G}}$, then Q is $\mathcal{G}$-conditional belief at f.

Proof: (i) $\Rightarrow$ (ii) Suppose that Q is a $\mathcal{G}$-conditional belief at a $\mathcal{G}$-measurable act f and that (4) does not hold. Then there exists an act g such that $\mathbb{E}_Q[g|\mathcal{G}] = f$ and $g \succ f$. Since Q is $\mathcal{G}$-concordant with a subjective belief P at f, it follows that $\mathbb{E}_P[g|\mathcal{G}] = f$. This implies $\mathbb{E}_P(g) = \mathbb{E}_P(f)$ which together with $g \succ f$ contradicts P being a subjective belief at f upon recalling that $\succsim$ is continuous.
(i) $\Rightarrow$ (ii) Suppose that (4) holds. Let $A = \{h \in \mathcal{F} : h \succsim f\}$ and $B = \{g \in \mathcal{F} : \mathbb{E}_Q[g|\mathcal{G}] = f\}$. Note that A is a convex set and $\operatorname{ri} A = \operatorname{int} A \subseteq \{h \in \mathcal{F} : h \succ f\}$. Also B is a convex set and by (4), we have that $B \subseteq \{g \in \mathcal{F} : f \succsim g\}$. Hence, $\operatorname{ri} A \cap \operatorname{ri} B = \emptyset$. By Theorem 11.3 of Rockafellar [20], there exists a probability measure $P \in \Delta$ such that $\mathbb{E}_P(h) \geq \mathbb{E}_P(f)$ for every $h \succsim f$ and $\mathbb{E}_P(f) \geq \mathbb{E}_P(g)$ for every g such that $\mathbb{E}_Q[g|\mathcal{G}] = f$. It follows that P is a subjective belief at f. Suppose that P is not concordant with Q. Then there exists $g \in \mathcal{F}$ such that $\mathbb{E}_Q[g|\mathcal{G}] = f$ but $\mathbb{E}_P[g|\mathcal{G}](s) \geq f(s)$ for all $s \in S$ and there exists $G_j \in \mathcal{G}$ with $P(G_j) > 0$ and $\mathbb{E}_P[g|\mathcal{G}](s) > f(s)$ for all $s \in G_j$. By the law of iterated expectations $\mathbb{E}_P(g) > \mathbb{E}_p(f)$. Contradiction. $\square$

For $\mathcal{G}$-monotone and convex preference relation $\succsim$, condition (4) is equivalent to Q being $\mathcal{G}$-conditional belief at strictly positive and $\mathcal{G}$-measurable act f. Condition (4) can be written as $f \succsim f + \epsilon$ for every $\epsilon \in \mathcal{F}$ such that $\mathbb{E}_Q[\epsilon|\mathcal{G}] = 0$. It expresses the agent's *unwillingness to take $\mathcal{G}$-conditional bets*. Proposition 1 extends Proposition 1 in RSS ([17]).

Conditional subjective beliefs may differ across $\mathcal{G}$-measurable acts. For instance, conditional beliefs for expected utility with nondifferentiable von Neumann-Morgenstern utility function are usually different at points of differentiability of the utility function and at points where it is nondifferentiable. We define consistent conditional beliefs for partition $\mathcal{G}$ as follows

Definition 3. Probability measure $Q \in \overset{\circ}{\Delta}_{\mathcal{G}}$ is a *consistent conditional subjective belief* for partition $\mathcal{G}$ if Q is a conditional subjective belief for every strictly positive $\mathcal{G}$-measurable act f.

The restriction to strictly positive acts in the definition of consistency has twofold motivation. First, we are aiming at characterizing strictly positive Pareto optimal allocations. Second, our primary tool for deriving conditional belief for models of ambiguity aversion is the superdifferential which cannot be used for acts at the boundary of $\mathcal{F}$ without further complications. The set of all consistent $\mathcal{G}$-conditional beliefs is denoted by $\boldsymbol{\pi}_{\mathcal{G}}$. We say that $\mathcal{G}$-conditional beliefs are consistent if $\boldsymbol{\pi}_{\mathcal{G}} \neq \emptyset$

For concave and differentiable expected utility with probability measure P, the $\mathcal{G}$-conditional beliefs at every strictly positive $\mathcal{G}$-measurable act f are all probability measures in $\overset{\circ}{\Delta}_{\mathcal{G}}$ that are concordant with P. Consequently, all these measures are consistent $\mathcal{G}$-conditional beliefs. We show in Section 4 that the set of all probability measures that are $\mathcal{G}$-concordant with P is the set of consistent $\mathcal{G}$-conditional beliefs for nondifferentiable concave utility even though $\mathcal{G}$-conditional beliefs at some $\mathcal{G}$-measurable acts f may be a superset of that set.

Corollary 1.

(i) Suppose that $\succsim$ is $\mathcal{G}$-monotone and convex. A probability measure $Q \in \overset{\circ}{\Delta}_{\mathcal{G}}$ is a consistent $\mathcal{G}$-conditional belief if and only if

$$\mathbb{E}_Q[g|\mathcal{G}] \succsim g \quad \textit{for every strictly positive act } g \in \mathcal{F}. \tag{5}$$

(ii) For arbitrary $\succsim$, if $Q \in \overset{\circ}{\Delta}_{\mathcal{G}}$ is a consistent $\mathcal{G}$-conditional belief then (5) holds.

Proof: Condition (4) of Proposition 1 can be written as $\mathbb{E}_Q[g|\mathcal{G}] \succsim g$ for every $g \in \mathcal{F}$ such that $\mathbb{E}_Q[g|\mathcal{G}] = f$. It follows that $Q \in \overset{\circ}{\Delta}_{\mathcal{G}}$ is a consistent $\mathcal{G}$-conditional belief for $\mathcal{G}$-monotone and convex $\succsim$ if and only if (5) holds for every g such that $\mathbb{E}_Q[g|\mathcal{G}]$ is strictly positive. An inspection of the proof of Proposition 1 reveals that the equivalence remains true with (5) required to hold only for strictly positive acts g. Part (ii) follows from Proposition 1 (i). □

Condition (5) in Corollary 1 expresses preference for $\mathcal{G}$-conditional expectations under Q. This condition is satisfied for concave expected utility for every probability measure Q that is $\mathcal{G}$-concordant with the subjective probability measure P, in particular, for P. Thus, the preference for $\mathcal{G}$-conditional expectations holds under P for every partition $\mathcal{G}$. That is to say, P is a consistent $\mathcal{G}$-conditional belief for concave expected utility with P for every $\mathcal{G}$. The same holds for every preference relation that is monotone decreasing with respect to second-order stochastic dominance, which we refer to as strong risk aversion. Strong risk-aversion under probability measure P implies preference for $\mathcal{G}$-conditional expectations under P for every $\mathcal{G}$. Therefore P is a consistent $\mathcal{G}$-conditional belief for such preferences. Examples of preferences that are strongly risk averse include rank-dependent expected utilities of Quiggin [18] (see Chew, Karni and Safra [7]) and mean-variance preferences. An extensive discussion of the property of preference for conditional expectations and its relation to aversion to risk can be found in Werner ([25]). It is worth pointing out that for all strongly risk averse preferences, including concave expected utility, condition (5) holds for all acts f, strictly positive or not.

3 Optimal Risk Sharing

Suppose that there are I agents indexed by $i = 1, \ldots, I$. Agent i is endowed with a preference relation $\succsim_i$ on the set of acts $\mathcal{F}$ and her consumption set is also $\mathcal{F}$. The aggregate endowment available to the agents is $w \in \mathbb{R}^S_{++}$. A feasible allocation is a collection of consumption plans $\{f_i\}_{i=1}^I$ such that $f_i \in \mathcal{F}$ for every i and $\sum_{i=1}^I f_i(s) = w(s)$ for each $s \in S$. We shall consider only feasible allocations and refer to them as allocations dropping the adjective feasible. An allocation $\{f_i\}$ is *Pareto optimal* if there is no other allocation $\{g_i\}$, such that $g_i \succsim_i f_i$ for all i and $g_j \succ_j f_j$ for some j.

We consider two properties of risk sharing that Pareto optimal allocations may have: measurability with respect to the aggregate endowment, and comonotonicity. We first explain the property of measurability. The aggregate endowment w induces a partition of states

$\mathcal{E} = \{E_1, \ldots, E_K\}$ such that $w(s) = w(s')$ for $s \neq s'$ if and only if $s, s' \in E_k$ for some k. The partition $\mathcal{E}$ is a (crude) description of the aggregate risk in the economy. For each event $E \in \mathcal{E}$, there is no aggregate risk conditional on E. The coarser the partition $\mathcal{E}$, the less aggregate risk in this sense. If the partition is the trivial partition $\mathcal{E} = \{S\}$, then there is no aggregate risk, as w is constant. An allocation $\{f_i\}$ is *$\mathcal{E}$-measurable,* if every consumption plan f_i is $\mathcal{E}$-measurable. If an allocation is $\mathcal{E}$-measurable, then there is no individual risk conditional on every event on which there is no aggregate risk.

We turn now to comonotonicity. Two acts f and g are *comonotone* if $[f(s)-f(s')][g(s)-g(s')] \geq 0$ for every pair of states s and s'. An allocation $\{f_i\}$ is comonotone if f_i and f_j are comonotone for every i and j. One can show (see Chateauneuf, Dana and Tallon [4]) that an allocation $\{f_i\}$ is comonotone if and only if there exist non-decreasing functions $F_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f_i(s) = F^i(w(s))$, for every i. It follows that every comonotone allocation is $\mathcal{E}$-measurable; however, the converse is not true.

If agents have strictly concave expected utility, then a sufficient condition for $\mathcal{E}$-measurability of Pareto optimal allocations is that agents' probability beliefs be *$\mathcal{E}$-concordant*. If there is no aggregate risk so that $\mathcal{E}$ is the trivial partition, then beliefs are concordant if and only if they are the same for all agents. A sufficient condition for comonotonicity of Pareto optimal allocations for strictly concave expected utility is that probability beliefs be the same for all agents (see Theorem 15.5.1 in LeRoy and Werner [13]).

Billot et al [2] show that having at least one common prior is sufficient for w-measurability of optimal allocations if there is no aggregate risk (i.e., w is constant) and agents have concave multiple-prior expected utilities. RSS [17] extended this result to other models of ambiguity aversion using unconditional subjective beliefs in place of prior beliefs.

We begin with an example demonstrating that the existence of a common prior is not sufficient for $\mathcal{E}$-measurability of Pareto optimal allocations, and hence not sufficient for comonotonicity, if the aggregate endowment is risky and agents have concave multiple-prior expected utilities. This example clearly indicates the importance of conditional subjective beliefs for a characterization of optimal allocations.

Example 1. There are three states of nature and two agents. Agent 1 has multiple-prior expected utility function $\min_{P \in \mathcal{P}_1} \mathbb{E}_P[v_1(f)]$ with the set of priors being a circle around the center of probability simplex Δ^3 shown in Figure 1. Agent 2 has the standard expected

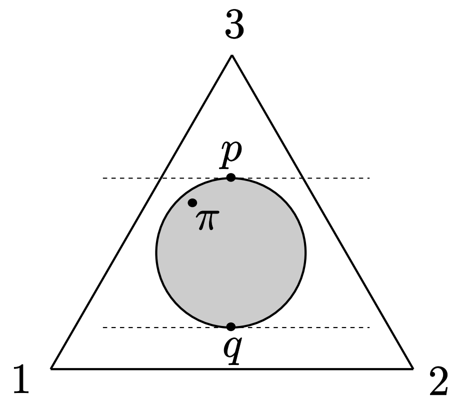

Figure 1: Priors of agents 1 and 2.

utility $\mathbb{E}_\pi[v_2(f)]$ with a unique prior $\pi = (\pi_1, \pi_2, \pi_3)$ such that $\pi_1 \neq \pi_2$. It holds $\pi \in \mathcal{P}_1$ so that π is the common prior. The von-Neumann-Morgenstern utility functions v_1 and v_2 are strictly concave, differentiable, and satisfy the Inada condition.

If the aggregate endowment $w = (w(1), w(2), w(3))$ is risk-free, then it follows from Billot et al [2] that all Pareto optimal allocations are risk-free, that is, measurable with respect to the trivial partition. Suppose that w is such that $w(1) = w(2) > w(3) > 0$. The induced partition is $\mathcal{E} = \{\{1, 2\}, 3\}$. An allocation (f_1, f_2) with $f_i \in \mathbb{R}^3_+$ is $\mathcal{E}$-measurable if and only if $f_i(1) = f_i(2)$ for $i = 1, 2$. We claim that there are no $\mathcal{E}$-measurable Pareto optimal allocations other than the two extreme allocations $(0, w)$ and $(w, 0)$.

Because of the Inada condition, all Pareto optimal allocations, other than the extreme allocations, are interior. Consider an allocation such that $f_1(1) = f_1(2) > f_1(3) > 0$. The prior that gives the minimum expected utility of f_1 is p, see Figure 1. Agent's 1 multiple-prior utility is differentiable at f_1 and the marginal rate of substitution between consumption in states 1 and 2 is $p_1/p_2 = 1$. The respective marginal rate of substitution for agent 2 at f_2 is π_1/π_2, which is different from 1. Such allocation (f_1, f_2) cannot be Pareto optimal. Next, consider (f_1, f_2) such that $0 < f_1(1) = f_1(2) < f_1(3)$. The prior that gives the minimum expected utility of f_1 is q, see Figure 1. The marginal rates of substitution between consumption in states 1 and 2 are again 1 for agent 1 and π_1/π_2 for agent 2. Such allocation cannot be Pareto optimal either. Finally, consider $f_1(1) = f_1(2) = f_1(3) > 0$ so that f_1 is risk-free. Agent's 1 utility is not differentiable at f_1. The superdifferential of the multiple-prior utility of agent 1 at f_1 are all probability priors in $\mathcal{P}_1$ rescaled by the marginal utility $v_1'(f_1)$. The vector of marginal utilities for agent 2 at f_2 is $(\pi_1 v_2'(f_2(1)), \pi_2 v_2'(f_2(2)), \pi_3 v_2'(f_2(3)))$. Since $f_2(3) - f_2(1) = w(3) - w(1)$, one can choose utility function v_2 so that this vector lies outside of the superdifferential for agent 1, for any such (f_1, f_2). It follows from Theorem 7 in RSS [17] that such allocations cannot be Pareto optimal. Thus there is no $\mathcal{E}$-measurable optimal allocations other than the extreme allocations.

3.1 Risk Sharing with no Aggregate Conditional Risk

In this section we provide necessary and sufficient conditions for $\mathcal{E}$-measurability of Pareto optimal allocations for general preferences using the concept of conditional subjective beliefs. We shall use a slightly weaker notion of essential $\mathcal{E}$-measurability in our results. An allocation $\{f_i\}$ is *essentially $\mathcal{E}$-measurable*, if there exists a $\mathcal{E}$-measurable allocation $\{\hat{f}_i\}$ such that $f_i \sim_i \hat{f}_i$, for every i. Clearly, if every agent's preference relation is strictly convex, then a Pareto optimal allocation is essentially $\mathcal{E}$-measurable if and only if it is $\mathcal{E}$-measurable.

Theorem 1. *Suppose that each agent's $\mathcal{E}$-conditional beliefs are consistent. If agents have at least one common consistent $\mathcal{E}$-conditional belief, i.e.,*

$$\bigcap_{i=1}^{I} \boldsymbol{\pi}_{\mathcal{E}}^{i} \neq \emptyset \tag{6}$$

then every interior Pareto optimal allocation is essentially $\mathcal{E}$-measurable.

Proof: Let $\{f_i\}$ be a Pareto optimal allocation such that f_i is strictly positive for every i, and let Q be a probability measure in $\cap_{i=1}^{I} \boldsymbol{\pi}_{\mathcal{E}}^{i}$. Consider an allocation $\{\tilde{f}_i\}$ defined by

$$\tilde{f}_i = \mathbb{E}_Q[f_i|\mathcal{E}],$$

for every i. The allocation $\{\tilde{f}_i\}$ is feasible and $\mathcal{E}$-measurable. By Corollary 1 (ii), $\tilde{f}_i \succsim_i f_i$, for every i. Since the allocation $\{f_i\}$ is Pareto optimal, it follows that $f_i \sim_i \tilde{f}_i$ for every i. Therefore $\{f_i\}$ is essentially $\mathcal{E}$-measurable. $\square$

If agents have concave expected utilities with subjective probability measures $P_i \in \mathring{\Delta}_{\mathcal{E}}$, then condition (6) holds if and only if probability measures P_i are $\mathcal{E}$-concordant

A converse result to Theorem 1 holds under strong consistency of beliefs and convexity of preferences. We say that $\mathcal{G}$-conditional beliefs are *strongly consistent* if $\boldsymbol{\pi}_{\mathcal{G}}(f) = \boldsymbol{\pi}_{\mathcal{G}}(g) \neq \emptyset$ for all $\mathcal{G}$-measurable strictly positive acts f, g. Expected utility provides a good illustration of the difference between consistency and strong consistency of conditional beliefs. Consistency holds for every concave expected utility function with the set of consistent $\mathcal{G}$-conditional beliefs equal to all probability measures in $\mathring{\Delta}_{\mathcal{G}}$ that are $\mathcal{G}$-concordant with probability measure P. Strong consistency holds for concave expected utility if and only if the utility function is differentiable. In this case, all probability measures in $\mathring{\Delta}_{\mathcal{G}}$ that are concordant with P are G-conditional beliefs at every strictly positive and $\mathcal{G}$-measurable act (see Section 4.1).

Theorem 2. *Suppose that each agent's preferences are $\mathcal{E}$-monotone and convex, and her $\mathcal{E}$-conditional beliefs are strongly consistent. If there exists an interior Pareto optimal allocation that is $\mathcal{E}$-measurable, then there exists at least one common consistent $\mathcal{E}$-conditional belief, i.e., condition (6) holds.*

Proof: Consider an interior $\mathcal{E}$-measurable Pareto optimal allocation. By the separation argument as in the standard proof of the Second Welfare Theorem, there exists a probability measure $Q \in \Delta$ such that $\mathbb{E}_Q(g_i) \geq \mathbb{E}_Q(f_i)$ whenever $g_i \succsim_i f_i$ for every i. Hence, Q is a subjective belief at f_i for agent i. By $\mathcal{E}$-monotonicity of $\succsim$, it follows that $Q \in \mathring{\Delta}_{\mathcal{E}}$ and therefore it is a $\mathcal{E}$-conditional subjective belief at f. Since $\mathcal{E}$-conditional beliefs are strongly consistent for each i, it follows that $Q \in \bigcap_{i=1}^{I} \boldsymbol{\pi}_{\mathcal{E}}^{i}$. □

Theorems 1 and 2 imply the following corollary that extends the main result of RSS [17] from a constant aggregate endowment w (no aggregate risk) to arbitrary w.

Corollary 2. *Suppose that each agent's preferences are $\mathcal{E}$-monotone and strictly convex, and her subjective beliefs are strongly consistent. The following conditions are equivalent:*

(i) There exists an interior $\mathcal{E}$-measurable Pareto optimal allocation

(ii) All interior Pareto optimal allocations are $\mathcal{E}$-measurable

(iii) $\bigcap_{i=1}^{I} \boldsymbol{\pi}_{\mathcal{E}}^{i} \neq \emptyset$

In RSS [17] it is assumed that aggregate endowment is constant, (unconditional) beliefs are strongly consistent (their Axiom 7), and preferences are strictly convex. Their Proposition 9 states that agents have at least one common consistent unconditional belief if and only if all interior Pareto optimal allocations are constant, which in turn is equivalent to the existence of a constant interior optimal allocation.

3.2 Comonotone Risk Sharing

In this section we provide sufficient conditions for comonotonicity of Pareto optimal allocations. These conditions involve a greater degree of agreement of conditional beliefs across agents. We consider a collection of partitions of S that are coarser than the partition $\mathcal{E}$ induced by the aggregate endowment. We denote this set of partitions by Σ_c.

Theorem 3. *Suppose that every agent's preferences are strictly convex, and her $\mathcal{G}$-conditional beliefs are consistent for every $\mathcal{G} \in \Sigma_c$. If agents have at least one common consistent $\mathcal{G}$-conditional belief for every $\mathcal{G} \in \Sigma_c$, i.e.,*

$$\bigcap_{i=1}^{I} \boldsymbol{\pi}^i_{\mathcal{G}} \neq \emptyset \tag{7}$$

for all $\mathcal{G} \in \Sigma_c$, then every interior Pareto optimal allocation is comonotone.

Proof: Let $\{f_i\}$ be a Pareto optimal allocation such that f_i is strictly positive for every i. Theorem 1 and strict convexity of preferences imply that f_i is $\mathcal{E}$-measurable for every i. Suppose that there are i and i' such that f_i and $f_{i'}$ are not comonotone. Then there exist events E_j and E_k in the partition $\mathcal{E}$ such that $f_i(E_j) < f_i(E_k)$ and $f_{i'}(E_j) > f_{i'}(E_k)$.

Let $\mathcal{E}_{jk}$ denote the partition obtained from partition $\mathcal{E}$ by replacing two cells E_j and E_k by their union $E_j \cup E_k$. Since $\mathcal{E}_{jk} \in \Sigma_c$, there exists $Q \in \bigcap_{i=1}^{I} \boldsymbol{\pi}^i_{\mathcal{E}_{jk}}$. Let $\tilde{f}_i = \mathbb{E}_Q[f_i|\mathcal{E}_{jk}]$. Act $\tilde{f}_i$ differs from f_i in that consumptions states belonging to event $E_j \cup E_k$ are replaced by their expectation under Q conditional on $E_j \cup E_k$. Further, let $\tilde{f}_{i'} = \mathbb{E}_Q[f_{i'}|\mathcal{E}_{jk}]$ and $\varepsilon_i = f_i - \tilde{f}_i$ and $\varepsilon_{i'} = f_{i'} - \tilde{f}_{i'}$.

Since $\mathbb{E}_Q(\varepsilon_i) = \mathbb{E}_Q(\varepsilon_{i'}) = 0$ and acts ε_i and $\varepsilon_{i'}$ differ only in two states, it holds $\varepsilon_j = -\lambda\varepsilon_i$, for some $\lambda > 0$. Suppose that $\lambda \geq 1$. We will show that transferring ε_i from agent i to agent j makes both of them strictly better off. Transferring ε_i from agent i leaves him with $\tilde{f}_i$. Corollary 1 (ii) implies that $\tilde{f}_i \succsim_i f_i$. Since $\succ_i$ is strictly convex and $\tilde{f}_i \neq f_i$, we actually have that $\tilde{f}_i \succ_i f_i$. Transferring ε_i to agent j leaves him with $f_j + \varepsilon_i = \left(\frac{1}{\lambda}\right)\tilde{f}_j + \left(\frac{\lambda-1}{\lambda}\right)f_j$. As for agent i, it holds that $\tilde{f}_j \succ_j f_j$. Using strict convexity we obtain $f_j + \varepsilon_i \succ_j f_j$.

If $\lambda < 1$, then transferring ε_j from agent j to agent i makes both agents better off. Thus we obtained a contradiction to Pareto optimality of allocation $\{f_i\}$. $\square$

The proof of Theorem 3 clearly shows that the hypothesis remains true if the condition of agreement of consistent conditional beliefs across agents is required only for a subset of partitions coarser than $\mathcal{E}$, namely those that can be obtained by merging arbitrary two elements of the partition $\mathcal{E}$.

If agents have concave expected utilities with subjective probability measures $P_i \in \mathring{\Delta}_{\mathcal{E}}$, then condition (7) holds if and only if P_i are the same. The requirement of an agreement of conditional beliefs on the partition $\mathcal{E}$ and on coarser partitions is equivalent to probability measures P_i being the same.

4 Conditional Beliefs under Ambiguity Aversion

4.1 Multiple-Prior Expected Utility

One of the most popular alternatives to expected utility is the multiple-prior (or MaxMin) expected utility (MPEU, for short). Under the MPEU specification, the agent has a set of probability measures on states—multiple priors—and makes her decisions by considering the expected utility under the prior that gives the lowest value of expected utility. Such preferences are most appealing in situations of so-called ambiguity when, as in the Ellsberg paradox, there is insufficient information for an agent to form a unique probabilistic belief. The axiomatization of multiple-prior expected utility is due to Gilboa and Schmeidler [10].

The multiple-prior expected utility takes the form

$$\min_{P \in \mathcal{P}} \mathbb{E}_P[v(f)], \tag{8}$$

for some strictly increasing and continuous utility function $v : \mathbb{R}_+ \to \mathbb{R}$ and some convex and closed set $\mathcal{P} \subseteq \Delta$ of probability measures on S. We assume throughout this section that v is concave. Observe that the preference is $\mathcal{G}$-monotone if and only if $\mathcal{P} \subseteq \mathring{\Delta}_{\mathcal{G}}$.

Let $\mathcal{P}^v(f)$ denote the set of priors for which the minimum expected utility is attained.

$$\mathcal{P}^v(f) = \arg\min_{\bar{P} \in \mathcal{P}} \mathbb{E}_P[v(f)]. \tag{9}$$

Let $\mathcal{P}^v_{\mathcal{G}}(f)$ denote the set of probability measures in $\mathring{\Delta}_{\mathcal{G}}$ that are $\mathcal{G}$-concordant with some probability in $\mathcal{P}^v(f)$. The set $\mathcal{P}^v_{\mathcal{G}}(f)$ represents the $\mathcal{G}$-conditional probabilities induced by the minimizing probabilities at f. The sets of minimizing probabilities for the linear utility function $v(x) = x$ and the induced $\mathcal{G}$-conditional probabilities are denoted by $\mathcal{P}(f)$ and $\mathcal{P}_{\mathcal{G}}(f)$, respectively. They will be used later.

If the function v is differentiable at a strictly positive act f,[3] then the superdifferential of MPEU (8) at f is

$$\{\phi \in \mathcal{R}^S : \phi_s = v'(f(s))P(s), \ \forall s, \quad \text{for some } P \in \mathcal{P}^v(f)\}, \tag{10}$$

see Aubin ([1]). The normalized vectors in the superdifferential (10) are the subjective beliefs at the act f (see ([17]). If the act f is $\mathcal{G}$-measurable, then the marginal utility $v'(f)$ is also $\mathcal{G}$-measurable and every normalized vector in (10) is $\mathcal{G}$-concordant with some probability measure in $\mathcal{P}^v(f)$. It follows that

$$\pi_{\mathcal{G}}(f) = \mathcal{P}^v_{\mathcal{G}}(f). \tag{11}$$

If v is not differentiable at f, then only one inclusion holds: $\pi_{\mathcal{G}}(f) \supseteq \mathcal{P}^v_{\mathcal{G}}(f)$.

The minimizing probabilities (9) depend in general on the utility function v. Therefore, the conditional beliefs depend on v as well; however, this is not so for consistent beliefs. If Q is a consistent $\mathcal{G}$-conditional belief for MPEU with concave utility v, than it is a consistent $\mathcal{G}$-conditional belief for every concave utility, in particular, for the linear utility. This is demonstrated in the following

Proposition 2. *For every multiple-prior expected utility with concave utility, the set of consistent $\mathcal{G}$-conditional beliefs is*

[3]We say that v is differentiable at act f if it is differentiable at every $f(s)$ for $s \in S$.

$$\boldsymbol{\pi}_{\mathcal{G}} = \cap_{f\in\mathcal{F}_{\mathcal{G}}}\mathcal{P}_{\mathcal{G}}(f). \tag{12}$$

where $\mathcal{F}_{\mathcal{G}}$ is the set of all $\mathcal{G}$-measurable acts.

PROOF: The proof is straightforward if the function v is differentiable. We have from (11) that $\boldsymbol{\pi}_{\mathcal{G}} = \cap_{f\in\mathcal{F}_E}\mathcal{P}^v_{\mathcal{G}}(f)$. Upon observing that $\mathcal{P}^v(f) = \mathcal{P}(v(f))$, we obtain (12).

For an arbitrary concave v and $f \in \mathcal{F}_E$, it holds $\pi_{\mathcal{G}}(f) \supseteq \mathcal{P}^v_{\mathcal{G}}(f)$, with equality if v is differentiable at f. Since v has at most a countable set of points of non-differentiability, one can show that for every $\mathcal{G}$-measurable act f there exists a $\mathcal{G}$-measurable act g such that v is differentiable at g and $v(f) = \lambda v(g)$ for some scalar $\lambda > 0$. It holds $\mathcal{P}^v(f) = \mathcal{P}^v(g)$ and $\boldsymbol{\pi}_{\mathcal{G}}(g) = \mathcal{P}^v(g)$. Consequently, $\boldsymbol{\pi}_{\mathcal{G}}(f) \cap \boldsymbol{\pi}_{\mathcal{G}}(g) = \mathcal{P}^v(f)$. Using the same argument as in the case of differentiable v, we obtain (12). □

The following example illustrates consistent conditional beliefs for MPEU.

Example 2. Consider the set of priors $\mathcal{P} = \{(p_1, p_2, p_3) \in \Delta^3 : p_s \geq b \text{ for } s = 1, 2, 3\}$, where b is a lower bound on probabilities satisfying $0 < b < \frac{1}{3}$. Let the partition of states be $\mathcal{G} = \{\{1, 2\}, \{3\}\}$. Act f is $\mathcal{G}$-measurable if and only if $f(1) = f(2)$. The sets of minimizing probabilities at f are $\mathcal{P}(f) = \{(b, b, 1-2b)\}$ if $f(3) < f(1)$, $\mathcal{P}(f) = \{(q_1, q_2, b) : q_1 + q_2 = 1 - b, q_1 \geq b, q_2 \geq b\}$ if $f(3) < f(1)$, and $\mathcal{P}(f) = \mathcal{P}$ if $f(1) = f(3)$. The respective sets of $\mathcal{G}$-conditional beliefs are $\mathcal{P}_{\mathcal{G}}(f) = \{(q_1, q_2, q_3) \in \Delta^3 : q_1 = q_2\}$, $\mathcal{P}_{\mathcal{G}}(f) = \{(q_1, q_2, q_3) \in \Delta^3 : \frac{b}{1-2b} \leq \frac{q_2}{q_1} \leq \frac{1-2b}{b}\}$, and $\mathcal{P}_{\mathcal{G}}(f) = \mathcal{P}_{\mathcal{G}}$. It follows from (12) that the set of consistent $\mathcal{G}$-conditional beliefs is $\boldsymbol{\pi}_{\mathcal{G}} = \{(q_1, q_2, q_3) \in \Delta^3 : q_1 = q_2\}$, that is all measures with equal probabilities of states 1 and 2.

If $\mathcal{G}$ is the trivial partition, then the $\mathcal{G}$-measurable acts are simply the constant acts. The set of minimizing probabilities for every constant act is the whole set of priors $\mathcal{P}$. Conditional probabilities for trivial partition coincide with unconditional probabilities. Proposition 2 implies that conditional beliefs for the trivial partition (or unconditional beliefs, for short) are consistent and the set of consistent unconditional beliefs is the whole set $\mathcal{P}$. They are the subjective beliefs for constant acts, see RSS [17]. Consistency of conditional beliefs for other partitions is not always guaranteed. This is illustrated by the following.

Example 3. The set of priors arising in the context of the Ellsberg Paradox (with one urn and balls of 3 colors) is $\mathcal{P} = \{(p_1, p_2, p_3) \in \Delta^3 : p_1 \geq b, p_2 \geq b, p_3 = \frac{1}{3}\}$, where b is a lower bound such that $0 < b < \frac{1}{3}$. Consider the partition $\mathcal{G} = \{\{1\}, \{2, 3\}\}$. Subjective beliefs at $\mathcal{G}$-measurable acts are $\mathcal{P}(f) = \{(b, \frac{2}{3} - b, \frac{1}{3})\}$ if $f(1) < f(3)$, $\mathcal{P}(f) = \{(b, \frac{2}{3} - b, \frac{1}{3})\}$ if $f(1) > f(3)$, and $\mathcal{P}(f) = \mathcal{P}$ if $f(1) = f(3)$. The former two sets consist of single probability measures that are not $\mathcal{G}$-concordant with each other. Therefore, the set of consistent $\mathcal{G}$-conditional beliefs is empty.

We now present a characterization of consistent conditional beliefs for MPEU. For every pair of probability measures Q and P on S, define another probability measure $P_{\mathcal{G}}^{Q}$ on S by

$$P_{\mathcal{G}}^{Q}(A) = \sum_{i=1}^{k} Q(A|G_i)P(G_i) \tag{13}$$

for every $A \subseteq S$. The probability measure $P_{\mathcal{G}}^{Q}$ coincides with P on elements of partition $\mathcal{G}$ and has conditional probabilities of Q within each element of the partition.

Proposition 3. *For every multiple-prior expected utility with concave utility and set of priors* $\mathcal{P} \subseteq \mathring{\Delta}_{\mathcal{G}}$*, the following conditions are equivalent for* $\mathcal{G}$ *and* $Q \in \mathring{\Delta}_{\mathcal{G}}$*.*

(i) $Q \in \boldsymbol{\pi}_{\mathcal{G}}$

(ii) $P_{\mathcal{G}}^{Q} \in \mathcal{P}$ *for every* $P \in \mathcal{P}$*.*

Proof: It suffices to show equivalence of (i) and (ii) for linear utility since the set $\boldsymbol{\pi}_{\mathcal{G}}$ does not depend on the utility function by Proposition 2. We first prove that (ii) implies (i). By Corollary 1 (i), it suffices to show that the multiple-prior expected utility with linear utility and set of priors $\mathcal{P}$ satisfying (ii) exhibits preference for $\mathcal{G}$-conditional expectations (5).

For every $f \in \mathcal{F}$, we have $\mathbb{E}_P[\mathbb{E}_Q[f|\mathcal{G}]] = \mathbb{E}_{P_{\mathcal{G}}^{Q}}[f]$. Therefore

$$\min_{P \in \mathcal{P}} \mathbb{E}_P[[\mathbb{E}_Q[f|\mathcal{G}]] = \min_{P \in \mathcal{P}} \mathbb{E}_{P_{\mathcal{G}}^{Q}}[f] \geq \min_{P \in \mathcal{P}} \mathbb{E}_P[f] \tag{14}$$

where we used (ii). Inequality (14) implies that multiple-prior expected utility with set of priors $\mathcal{P}$ and linear utility exhibits preference for conditional expectation under Q.

To show that (i) implies (ii), suppose by contradiction that $\bar{P}_{\mathcal{G}}^{Q} \notin \mathcal{P}$ for some $\bar{P} \in \mathcal{P}$. By the separation theorem, there exists $\hat{f} \in \mathbb{R}^S$ such that

$$\mathbb{E}_{\bar{P}^Q_{\mathcal{G}}}(\hat{f}) < \min_{P \in \mathcal{P}} \mathbb{E}_P(\hat{f}). \tag{15}$$

Since adding any constant act to $\hat{f}$ would not change inequality (15) we can assume that $\hat{f} \in \mathcal{F}$. Using $\mathbb{E}_P[\mathbb{E}_Q[\hat{f}|\mathcal{G}]] = \mathbb{E}_{P^Q_{\mathcal{G}}}[\hat{f}]$ and (15) we obtain

$$\min_{P \in \mathcal{P}} \mathbb{E}_P[\mathbb{E}_Q[\hat{f}|\mathcal{G}]] < \min_{P \in \mathcal{P}} \mathbb{E}_P[\hat{f}] \tag{16}$$

This contradicts the preference for $\mathcal{G}$-conditional expectations under Q for linear multiple-prior expected utility with $\mathcal{P}$, and hence contradicts (i). □

As defined in Section 3.1, $\mathcal{G}$-conditional beliefs are strongly consistent if they are the same for every $\mathcal{G}$-measurable strictly positive act. For the trivial partition, unconditional beliefs are strongly consistent for MPEU with concave utility v and set of priors $\mathcal{P}$ if and only if v is differentiable. Unconditional beliefs at every strictly positive constant act are then the set $\mathcal{P}$. Since constant acts are $\mathcal{G}$-measurable for every partition $\mathcal{G}$, it follows that, if $\mathcal{G}$-conditional beliefs are strongly consistent, then they must equal the set $\mathcal{P}_{\mathcal{G}}$ of all probability measures in $\mathring{\Delta}_{\mathcal{G}}$ that are $\mathcal{G}$-concordant with some probability measure in the set of priors $\mathcal{P}$. Using Proposition 3, we obtain the following

Proposition 4. *For every multiple-prior expected utility with concave and differentiable utility and set of priors $\mathcal{P} \subseteq \mathring{\Delta}_{\mathcal{G}}$, the $\mathcal{G}$-conditional beliefs are strongly consistent if and only if the set $\mathcal{P}$ has the following property:*

$$P^Q_{\mathcal{G}} \in \mathcal{P} \text{ for every } P, Q \in \mathcal{P}. \tag{17}$$

The set of strongly consistent $\mathcal{G}$-conditional beliefs $\boldsymbol{\pi}_{\mathcal{G}}$ is equal to $\mathcal{P}_{\mathcal{G}}$.[4]

Proof: If $\mathcal{G}$-conditional beliefs are strongly consistent, then, as already noted, it holds $\boldsymbol{\pi}_{\mathcal{G}} = \mathcal{P}_{\mathcal{G}}$. Condition (17) follows then from Proposition 3.

[4]Although our setup is static, it is worthwhile observing that condition (17) is the requirement of rectangularity of P with respect to $\mathcal{G}$, see Epstein and Schneider [8], which is equivalent to dynamic consistency of preferences.

For the converse implication, suppose that condition (17) is satisfied. Using Proposition 3, we have $\boldsymbol{\pi}_{\mathcal{G}} = \mathcal{P}_{\mathcal{G}}$. We have to show that $\pi_{\mathcal{G}}(f) = \mathcal{P}_{\mathcal{G}}$ for every $f \in \mathcal{F}_{\mathcal{G}}$. Since $\pi_{\mathcal{G}} \subseteq \pi_{\mathcal{G}}(f)$, it follows $\mathcal{P}_{\mathcal{G}} \subseteq \pi_{\mathcal{G}}(f)$. Further, since v is differentiable, it holds $\pi_{\mathcal{G}}(f) = \mathcal{P}^v_{\mathcal{G}}(f)$. Since $\mathcal{P}^v(f) \subseteq \mathcal{P}$, it follows that $\mathcal{P}^v_{\mathcal{G}}(f) \subseteq \mathcal{P}_{\mathcal{G}}$, and hence the conclusion. □

An important example of MPEU preferences with strongly consistent conditional beliefs arises when the partition consists of unambiguous events.

Definition 4. An event $E \subseteq S$ is *unambiguous* if $P(E) = Q(E)$ for every $P, Q \in \mathcal{P}$. A partition $\mathcal{G}$ is *unambiguous* if it consists of unambiguous events.

If there exists an unambiguous event other than S or $\emptyset$, then there exits a non-trivial unambiguous partition. Ghirardato and Marinacci [9] and Nehring [16] provide axiomatic characterizations of MPEU with sets of priors that have unambiguous events. If $\mathcal{G}$ is an unambiguous partition and utility function v is differentiable, then the set of minimizing probabilities for every $\mathcal{G}$-measurable act is the set of priors $\mathcal{P}$. $\mathcal{G}$-conditional beliefs are then the set $\mathcal{P}$. Further, it is easy to see that condition (17) holds when $\mathcal{G}$ is unambiguous. Therefore Proposition 4 implies the following

Corollary 3. *For every multiple-prior expected utility with concave and differentiable utility and set of priors $\mathcal{P} \subseteq \mathring{\Delta}_{\mathcal{G}}$, if the partition $\mathcal{G}$ is unambiguous, then the $\mathcal{G}$-conditional beliefs are strongly consistent and $\boldsymbol{\pi}_{\mathcal{G}} = \mathcal{P}_{\mathcal{G}}$.*

Theorem 3 requires that conditional beliefs be consistent for the collection Σ_c of partitions, not merely for one partition. If the probability measure $P^Q_{\mathcal{G}}$ lies in $\mathcal{P}$ for every $P \in \mathcal{P}$ and every partition of states $\mathcal{G}$, then, by Proposition 3, the measure Q is a consistent $\mathcal{G}$-conditional belief for every $\mathcal{G}$. Sets of priors with this property are called Q-*stable*, see Werner ([25]). One class of examples of Q-stable sets of priors are cores of convex distortions of Q. For an increasing and convex function $\varphi : [0,1] \to [0,1]$ that satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$ the core of the distortion φ of Q is

$$\{P \in \Delta : P(A) \geq \varphi(Q(A)) \text{ for every } A \subseteq S\}. \tag{18}$$

Examples are sets of priors with lower bound $\{P \in \Delta : P \geq \gamma Q\}$ or with upper bound $\{P \in \Delta : P \leq \lambda Q\}$, for $\gamma, \lambda \in [0,1]$. The set $\mathcal{P}$ in Example (2) is a core of a convex distortion of $Q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Another class of examples of Q-stable sets are neighborhoods of Q in a divergence distance. A divergence distance between two probability measures P and Q such that $Q \in \mathring{\Delta}$ is

$$d(P, Q) = \sum_{s \in S} \psi\big(\frac{P(s)}{Q(s)}\big) Q(s) \tag{19}$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function satisfying $\psi(1) = 0$, and $\lim_{t\to\infty} \psi(t)/t = \infty$, see [14]. The Kullback-Leibler divergence or relative entropy is a special case of (19) with $\psi(t) = t \ln(t)$. A neighborhood of Q in divergence distance is the set $\{P \in \Delta : d(P, Q) \leq \epsilon\}$ for some $\epsilon > 0$.

4.2 Variational Preferences

Variational preferences have a utility representation of the form

$$\min_{P \in \Delta} \big\{\mathbb{E}_P[v(f)] + c(P)\big\}, \tag{20}$$

for some strictly increasing and continuous utility function $v : \mathbb{R}_+ \to \mathbb{R}$, such that $v(\mathbb{R}_+) = \mathbb{R}_+$,[5] and some convex and lower semicontinuous function $c : \Delta \to [0, \infty]$ such that there exists $Q \in \Delta$ with $c(Q) = 0$. In this specification c is the cost (in terms of utility) of considering every belief. Observe that the preference is $\mathcal{G}$-monotone if and only if $\mathcal{P}^{\text{fin}} \subseteq \mathring{\Delta}_{\mathcal{G}}$ where $\mathcal{P}^{\text{fin}} = \{P \in \Delta : c(P) < \infty\}$. The axiomatization of variational preferences is due to [14]. Hansen and Sargent [11] considered variational preferences with a cost function c given by $c(P) = \theta R(P, Q)$, where Q is the agent's reference belief, R is the relative entropy measure and $\theta > 0$ is a scale parameter. Such variational preferences are called multiplier preferences. A more general subclass of variational preferences are divergence preferences, where $c(P) = \theta d(P, Q)$ is proportional to a divergence distance (see (19) in the previous section).

Let $\mathcal{P}^v(f)$ denote the set of priors for which the minimum in (20) is attained. That is,

$$\mathcal{P}^v(f) = \arg\min_{P \in \Delta} \big\{\mathbb{E}_P[v(f)] + c(P)\big\} \tag{21}$$

[5] The unboundedness of v is guaranteed by Axiom A7 of [14].

Let $\mathcal{P}^v_{\mathcal{G}}(f)$ denote the set of probability measures that are $\mathcal{G}$-concordant with some probability in $\mathcal{P}^v(f)$. Further, let $\mathcal{P}(f)$ and $\mathcal{P}_{\mathcal{G}}(f)$ be the sets of minimizing probabilities and the induced $\mathcal{G}$-conditional probabilities, respectively, for linear utility function.

If the function v is differentiable at a strictly positive act f, then the superdifferential of utility function (20) at f is (by Theorem 18 of [14])

$$\{\phi \in \mathbb{R}^S : \phi_s = v'(f(s))P(s),\ \forall s \in S, \ \text{ for some } P \in \mathcal{P}^v(f)\} \tag{22}$$

If f is $\mathcal{G}$-measurable, then the marginal utility $v'(f(s))$ is the same within each cell of the partition $\mathcal{G}$ and every normalized vector in the superdifferential (22) is $\mathcal{G}$-concordant with some measure in $\mathcal{P}^v(f)$. Therefore

$$\boldsymbol{\pi}_{\mathcal{G}}(f) = \mathcal{P}^v_{\mathcal{G}}(f) \tag{23}$$

for every $f \in \mathcal{F}_{\mathcal{G}}$. If v is not differentiable at f, then only one inclusion holds: $\pi_{\mathcal{G}}(f) \supset \mathcal{P}^v_{\mathcal{G}}(f)$.

Just like for MPEU, if Q is a consistent $\mathcal{G}$-conditional belief for variational preferences with some concave utility function v, then Q is a consistent $\mathcal{G}$-conditional belief for every concave utility, in particular, for the linear utility. We have the following proposition.

Proposition 5. *For every $\mathcal{G}$-monotone variational preference with concave utility, the set of consistent $\mathcal{G}$-conditional beliefs is*

$$\boldsymbol{\pi}_{\mathcal{G}} = \cap_{f\in\mathcal{F}_{\mathcal{G}}} \mathcal{P}_{\mathcal{G}}(f). \tag{24}$$

PROOF: The proof is straightforward if the function v is differentiable. We have from (11) that $\boldsymbol{\pi}_{\mathcal{G}} = \cap_{f\in\mathcal{F}_{\mathcal{G}}} \mathcal{P}^v_{\mathcal{G}}(f)$. Upon observing that $\mathcal{P}^v(f) = \mathcal{P}(v(f))$, we obtain the conclusion.

For an arbitrary concave v and $f \in \mathcal{F}_{\mathcal{G}}$, it holds $\pi_{\mathcal{G}}(f) \supseteq \mathcal{P}^v_{\mathcal{G}}(f)$, with equality if v is differentiable at f. Since v has at most a countable set of points of nondifferentiability, one can show that for every $f \in \mathcal{F}_{\mathcal{G}}$ there exists $g \in \mathcal{F}_{\mathcal{G}}$ such that v is differentiable at g and $v(f) = v(g) + k$ for some $k \in \mathbb{R}$. Because the set of minimizing probabilities is invariant to additive shifts, it holds $\mathcal{P}^v(f) = \mathcal{P}^v(g)$ and $\boldsymbol{\pi}_E(g) = \mathcal{P}^v(g)$. Consequently $\boldsymbol{\pi}_{\mathcal{G}}(f) \cap \boldsymbol{\pi}_{\mathcal{G}}(g) = \mathcal{P}^v(f)$. Using the same argument as in the case of differentiable v, we obtain (24). □

For a constant act f the set of minimizing probabilities (21) consists of all probability measures with zero cost, $\mathcal{P}^0 = \{Q \in \Delta : c(Q) = 0\}$. This implies that unconditional beliefs for variational preferences are consistent for every concave utility, and the set of consistent unconditional beliefs is $\mathcal{P}^0$ (see RSS [17]).

The following proposition is an analog of Proposition 3 for variational preferences.

Proposition 6. *For every $Q \in \overset{\circ}{\Delta}_{\mathcal{G}}$ and partition $\mathcal{G}$ of S if*

$$c(P^Q_{\mathcal{G}}) \leq c(P) \quad \text{for every } P \in \Delta, \tag{25}$$

then $Q \in \boldsymbol{\pi}_{\mathcal{G}}$ for variational preferences with concave utility.

Proof: It suffices to show that variational preferences with linear utility and cost function c satisfying (25) exhibits preference for $\mathcal{G}$-conditional expectations (5). The result follows then from Proposition 1 and the observation that the set $\boldsymbol{\pi}_{\mathcal{G}}$ does not depend on the utility function as long as it is concave (Proposition 5).

For every $f \in \mathcal{F}$, we have $\mathbb{E}_P[\mathbb{E}_Q[f|\mathcal{G}]] = \mathbb{E}_{P^Q_{\mathcal{G}}}[f]$. Using this and (25) we obtain

$$\min_{P \in \Delta} \left\{ \mathbb{E}_P[\mathbb{E}_Q[f|\mathcal{G}]] + c(P) \right\} \geq \min_{P \in \Delta} \left\{ \mathbb{E}_{P^Q_{\mathcal{G}}}[f] + c(P^Q_{\mathcal{G}}) \right\} \geq \min_{P \in \Delta} \left\{ \mathbb{E}_P[f] + c(P) \right\} \tag{26}$$

It follows from (26) that variational preferences with the cost function c satisfying (25) and linear utility exhibit preference for $\mathcal{G}$-conditional expectation under Q. $\square$

In close similarity with the MPEU preferences, conditional beliefs for partitions consisting of unambiguous events are strongly consistent. An event is unambiguous if all probability measures that the agent considers to be plausible assign the same probability to that event.

Definition 5. An event $E \subseteq S$ is *unambiguous* if $P(E) = Q(E)$ for every $P, Q \in \mathcal{P}^{\text{fin}}$. A partition $\mathcal{G}$ is *unambiguous* if it consists of unambiguous events.

If there exists an unambiguous event other than S or $\emptyset$, then there exits a non-trivial unambiguous partition. Strzalecki [24] provides a characterization of variational preferences cost functions that give rise to unambiguous events. If $\mathcal{G}$ is an unambiguous partition, then the set of minimizing probabilities for every $\mathcal{G}$-measurable act is the set $\mathcal{P}^0$. Using $\mathcal{P}^0_{\mathcal{G}}$ to denote the set of all $\mathcal{G}$-conditional probabilities induced by $\mathcal{P}^0$, Proposition 5 implies the following

Corollary 4. *For every $\mathcal{G}$-monotone variational preference with concave utility and every unambiguous partition $\mathcal{G}$, the $\mathcal{G}$-conditional beliefs are strongly consistent and $\boldsymbol{\pi}_{\mathcal{G}} = \mathcal{P}^0_{\mathcal{G}}$.*

An important class of cost functions where condition (25) of Proposition 6 holds are divergence measures (19) with Q being the reference measure.

Proposition 7. *For any divergence measure $d(P,Q)$ with $Q \in \mathring{\Delta}$, it holds*

$$d(P^Q_{\mathcal{G}}, Q) \leq d(P,Q), \tag{27}$$

for every $P \in \Delta$ and every partition $\mathcal{G}$ of S.

Proof: We have

$$\begin{aligned} d(P^Q_{\mathcal{G}}, Q) &= \sum_{j=1}^{K} \sum_{s \in G_j} \phi\left(\frac{Q(s)P(G_j)}{Q(s)Q(G_j)}\right) Q(s) = \\ &= \sum_{j=1}^{K} \phi\left(\frac{P(G_j)}{Q(G_j)}\right) Q(G_j) \ \leq \ \sum_{j=1}^{K} \sum_{s \in G_j} \phi\left(\frac{P(s)}{Q(s)}\right) Q(s) = d(P,Q) \end{aligned}$$

where the last inequality follows from Jensen's inequality. $\square$

4.3 Smooth Model of Ambiguity Aversion

The utility representation in the smooth model of Klibanoff, Marinacci and Mukherji [12] takes the form

$$\mathbb{E}_\mu[\phi(\mathbb{E}_P v(f))], \tag{28}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R}_+ \to \mathbb{R}$ are strictly increasing and concave functions that are differentiable in the interior of their domains. The probability measure μ is the second-order prior, that is, a probability distribution on the set of probability measures Δ. Observe that the preference is $\mathcal{G}$-monotone if and only if $\operatorname{supp}(\mu) \subseteq \mathring{\Delta}_{\mathcal{G}}$.

A measure that plays an important role in the analysis is the "average measure" $P^\mu \in \Delta$ defined as $P^\mu(s) = \mathbb{E}_\mu[P(s)]$ for every $s \in S$. As RSS [17] show, the measure P^μ is the subjective belief at every strictly positive constant act. More generally, Proposition 5 of RSS [17] implies that the utility representation (28) is differentiable at every strictly positive act f with the gradient being a vector whose sth coordinate for $s \in S$ is

$$v'(f(s))\mathbb{E}_\mu\big[\phi'(\mathbb{E}_P v(f))P(s)\big]. \tag{29}$$

Subjective belief at f is the gradient vector (29) normalized to be a probability measure. Conditional beliefs for partition $\mathcal{G}$ are all probability measures in $\mathring{\Delta}_\mathcal{G}$ that are $\mathcal{G}$-concordant with the subjective belief. In general, conditional beliefs in the smooth model need not be consistent.

Example 4. Let there be 3 states and let the second-order prior μ assign equal probabilities to two probability vectors in Δ^3: $(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$. Suppose that v is the linear utility and ϕ is strictly concave. Consider the partition $\mathcal{G} = \{\{1, 2\}, \{3\}\}$ and two $\mathcal{G}$-measurable acts $f = (7, 7, 1)$ and $g = (2, 2, 8)$. The unique subjective belief at f is $\big(2\phi'(4) + 3\phi'(5), \phi'(4) + \phi'(5), 3\phi'(4) + 2\phi'(5)\big)$ normalized to be a probability vector. The subjective belief at g is $\big(2\phi'(5) + 3\phi'(4), \phi'(5) + \phi'(4), 3\phi'(5) + 2\phi'(4)\big)$, normalized. These two subjective beliefs are not $\mathcal{G}$-concordant since ratios of probabilities of states 1 and 2 are different. Therefore, the set of consistent $\mathcal{G}$-conditional beliefs is empty.

A sufficient condition for consistency of conditional beliefs in the smooth model is concordancy of all measures in the support of the second-order prior μ. The following proposition is similar to Proposition 3 for MPEU and Proposition 6 for variational preferences, which also relate consistency to concordancy.

Proposition 8. *If all probability measures in the support of μ are $\mathcal{G}$-concordant, then $\mathcal{G}$-conditional beliefs are strongly consistent and equal to the set $\{P^\mu\}_\mathcal{G}$ of all measures that are $\mathcal{G}$-concordant with P^μ.*

Proof: Because all $P \in \text{supp}\,(\mu)$ are concordant, for any $G_i \in \mathcal{G}$ for any $s, s' \in G_i$ there exists $\alpha > 0$ such that $P(s) = \alpha P(s')$ for all $P \in \text{supp}\,(\mu)$. The ratio of subjective probabilities of states s and s' at any $\mathcal{G}$-measurable strictly positive act f is $\frac{\mathbb{E}_\mu\left\{(\phi'(\mathbb{E}_P v(f))\cdot v'(f(s))\cdot P(s))\right\}}{\mathbb{E}_\mu\left\{(\phi'(\mathbb{E}_P v(f))\cdot v'(f(s))\cdot P(s'))\right\}} = \frac{\mathbb{E}_\mu\left\{(\phi'(\mathbb{E}_P v(f))\cdot \alpha P(s'))\right\}}{\mathbb{E}_\mu\left\{(\phi'(\mathbb{E}_P v(f))\cdot P(s'))\right\}} = \alpha = \frac{P^\mu(s)}{P^\mu(s')}$. $\square$

Unambiguous events and partitions can be defined for the smooth model (see an axiomatic derivation in KMM [12]) and they lead to strongly consistent conditional beliefs.

Definition 6. An event $E \subseteq S$ is *unambiguous* if there exists a $\gamma \in [0,1]$ such that $P(E) = \gamma$, μ-almost-everywhere. A partition $\mathcal{G}$ is *unambiguous* if it consists of unambiguous events.

Conditional beliefs are strongly consistent if the partition $\mathcal{G}$ is unambiguous.

Proposition 9. *For every $\mathcal{G}$-monotone smooth ambiguity preference and every unambiguous partition $\mathcal{G}$, the $\mathcal{G}$-conditional beliefs are strongly consistent and equal to the set of all measures which are $\mathcal{G}$-concordant with P^μ.*

Proof: Under the assumptions of Proposition 9, it holds that $\mu\{P \in \Delta : P_{|\mathcal{G}} = P^\mu_{|\mathcal{G}}\} = 1$. Then, for any strictly positive $\mathcal{G}$-measurable act f, the expression $\phi'(\mathbb{E}_P v(f))$ does not depend on P. Thus, the gradient (29) is proportional to the vector with an sth coordinate equal to $v'(f(s))\mathbb{E}_\mu[P(s))]$. This implies that the subjective belief at any strictly positive $\mathcal{G}$-measurable act f is $\mathcal{G}$-concordant with P^μ. $\square$

5 Summary of the Results and Discussion

5.1 Summary of the Results

In Section 3 we studied the conditions of measurability and comonotonicity of Pareto optimal allocations with respect to the aggregate endowment. As the main tool of the analysis we introduced the notion of conditional subjective beliefs, defined in Section 2. In Section 4 we characterized the subjective conditional beliefs for the most frequently used classes of preferences. This section provides a discussion of the two properties of Pareto optimal allocations—$\mathcal{E}$-measurability and comonotonicity—when agents' preferences belong to one of the classes of preferences analyzed in Section 4. We apply Theorems 1 and 3 of Section 2, which identify sufficient conditions, and use our characterizations of consistent conditional subjective beliefs from Section 4. Our discussion is somewhat loose and focused on the main aspects, leaving out Theorem 2, which identifies necessary conditions.

The consistent $\mathcal{E}$-conditional beliefs for concave MPEU preferences consist of all probability measures $\mathcal{E}$-concordant with the minimizing probabilities for linear utility at $\mathcal{E}$-measurable acts. Pareto optimal allocations are $\mathcal{E}$-measurable if there exist consistent $\mathcal{E}$-conditional beliefs for all agents and those beliefs are $\mathcal{E}$-concordant, that is, if for every agent i there exist a probability measure $\overline{Q}_i \in \mathring{\Delta}_{\mathcal{G}}$ such that $P_{\mathcal{E}}^{\overline{Q}_i} \in \mathcal{P}_i$ for every $P \in \mathcal{P}_i$, and measures $\overline{Q}_i$ are $\mathcal{E}$-concordant. Examples of sets of priors that satisfy this condition are cores of convex distortions of probability measures $\overline{Q}_i$ and neighborhoods of $\overline{Q}_i$ in divergence distance, as well as many others. Comonotonicty of Pareto optimal allocations obtains if the above condition holds not only for the partition $\mathcal{E}$, but also for all partitions coarser than $\mathcal{E}$. This happens for example if the sets of priors are $\overline{Q}$-stable for a probability measure $\overline{Q}$ that is common to all agents. Examples of Q-stable sets are again cores of convex distortions of $\overline{Q}$ and neighborhoods of $\overline{Q}$ in divergence distance.

The results for variational preferences are quite similar to those for MPEU preferences. The consistent $\mathcal{E}$-conditional beliefs for concave variational preferences consist of all probability measures $\mathcal{E}$-concordant with minimizing probabilities for linear utility at $\mathcal{E}$-measurable acts. Pareto optimal allocations are $\mathcal{E}$-measurable if there exist consistent $\mathcal{E}$-conditional beliefs for all agents and they are are $\mathcal{E}$-concordant, that is, for every agent i there exist a probability measure $\overline{Q}_i \in \mathring{\Delta}_{\mathcal{G}}$ such that $c_i(P_{\mathcal{E}}^{\overline{Q}_i}) \leq c_i(P_i)$ for every $P \in \Delta$, and measures $\overline{Q}_i$ are $\mathcal{E}$-concordant. The most important examples of variational preferences that satisfy this condition are the multiplier preferences of Hansen and Sargent [11], or more generally, divergence preferences. Comonotonicty of Pareto optimal allocations obtains for divergence preferences if the reference probability measures $\overline{Q}_i$ are the same.

For smooth ambiguity preferences a sufficient condition for $\mathcal{E}$-measurability of Pareto optimal allocations is that all probability measures in the support of the second-order priors μ_i are concordant with the average measure $P_i^{\mu} \in \mathring{\Delta}_{\mathcal{G}}$, and measures P_i^{μ} are $\mathcal{E}$-concordant. If in addition the average measures P_i^{μ} are identical, then the optimal allocations are comonotone.

5.2 The Case of Unambiguous Aggregate Endowment

The aggregate endowment is unambiguous if the induced partition $\mathcal{E}$ is unambiguous. As Section 4 shows, the conditional subjective beliefs for unambiguous partitions are rather special, as they coincide with the subjective beliefs at constant acts. In this section we present three results on $\mathcal{E}$-measurability obtained from Theorems 1 and 2, or more exactly from Corollary 2, and the results of Section 4. We shall assume that each utility function v_i is strictly concave, differentiable, and satisfies the Inada condition, $\lim_{x\to 0} v_i'(x) = +\infty$. For MPEU preferences we have

Corollary 5. *Suppose that each agent has MPEU preferences that are $\mathcal{E}$-monotone. If the partition $\mathcal{E}$ is unambiguous for every agent, then the following conditions are equivalent:*

(i) There exists an interior $\mathcal{E}$-measurable Pareto optimal allocation.

(ii) All Pareto optimal allocations are $\mathcal{E}$-measurable.

(iii) $\bigcap_{i=1}^{I} (\mathcal{P}_i)_{\mathcal{E}} \neq \emptyset$

Corollary 5 has been proved earlier by de Castro and Chateauneuf ([6], Theorem 5.3). For variational preference, the result is

Corollary 6. *Suppose that each agent has variational preferences that are $\mathcal{E}$-monotone. If the partition $\mathcal{E}$ is unambiguous for every agent, then the following conditions are equivalent:*

(i) There exists an interior $\mathcal{E}$-measurable Pareto optimal allocation

(ii) All Pareto optimal allocations are $\mathcal{E}$-measurable

(iii) $\bigcap_{i=1}^{I} (\mathcal{P}_i^0)_{\mathcal{E}} \neq \emptyset$

Finally for smooth KMM preferences, we have

Corollary 7. *Suppose that each agent has smooth KMM preferences that are $\mathcal{E}$-monotone. If the partition $\mathcal{E}$ is unambiguous for every agent, then the following conditions are equivalent:*

(i) There exists an interior unambiguous Pareto optimal allocation

(ii) All Pareto optimal allocations are unambiguous

(iii) The measures P^{μ_i} are identical

The condition (*iii*) of Corollary 7 is simpler than of Corollaries 5 and 6 because of the single-valuedness of subjective beliefs for smooth KMM preferences. The requirement of common conditionals and common marginals (unambiguous partition) simply implies that the measures are identical.

5.3 Relation to the Literature

Chateauneuf, Dana and Tallon [4] study the properties of Pareto optimal allocations when agents have Choquet expected utilities, that is, expected utilities with nonadditive probabilities, or capacities, see Schmeidler [22]. Since the Choquet expected utility preferences with a convex capacity have a representation as multiple-prior expected utility with the set of priors equal to the core of the capacity, our results can be compared with theirs. One of the main results in Chateauneuf, Dana and Tallon [4] is Proposition 3.1 showing that Pareto optimal allocations are comonotone if agents have the same convex capacity. This result is not implied by our Theorem 3, the reason being that the cores of convex capacities need not in general generate consistent conditional beliefs. For example, the set of priors in Example 3 with an empty set of consistent conditional beliefs is the core of a convex capacity.

A special property of the Choquet expected utility preferences with convex capacity is the comonotonic independence, meaning that for each region of comonotonicity in the space of acts, there exists a probability measure such that preferences coincide with the expected utility preferences with this probability measure. This property plays a critical role in the Chateauneuf, Dana and Tallon [4] proof. They show that Pareto optimal allocations for the Choquet preferences are the same as for expected utility preferences with the common probability measure identified in the region of comonotonicity with the aggregate endowment.

Their method of deducing the comonotonicity of allocations from the comonotonic independence of preferences applies only to the CEU model. On the other hand, we study a very general class of models, many of which violate comonotonic independence, and for this reason the method of proof of Chateauneuf, Dana and Tallon [4] does not apply here.

Our observation from Section 5.1 that Pareto optimal allocations are comonotone when sets of priors for MPEU preferences are cores of convex distortions of a common probability measure can be found in Chateauneuf, Dana and Tallon ([4], Proposition 4.1). Cores of convex distortions of probability measures do generate consistent conditional beliefs.

References

[1] J-P. Aubin. *Optima and Equilibria, An Introduction to Nonlinear Analysis*, Springer. 1998.

[2] A. Billot, A. Chateauneuf, I. Gilboa and J-M. Tallon. Sharing Beliefs: Between Agreeing and Disagreeing. *Econometrica*, **68**, No. 4, 685-694, (2000)

[3] D. Cass and K. Shell, Do Sunspots Matter? *Journal of Political Economy*, 91(2), (1983), pp. 193-227.

[4] A. Chateauneuf, R.A. Dana and J.M. Tallon. Risk sharing rules and Equilibria with non-additive expected utilities, *Journal of Mathematical Economics*, **34**, pp. 191–214, (2000).

[5] A. Chateauneuf and J. H. Faro. Ambiguity through Confidence Functions, *Journal of Mathematical Economics*, forthcoming.

[6] L. de Castro and A. Chateauneuf. Ambiguity Aversion and Trade, mimeographed, March 2008.

[7] Chew, S.H, E. Karni and Z. Safra. Risk aversion in the theory of expected utility with rank dependent preferences. *Journal of Economic Theory*, (1987), pp. 370-381.

[8] Epstein, L.G. and Schneider, M. Recursive multiple-priors, *Journal of Economic Theory*, 113, (2003), pp. 1-31.

[9] Ghirardato, P. and Marinacci, M. Ambiguity Made Precise: A Comparative Foundation, *Journal of Economic Theory*, 102, (2002), 251-289.

[10] I. Gilboa and D. Schmeidler. Maxmin expected utility with non unique prior, *Journal of Mathematical Economics*, 18, (1989), 141-153.

[11] Hansen, L. and T. Sargent. "Robust Control and Model Uncertainty," *American Economic Review*, **91**, pp 60-66, (2001).

[12] P. Klibanoff, M. Marinacci and S. Mukherji. A Smooth Model of Decision Making under Ambiguity *Econometrica*, **73**, No. 6, 1849-92, (2005)

[13] LeRoy, S. and J. Werner. *Principles of Financial Economics*, Cambridge University Press, 2001.

[14] Maccheroni, F, Marinacci M. and A. Rustichini. Ambiguity Aversion, Robustness, and the Variational Representation of Preferences, *Econometrica*, **60**, No. 4, 745-780, (2006)

[15] Milgrom, P. and N. Stokey. Information, Trade and Common Knowledge. *Journal of Economic Theory*, 26, 1982, 17–27.

[16] Nehring, K. Capacities and Probabilistic Beliefs: A Precarious Coexistence, *Mathematical Social Sciences*, **38**, 197-213, (1999)

[17] Rigotti, L., Ch. Shannon and T. Strzalecki. Subjective Beliefs and Ex-Ante Trade, *Econometrica*, 76, (2008), 1176-1190

[18] Quiggin, J. A theory of anticipated utility, *Journal of Economic Behavior and Organization*, 3, pp. 323-343, (1982)

[19] Rockafellar, T. *The Theory of Subgradients and Its Applications to Problems of Optimization.* Helderman, 1981.

[20] Rockafellar, T. *Convex analysis.* Princeton university press, 1997.

[21] Rothschild, M. and J.E. Stiglitz, Increasing Risk. I: A Definition, *Journal of Economic Theory*, 2, (1970), 225-243.

[22] Schmeidler, D. Subjective Probability and Expected Utility Without Additivity, *Econometrica*, 57, (1989), 571-587.

[23] Strzalecki, T. Axiomatic Foundations of Multiplier Preferences, mimeographed, 2008

[24] Strzalecki, T. Probabilistic Sophistication and Variational Preferences, mimeographed, 2009

[25] Werner J. Risk Aversion for Multiple-Prior Expected Utility, mimeographed, 2006.

[26] Yaari, M. Some remarks on measures of risk aversion and on their uses. *Journal of Economic Theory*, 55:95–115, 1969.