Efficient Allocations under Ambiguity[*]

Tomasz Strzalecki

Harvard University

Jan Werner

University of Minnesota

May 2010

Abstract

An important implication of the expected utility model under risk aversion is that if agents have the same probability belief, then the efficient allocations under uncertainty are comonotone with the aggregate endowment, and if their beliefs are concordant, then the efficient allocations are measurable with respect to the aggregate endowment. We study these two properties of efficient allocations for models of preferences that exhibit ambiguity aversion using the concept of conditional belief, which we introduce in this paper. We provide characterizations of such conditional beliefs for the standard models of preferences used in applications.

1 Introduction

The hypotheses of expected utility and risk aversion have strong implications for risk sharing among multiple agents. In the case of no aggregate risk, that is, when aggregate resources are state independent, the consumption plans in any Pareto optimal allocation are risk free, provided that all of agents' probability beliefs are the same. In this case, agents are unwilling to bet against each other. This is, of course, the well known result that no aggregate risk implies no individual risk in an efficient allocation.

Billot et al [6] extended the no-individual-risk result to multiple-priors (or MaxMin) expected utility of Gilboa and Schmeidler [15]. They show that if agents have at least one prior in common and their von Neumann-Morgenstern utility functions are concave, then the consumption plans in Pareto optimal allocations are risk free. Rigotti, Shannon and Strzalecki [28] provide further extensions of that result to models of ambiguity aversion such as variational preferences of Maccheroni, Marinacci and Rustichini [23] and the smooth ambiguity model of Klibanoff, Marinacci and Mukherji [19]. They introduce the concept of *subjective beliefs* revealed by agents' unwillingness to take fair bets and show that no aggregate risk implies no individual risk if agents have and at least one common subjective belief.

In this paper we study stronger properties of optimal risk sharing such as *measurability* and *comonotonicity* of individual consumption plans with respect to the aggregate endowment, which apply to economies where the aggregate uncertainty is present. The former property asserts that consumption plans are state independent in every event in which the aggregate endowment is state independent (no individual risk conditional on every event in which there is no aggregate risk). A sufficient condition for this property under risk averse expected utility is that agents' probability beliefs are *concordant* (Milgrom and Stokey [24])[1] that is, beliefs conditional on every event in which there is no aggregate risk are the same. The property of comonotonicity asserts that the consumption plans are non-decreasing functions of the aggregate endowment, i.e., the greater the aggregate resources, the greater each agent's consumption. A sufficient condition for this property under risk averse expected util-

---

[1]Cass and Shell [7] prove the same result in the context of sunspot uncertainty,

ity is that agents' probability beliefs are identical (see LeRoy and Werner [21])[2]. Of course, comonotonicity of consumption plans implies their measurability, which in turn implies no individual risk when there is no aggregate risk.

We extend the approach of Rigotti, Shannon and Strzalecki [28] by introducing the novel notion of *conditional beliefs*. These are the probability beliefs revealed by agents' unwillingness to take fair bets conditional on an event. We consider complete preferences[3] and show that a necessary and sufficient condition for measurability of Pareto optimal allocations with respect to the aggregate endowment is that agents have at least one conditional belief in common for every event in the partition induced by the aggregate endowment. This condition is a generalization of the concordancy of probability beliefs under expected utility. The comonotonicity of consumption plans with aggregate endowment requires a stronger condition: we show that if there is at least one common conditional belief for every event in each partition coarser than the one induced by the aggregate endowment, then agents' consumption plans in all Pareto optimal allocations are comonotone with the aggregate endowment.

We provide characterizations of conditional beliefs for the most important models of ambiguity aversion, such as the multiple-priors (or maxmin) expected utility of Gilboa and Schmeidler [15], the variational preferences of Marinacci et al [23], and the smooth ambiguity model of Klibanoff, Marinacci and Mukherji [19]. For the multiple-priors expected utility the conditional beliefs for an event are the conditional probabilities derived from the priors at which the minimum expected utility is attained for consumption plans that are state independent conditional on that event. A similar result holds for variational preferences with the only difference being that, instead of the minimum of expected utility over the set of priors, one has to take the minimum of expected utility plus the cost of a probability measure over the set of all probability measures. Then, we use these characterizations of conditional beliefs to derive conditions for measurability and comonotonicity properties of risk sharing specific to each model of ambiguity aversion. In particular, we identify sets of priors for multiple-priors utility and cost functions for variational preferences which guarantee those properties of risk sharing. Conditions for comonotonicity of consumption

---

[2]This result has been known much earlier. For references, see Chateauneuf, Dana and Tallon [8].

[3]For incomplete preferences of Bewley [5] arising under Knightian uncertainty, characterizations of Pareto optimal allocations can be found in Bewley [4] and Rigotti and Shannon [27].

plans in Pareto optimal allocations are quite strong; much more so than those established by Rigotti, Shannon and Strzalecki [28] for no individual risk when there is no aggregate risk.

Properties of optimal risk sharing have attracted considerable interest in macroeconomics. Many authors, starting with Aiyagari [1], have reported an empirical observation that individual consumption plans often deviate from positive correlation with the aggregate consumption or endowment. Positive correlation is implied by comonotonicity. The so-called Aiyagari-Bewley models that have their foundations in Bewley [3] have been proposed to address this issue. These models have incomplete markets and therefore equilibrium consumption allocations need not be Pareto optimal, and hence not comonotone with the aggregate endowment, despite agents having common beliefs (and expected utilities). Another class of models introduces limited enforcement of trades that leads to non-optimal and not comonotone equilibrium allocations despite complete markets and common beliefs. (See Krueger and Perri [20] for a comprehensive study of risk sharing in these two classes of models.) More recently, models with agents' having different subjective beliefs have been introduced (e.g, Cogley and Sargent [12]). The results of this paper suggest that non-expected utilities could provide an alternative approach. For instance, it possible that Pareto optimal allocations are not comonotone when agents have multiple-priors expected utilities with sets of prior beliefs that have non-empty intersection. This happens if the more stringent condition of common conditional beliefs for all partitions coarser than the one induced by the aggregate endowment does not hold. We show this in Example 1.

The paper is organized as follows: We introduce the conditional beliefs in Section 2. In Section 3 we prove our main results about optimal risk sharing. Characterizations of conditional beliefs for various models of ambiguity averse preferences are presented in Section 4. In Section 5 we discuss the relation of our results to the literature and provide some remarks. In particular, we relate our results to those of Chateauneuf, Dana and Tallon [8] for the Choquet expected utility and to the recent work of de Castro and Chateauneuf [10] on optimal allocations for multiple-priors and Choquet expected utility when the aggregate endowment is unambiguous.

2   Conditional Beliefs

Uncertainty is described by a finite set of *states* S.[4] The set of consequences is $\mathbb{R}_+$, which we interpret as monetary payoffs. *Acts* are functions from states to consequences and can be identified with vectors in $\mathbb{R}_+^S$. Acts are denoted by f, g or h. *Constant acts* are acts that do not depend on the state, i.e,. f is constant if $f(s) = f(s')$ for all $s, s' \in S$. The set of all acts is $\mathcal{F} = \mathbb{R}_+^S$. The set of probability measures on S is denoted by Δ and $\mathring{\Delta} := \{P \in \Delta \mid P(s) > 0$ for all $s \in S\}$ is the set of probability measures that assign strictly positive probability to each state.

Let $\mathcal{G}$ be a partition of the set of states S consisting of K subsets G_j for $j = 1, \ldots, K$. An act f is $\mathcal{G}$-*measurable*, if $f(s) = f(s')$ for every $s, s' \in G_j$, for every j. Let $\mathcal{F}_G$ be the set of all $\mathcal{G}$-measurable acts and $\mathcal{F}_{\mathcal{G}}^+$ be the set of all strictly positive $\mathcal{G}$-measurable acts. Let $\mathring{\Delta}_{\mathcal{G}} = \{P \in \Delta : P(G_j) > 0$ for every $j\}$ denote the set of probability measures that assign strictly positive probability to each cell in the partition $\mathcal{G}$.

Definition 1. Two probability measures $P, Q \in \mathring{\Delta}_{\mathcal{G}}$ are $\mathcal{G}$-*concordant* if they induce the same conditional probabilities on $\mathcal{G}$, that is

$$\frac{P(s)}{P(G_j)} = \frac{Q(s)}{Q(G_j)}, \quad \forall s \in G_j, \ \forall j. \tag{1}$$

$\mathcal{G}$-concordancy is an equivalence relation on $\mathring{\Delta}_{\mathcal{G}}$ and it identifies classes of probability measures with the same $\mathcal{G}$-conditional probabilities. We will often use conditional expectation of an act on a partition of states. We write $\mathbb{E}_P[f|\mathcal{G}]$ to denote a $\mathcal{G}$-measurable act in $\mathcal{F}$ that is equal to the conditional expectation $\mathbb{E}_P[f|G_j]$ in each state $s \in G_j$. Note that if P and Q are $\mathcal{G}$-concordant, then $\mathbb{E}_P[f|\mathcal{G}] = \mathbb{E}_Q[f|\mathcal{G}]$ for every act f. For any set $\mathcal{P} \subseteq \Delta$ let $\mathcal{P}_{\mathcal{G}} := \{P \in \Delta \mid P$ is $\mathcal{G}$-concordant with some $Q \in \mathcal{P}\}$.

An agent's preferences on acts are described by a binary relation $\succsim$ on $\mathcal{F}$. We assume throughout that $\succsim$ is complete, transitive and continuous. Additional relevant properties that $\succsim$ may have are: *monotonicity* (for all $f, g \in \mathcal{F}$, if $f(s) > g(s)$ for every $s \in S$, then

---

[4]We make the finiteness assumption for tractability and ease of exposition. Rigotti, Shannon, and Strzalecki [28] extend their results from a finite to an infinite set of states. An important step is a continuity condition that guarantees the existence of Pareto optimal allocations. This approach could be applied in our setting as well.

$f \succ g$), $\mathcal{G}$-*monotonicity* (for all $f, g \in \mathcal{F}$, if $f \geq g$ and $f(s) > g(s)$ for every $s \in G_j$ for some j, then $f \succ g$), *convexity* (for all $f \in \mathcal{F}$, the set $\{g \in \mathcal{F} : g \succsim f\}$ is convex), and *strict convexity* (for all $f \neq g$ and $\alpha \in (0, 1)$, if $f \succsim g$, then $\alpha f + (1-\alpha)g \succ g$).

Rigotti, Shannon and Strzalecki ([28]; RSS henceforth) define subjective beliefs at an act $f \in \mathcal{F}$ as follows

Definition 2. A probability measure $P \in \Delta$ is a *subjective belief* at an act $f \in \mathcal{F}$ if

$$\mathbb{E}_P(g) \geq \mathbb{E}_P(f) \quad \text{for every } g \in \mathcal{F} \text{ such that } g \succsim f. \tag{2}$$

Subjective beliefs at f correspond to hyperplanes supporting upper contour set of f. The idea of relating subjective beliefs to supporting hyperplanes was proposed by Yaari [36]. If a preference relation $\succsim$ has a concave utility representation U, then it follows from a standard result in the theory of superdifferentials (see Rockafellar ([29]) and Aubin ([2])) that subjective beliefs at an interior act f are normalized supergradients of U at f. More precisely, a probability measure P is a subjective belief at a strictly positive act f if $P = \lambda\phi$ for some $\phi \in \partial U(f)$ and $\lambda > 0$, where $\partial U(f)$ denotes the superdifferential of U at f. Superdifferential $\partial U(f)$ is the set of all vectors $\phi \in \mathbb{R}^S$ (supergradients) such that

$$U(g) \leq U(f) + \phi(g - f) \quad \text{for every } g \in \mathcal{F}. \tag{3}$$

If the utility representation U is differentiable, then the superdifferential is the gradient vector $DU(f)$ in the usual sense.

RSS [28] provide characterizations of subjective beliefs for the most important models of preferences under uncertainty. Particularly important are beliefs at constant acts as they play a critical role in their study of optimal risk sharing with no aggregate risk. For the expected utility with a differentiable von Neumann-Morgenstern utility function, the subjective belief at a constant act is simply the probability measure of the expected utility representation. For multiple-priors utility of Gilboa and Schmeidler [15] (with differentiable utility), subjective beliefs at constant acts are the set of all probability priors. For variational preferences of [23], they are all probability measures with zero cost. For smooth preferences of [19], they are the average subjective probability measure.

Our focus in this paper is on conditional probabilities induced by subjective beliefs at acts that are measurable with respect to a partition of states. We identify conditional probabilities from subjective beliefs using the relation of concordancy.

Definition 3. Probability measure $Q \in \mathring{\Delta}_{\mathcal{G}}$ is a $\mathcal{G}$-*conditional belief* at an act $f \in \mathcal{F}$ if Q is $\mathcal{G}$-concordant with some subjective belief P at f such that $P \in \mathring{\Delta}_{\mathcal{G}}$.[5]

The set of all $\mathcal{G}$-conditional beliefs at an act f is denoted by $\boldsymbol{\pi}_{\mathcal{G}}(f)$. Clearly, every subjective belief that lies in $\mathring{\Delta}_{\mathcal{G}}$ is a conditional belief.

Particularly important are $\mathcal{G}$-conditional beliefs at $\mathcal{G}$-measurable acts. For concave expected utility with probability measure $P \in \mathring{\Delta}_{\mathcal{G}}$, every measure in $\mathring{\Delta}_{\mathcal{G}}$ that is $\mathcal{G}$-concordant with P is a $\mathcal{G}$-conditional belief at every $\mathcal{G}$-measurable act f. The following is an important characterization of $\mathcal{G}$-conditional beliefs at $\mathcal{G}$-measurable acts.

Proposition 1. *The following hold for every $\mathcal{G}$-measurable act f:*

(i) If $Q \in \mathring{\Delta}_{\mathcal{G}}$ is a $\mathcal{G}$-conditional belief at f, then

$$f \succsim g \quad \textit{for every } g \in \mathcal{F} \textit{ such that } \mathbb{E}_{Q}[g|\mathcal{G}] = f. \tag{4}$$

(ii) Conversely, if $\succsim$ is $\mathcal{G}$-monotone and convex, and (4) holds for a strictly positive $\mathcal{G}$-measurable act f and $Q \in \mathring{\Delta}_{\mathcal{G}}$, then Q is a $\mathcal{G}$-conditional belief at f.

PROOF: See Appendix.

For a $\mathcal{G}$-monotone and convex preference relation, condition (4) is equivalent to probability measure Q being a $\mathcal{G}$-conditional belief at a strictly positive and $\mathcal{G}$-measurable act f. Condition (4) can be written as $f \succsim f + \epsilon$ for every $\epsilon \in \mathbb{R}^{S}$ such that $\mathbb{E}_{Q}[\epsilon|\mathcal{G}] = 0$. It expresses the agent's *unwillingness to take $\mathcal{G}$-conditional bets*. Therefore Proposition 1 extends Proposition 1 in RSS ([28]).

Conditional beliefs may differ across $\mathcal{G}$-measurable acts. For instance, conditional beliefs for expected utility with nondifferentiable von Neumann-Morgenstern utility function are usually different at points of differentiability of the utility function and at points where it is nondifferentiable. We define consistent conditional beliefs for a partition $\mathcal{G}$ as follows

---

[5]We use the modifier "subjective" only when talking about the unconditional beliefs, a la RSS [28], and suppress it when talking about conditional beliefs, as many other modifiers will be used in the sequel.

Definition 4. Probability measure $Q \in \mathring{\Delta}_{\mathcal{G}}$ is a *consistent conditional belief* for partition $\mathcal{G}$ if Q is a $\mathcal{G}$-conditional belief for every strictly positive $\mathcal{G}$-measurable act, that is, $Q \in \bigcap_{f \in \mathcal{F}_{\mathcal{G}}^+} \boldsymbol{\pi}_{\mathcal{G}}(f)$. If such probability measure exists, we say that $\mathcal{G}$-*conditional beliefs are consistent*. Further, we say that $\mathcal{G}$-*conditional beliefs are strongly consistent* if all $\mathcal{G}$-conditional beliefs at every strictly positive $\mathcal{G}$-measurable act are consistent, that is, $\boldsymbol{\pi}_{\mathcal{G}}(f) = \boldsymbol{\pi}_{\mathcal{G}}(g)$ for all $f, g \in \mathcal{F}_{\mathcal{G}}^+$.

The set of all consistent $\mathcal{G}$-conditional beliefs is denoted by $\boldsymbol{\pi}_{\mathcal{G}}$. The restriction to strictly positive acts in the definition of consistency has a twofold motivation. First, we are aiming at characterizing strictly positive Pareto optimal allocations. Second, our primary tool for deriving conditional belief for models of ambiguity aversion is the superdifferential which cannot be used for acts at the boundary of $\mathcal{F}$ without further complications.

Expected utility provides a good illustration of the difference between consistency and strong consistency of conditional beliefs. Consistency holds for every concave expected utility function with the set of consistent $\mathcal{G}$-conditional beliefs being equal to all probability measures in $\mathring{\Delta}_{\mathcal{G}}$ that are $\mathcal{G}$-concordant with the probability measure P. Strong consistency holds for concave expected utility if and only if the utility function is differentiable. In this case, all probability measures in $\mathring{\Delta}_{\mathcal{G}}$ that are concordant with P are G-conditional beliefs at every strictly positive and $\mathcal{G}$-measurable act (see Section 4.1).

Corollary 1.

(i) *Suppose that $\succsim$ is $\mathcal{G}$-monotone and convex. A probability measure $Q \in \mathring{\Delta}_{\mathcal{G}}$ is a consistent $\mathcal{G}$-conditional belief if and only if*

$$\mathbb{E}_Q[g|\mathcal{G}] \succsim g \quad \text{for every strictly positive act } g \in \mathcal{F}. \tag{5}$$

(ii) *For arbitrary $\succsim$, if $Q \in \mathring{\Delta}_{\mathcal{G}}$ is a consistent $\mathcal{G}$-conditional belief then (5) holds.*

PROOF: See Appendix.

Condition (5) in Corollary 1 expresses preference for $\mathcal{G}$-conditional expectations under Q. This condition is satisfied for concave expected utility for every probability measure Q that is

$\mathcal{G}$-concordant with the prior P, in particular, for P. Thus, the preference for $\mathcal{G}$-conditional expectations holds under P for every partition $\mathcal{G}$. That is to say, P is a consistent $\mathcal{G}$-conditional belief for concave expected utility with prior P for every partition $\mathcal{G}$. The same holds for every preference relation that is monotone decreasing with respect to second-order stochastic dominance, which we refer to as strong risk aversion. Strong risk-aversion under probability measure P implies preference for $\mathcal{G}$-conditional expectations under P for every $\mathcal{G}$. Therefore P is a consistent $\mathcal{G}$-conditional belief for such preferences. Examples of preferences that are strongly risk averse include rank-dependent expected utilities of Quiggin [26] (see Chew, Karni and Safra [11]) and mean-variance preferences that are variance averse. An extensive discussion of the property of preference for conditional expectations and its relation to aversion to risk can be found in Werner ([35]). It is worth pointing out that for all strongly risk averse preferences, including concave expected utility, condition (5) holds for all acts f, strictly positive or not.

3   Optimal Risk Sharing

Suppose that there are I agents indexed by $i = 1, \ldots, I$. Agent i is endowed with a preference relation $\succsim_i$ on the set of acts $\mathcal{F}$ and her consumption set is also $\mathcal{F}$. The aggregate endowment available to the agents is $w \in \mathbb{R}_{++}^S$. A feasible allocation is a collection of consumption plans $\{f_i\}_{i=1}^I$ such that $f_i \in \mathcal{F}$ for every i and $\sum_{i=1}^I f_i(s) = w(s)$ for each $s \in S$. We shall consider only feasible allocations and refer to them as allocations dropping the adjective feasible. An allocation $\{f_i\}$ is *Pareto optimal* if there is no other allocation $\{g_i\}$, such that $g_i \succsim_i f_i$ for all i and $g_j \succ_j f_j$ for some j.

We consider two properties of risk sharing that Pareto optimal allocations may have: measurability with respect to the aggregate endowment, and comonotonicity. We first explain the property of measurability. The aggregate endowment w induces a partition of states $\mathcal{E} = \{E_1, \ldots, E_K\}$ such that $w(s) = w(s')$ for $s \neq s'$ if and only if $s, s' \in E_k$ for some k. The partition $\mathcal{E}$ is a (crude) description of the aggregate risk in the economy. For each event $E \in \mathcal{E}$, there is no aggregate risk conditional on E. The coarser the partition $\mathcal{E}$, the less aggregate risk in this sense. If the partition is the trivial partition $\mathcal{E} = \{S\}$, then there is no aggregate risk, as w is constant. An allocation $\{f_i\}$ is *$\mathcal{E}$-measurable,* if every consumption

plan f_i is $\mathcal{E}$-measurable. If an allocation is $\mathcal{E}$-measurable, then there is no individual risk conditional on every event on which there is no aggregate risk.

We turn now to comonotonicity. Two acts f and g are *comonotone* if $[f(s) - f(s')][g(s) - g(s')] \geq 0$ for every pair of states s and s'. An allocation $\{f_i\}$ is comonotone if f_i and f_j are comonotone for every i and j. One can show (see Chateauneuf, Dana and Tallon [8]) that an allocation $\{f_i\}$ is comonotone if and only if there exist non-decreasing functions $F_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f_i(s) = F_i(w(s))$, for every i. It follows that every comonotone allocation is $\mathcal{E}$-measurable; however, the converse is not true.

If agents have strictly concave expected utility, then a sufficient condition for $\mathcal{E}$-measurability of Pareto optimal allocations is that agents' probability beliefs be $\mathcal{E}$-*concordant*. If there is no aggregate risk so that $\mathcal{E}$ is the trivial partition, then beliefs are $\mathcal{E}$-concordant if and only if they are the same for all agents. A sufficient condition for comonotonicity of Pareto optimal allocations for strictly concave expected utility is that probability beliefs be the same for all agents (see Theorem 15.5.1 in LeRoy and Werner [21]).

Billot et al [6] show that having at least one common prior is sufficient for $\mathcal{E}$-measurability of optimal allocations if there is no aggregate risk (i.e., w is constant) and agents have concave multiple-priors utilities. RSS [28] extended this result to other models of ambiguity aversion using unconditional subjective beliefs in place of prior beliefs.

We begin with an example demonstrating that the existence of a common prior is not sufficient for $\mathcal{E}$-measurability of Pareto optimal allocations, and hence not sufficient for comonotonicity, if the aggregate endowment is risky and agents have concave multiple-priors utilities. This example clearly indicates the importance of conditional beliefs for a characterization of optimal allocations.

Example 1. There are three states of nature and two agents. Agent 1 has multiple-priors utility function $\min_{P \in \mathcal{P}_1} \mathbb{E}_P[v_1(f)]$ with the set of priors being a circle around the center of probability simplex Δ^3 shown in Figure 1. Agent 2 has the standard expected utility $\mathbb{E}_\pi[v_2(f)]$ with a unique prior $\pi = (\pi_1, \pi_2, \pi_3)$ such that $\pi_1 \neq \pi_2$. It holds $\pi \in \mathcal{P}_1$ so that π is the common prior. The von-Neumann-Morgenstern utility functions v_1 and v_2 are strictly concave, differentiable, and satisfy the Inada condition.

If the aggregate endowment w is risk-free, then it follows from Billot et al [6] that all Pareto optimal allocations are risk-free, that is, measurable with respect to the trivial par-
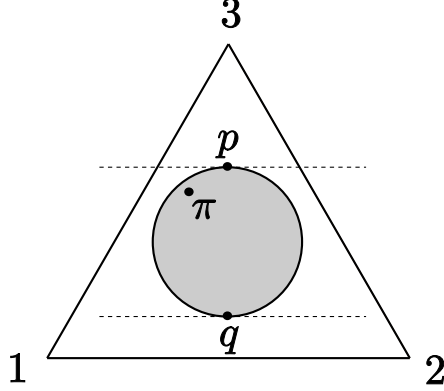
Figure 1: Priors of agents 1 and 2.

tition. Suppose that w is such that $w(1) = w(2) > w(3) > 0$. The induced partition is $\mathcal{E} = \{\{1,2\},3\}$. An allocation (f_1, f_2) with $f_i \in \mathbb{R}_+^3$ is $\mathcal{E}$-measurable if and only if $f_i(1) = f_i(2)$ for $i = 1, 2$. We claim that there are no $\mathcal{E}$-measurable Pareto optimal allocations other than the two extreme allocations $(0, w)$ and $(w, 0)$. Because of the Inada condition, all Pareto optimal allocations, other than the extreme allocations, are interior. Consider an allocation such that $f_1(1) = f_1(2) > f_1(3) > 0$. The prior that gives the minimum expected utility of f_1 is p, see Figure 1. Agent's 1 multiple-priors utility is differentiable at f_1 and the marginal rate of substitution between consumption in states 1 and 2 is $p_1/p_2 = 1$. The respective marginal rate of substitution for agent 2 at f_2 is π_1/π_2, which is different from 1. Such allocation (f_1, f_2) cannot be Pareto optimal. Next, consider (f_1, f_2) such that $0 < f_1(1) = f_1(2) < f_1(3)$. The prior that gives the minimum expected utility of f_1 is q, see Figure 1. The marginal rates of substitution between consumption in states 1 and 2 are again 1 for agent 1 and π_1/π_2 for agent 2. Such allocation cannot be Pareto optimal either. Finally, consider $f_1(1) = f_1(2) = f_1(3) > 0$ so that f_1 is risk-free. Agent's 1 utility is not differentiable at f_1. The superdifferential of the multiple-priors utility of agent 1 at f_1 are all probability prior in $\mathcal{P}_1$ rescaled by the marginal utility $v_1'(f_1)$. The vector of marginal utilities for agent 2 at f_2 is $(\pi_1 v_2'(f_2(1)), \pi_2 v_2'(f_2(2)), \pi_3 v_2'(f_2(3)))$. Since $f_2(3) - f_2(1) = w(3) - w(1)$, one can choose utility function v_2 so that this vector lies outside of the superdifferential for agent 1, for any such (f_1, f_2). It follows from Theorem 7 in RSS [28] that such allocations

11

cannot be Pareto optimal.

Thus there is no $\mathcal{E}$-measurable optimal allocations other than the extreme allocations.

3.1 Risk Sharing with no Aggregate Conditional Risk

In this section we provide necessary and sufficient conditions for $\mathcal{E}$-measurability of Pareto optimal allocations for general preferences using the concept of conditional beliefs. We shall use a slightly weaker notion of essential $\mathcal{E}$-measurability in our results. An allocation $\{f_i\}$ is *essentially $\mathcal{E}$-measurable*, if there exists a $\mathcal{E}$-measurable allocation $\{\hat{f}_i\}$ such that $f_i \sim_i \hat{f}_i$, for every i. Clearly, if every agent's preference relation is strictly convex, then a Pareto optimal allocation is essentially $\mathcal{E}$-measurable if and only if it is $\mathcal{E}$-measurable.

Theorem 1. *Suppose that each agent's $\mathcal{E}$-conditional beliefs are consistent. If agents have at least one common consistent $\mathcal{E}$-conditional belief, i.e.,*

$$\bigcap_{i=1}^{I} \boldsymbol{\pi}_{\mathcal{E}}^{i} \neq \emptyset \tag{6}$$

then every interior Pareto optimal allocation is essentially $\mathcal{E}$-measurable.

PROOF: Let $\{f_i\}$ be a Pareto optimal allocation such that f_i is strictly positive for every i, and let Q be a probability measure in $\cap_{i=1}^{I} \boldsymbol{\pi}_{\mathcal{E}}^{i}$. Consider an allocation $\{\tilde{f}_i\}$ defined by

$$\tilde{f}_i = \mathbb{E}_Q[f_i|\mathcal{E}],$$

for every i. The allocation $\{\tilde{f}_i\}$ is feasible and $\mathcal{E}$-measurable. By Corollary 1 (ii), $\tilde{f}_i \succsim_i f_i$, for every i. Since the allocation $\{f_i\}$ is Pareto optimal, it follows that $f_i \sim_i \tilde{f}_i$ for every i. Therefore $\{f_i\}$ is essentially $\mathcal{E}$-measurable. $\square$

If agents have concave expected utilities with prior $P_i \in \mathring{\Delta}_{\mathcal{E}}$, then condition (6) holds if and only if probability measures P_i are $\mathcal{E}$-concordant

A converse result to Theorem 1 holds under strong consistency of beliefs and convexity of preferences.

Theorem 2. *Suppose that each agent's preferences are $\mathcal{E}$-monotone and convex, and her $\mathcal{E}$-conditional beliefs are strongly consistent. If there exists an interior Pareto optimal allocation that is $\mathcal{E}$-measurable, then there exists at least one common consistent $\mathcal{E}$-conditional belief, i.e., condition (6) holds.*

PROOF: Consider an interior $\mathcal{E}$-measurable Pareto optimal allocation. By the separation argument as in the standard proof of the Second Welfare Theorem, there exists a probability measure $Q \in \Delta$ such that $\mathbb{E}_Q(g_i) \geq \mathbb{E}_Q(f_i)$ whenever $g_i \succsim_i f_i$ for every i. Hence, Q is a subjective belief at f_i for agent i. By $\mathcal{E}$-monotonicity of $\succsim_i$, it follows that $Q \in \mathring{\Delta}_{\mathcal{E}}$ and therefore it is a $\mathcal{E}$-conditional belief at f_i for every i. Since $\mathcal{E}$-conditional beliefs are strongly consistent for each i, it follows that $Q \in \bigcap_{i=1}^{I} \boldsymbol{\pi}_{\mathcal{E}}^i$. $\square$

Theorems 1 and 2 imply the following corollary that extends the main result of RSS [28] from a constant aggregate endowment w (no aggregate risk) to arbitrary w.

Corollary 2. *Suppose that each agent's preferences are $\mathcal{E}$-monotone and strictly convex, and her $\mathcal{E}$-conditional beliefs are strongly consistent. The following conditions are equivalent:*

(i) *There exists an interior $\mathcal{E}$-measurable Pareto optimal allocation*

(ii) *All interior Pareto optimal allocations are $\mathcal{E}$-measurable*

(iii) $\bigcap_{i=1}^{I} \boldsymbol{\pi}_{\mathcal{E}}^i \neq \emptyset$

In RSS [28] it is assumed that aggregate endowment is constant, (unconditional) beliefs are strongly consistent (their Axiom 7), and preferences are strictly convex. Their Proposition 9 states that agents have at least one common consistent unconditional belief if and only if all interior Pareto optimal allocations are constant, which in turn is equivalent to the existence of a constant interior optimal allocation.

3.2   Comonotone Risk Sharing

In this section we provide sufficient conditions for comonotonicity of Pareto optimal allocations. These conditions involve a greater degree of agreement of conditional beliefs across

agents. We consider a collection of partitions of S that are coarser[6] than the partition $\mathcal{E}$ induced by the aggregate endowment. We denote this set of partitions by Σ_c.

Theorem 3. *Suppose that every agent's preferences are strictly convex, and her $\mathcal{G}$-conditional beliefs are consistent for every $\mathcal{G} \in \Sigma_c$. If agents have at least one common consistent $\mathcal{G}$-conditional belief for every $\mathcal{G} \in \Sigma_c$, i.e.,*

$$\bigcap_{i=1}^{I} \boldsymbol{\pi}_{\mathcal{G}}^{i} \neq \emptyset \tag{7}$$

for all $\mathcal{G} \in \Sigma_c$, then every interior Pareto optimal allocation is comonotone.

The proof of Theorem 3 in the Appendix shows that the statement of the theorem remains true if the condition of agreement of consistent conditional beliefs across agents is required only for a subset of partitions coarser than $\mathcal{E}$, namely those that can be obtained by merging arbitrary two elements of the partition $\mathcal{E}$.

If agents have concave expected utilities, then condition (7) holds if and only if their prior P_i are the same.

4 Conditional Beliefs under Ambiguity Aversion

4.1 Multiple-Prior Expected Utility

One of the most popular alternatives to expected utility is the multiple-priors model. Under the multiple-priors specification, the agent has a set of probability measures on states—multiple priors—and makes her decisions by considering the expected utility under the prior that gives the lowest value of expected utility. Such preferences are most appealing in situations of so-called ambiguity when, as in the Ellsberg paradox, there is insufficient information for an agent to form a unique probabilistic belief. The axiomatization of multiple-priors utility is due to Gilboa and Schmeidler [15].

---

[6]Partition $\mathcal{G}'$ is coarser than partition $\mathcal{G}$ iff any element of $\mathcal{G}$ is a subset of some element of $\mathcal{G}'$.

The multiple-priors expected utility takes the form

$$\min_{P \in \mathcal{P}} \mathbb{E}_P[v(f)], \tag{8}$$

for some strictly increasing and continuous utility function $v : \mathbb{R}_+ \to \mathbb{R}$ and some convex and closed set $\mathcal{P} \subseteq \Delta$ of probability measures on S. We assume throughout this section that v is concave. Observe that the preference is $\mathcal{G}$-monotone if and only if $\mathcal{P} \subseteq \mathring{\Delta}_{\mathcal{G}}$.

Let $\mathcal{P}^v(f)$ denote the set of prior for which the minimum expected utility is attained.

$$\mathcal{P}^v(f) = \arg\min_{P \in \mathcal{P}} \mathbb{E}_P[v(f)]. \tag{9}$$

Let $\mathcal{P}^v_{\mathcal{G}}(f)$ denote the set of probability measures in $\mathring{\Delta}_{\mathcal{G}}$ that are $\mathcal{G}$-concordant with some probability in $\mathcal{P}^v(f)$. The set $\mathcal{P}^v_{\mathcal{G}}(f)$ represents the $\mathcal{G}$-conditional probabilities induced by the minimizing probabilities at f. The set of minimizing probabilities for the linear utility function $v(x) = x$ and the set of induced $\mathcal{G}$-conditional probabilities are denoted by $\mathcal{P}(f)$ and $\mathcal{P}_{\mathcal{G}}(f)$, respectively. They will be used later.

If the function v is differentiable at a strictly positive act f,[7] then the superdifferential of (8) at f is

$$\{\phi \in \mathcal{R}^S : \phi_s = v'(f(s))P(s), \ \forall s, \ \text{for some } P \in \mathcal{P}^v(f)\}, \tag{10}$$

see Aubin ([2]). The normalized vectors in the superdifferential (10) are the subjective beliefs at the act f (see RSS ([28])). If the act f is $\mathcal{G}$-measurable, then the marginal utility $v'(f)$ is also $\mathcal{G}$-measurable and every normalized vector in (10) is $\mathcal{G}$-concordant with some probability measure in $\mathcal{P}^v(f)$. It follows that

$$\pi_{\mathcal{G}}(f) = \mathcal{P}^v_{\mathcal{G}}(f). \tag{11}$$

If v is not differentiable at f, then only one inclusion holds: $\pi_{\mathcal{G}}(f) \supseteq \mathcal{P}^v_{\mathcal{G}}(f)$.

The minimizing probabilities (9) depend in general on the utility function v. Therefore, the conditional beliefs depend on v as well; however, this is not so for consistent beliefs. If Q is a consistent $\mathcal{G}$-conditional belief for the multiple-priors model with concave utility v, then it is a consistent $\mathcal{G}$-conditional belief for every concave utility, in particular, for the linear utility. This is demonstrated in the following

---

[7]We say that v is differentiable at act f if it is differentiable at every $f(s)$ for $s \in S$.

Proposition 2. *For every multiple-priors utility with concave utility, the set of consistent $\mathcal{G}$-conditional beliefs is*

$$\boldsymbol{\pi}_{\mathcal{G}} = \cap_{f \in \mathcal{F}_{\mathcal{G}}} \mathcal{P}_{\mathcal{G}}(f). \tag{12}$$

where $\mathcal{F}_{\mathcal{G}}$ is the set of all strictly positive $\mathcal{G}$-measurable acts.

PROOF: See Appendix.

The following example illustrates consistent conditional beliefs for the multiple-priors model.

Example 2. Consider the set of priors $\mathcal{P} = \{(p_1, p_2, p_3) \in \Delta^3 : p_s \geq b \text{ for } s = 1, 2, 3\}$, where b is a lower bound on probabilities satisfying $0 < b < \frac{1}{3}$. Let the partition of states be $\mathcal{G} = \{\{1, 2\}, \{3\}\}$. Act f is $\mathcal{G}$-measurable if and only if $f(1) = f(2)$. The sets of minimizing probabilities at f are $\mathcal{P}(f) = \{(b, b, 1 - 2b)\}$ if $f(3) < f(1)$, $\mathcal{P}(f) = \{(q_1, q_2, b) : q_1 + q_2 = 1 - b, q_1 \geq b, q_2 \geq b\}$ if $f(3) > f(1)$, and $\mathcal{P}(f) = \mathcal{P}$ if $f(1) = f(3)$. The respective sets of $\mathcal{G}$-conditional beliefs are $\mathcal{P}_{\mathcal{G}}(f) = \{(q_1, q_2, q_3) \in \Delta^3 : q_1 = q_2\}$, $\mathcal{P}_{\mathcal{G}}(f) = \{(q_1, q_2, q_3) \in \Delta^3 : \frac{b}{1-2b} \leq \frac{q_2}{q_1} \leq \frac{1-2b}{b}\}$, and $\mathcal{P}_{\mathcal{G}}(f) = \mathcal{P}_{\mathcal{G}}$. It follows from (12) that the set of consistent $\mathcal{G}$-conditional beliefs is $\boldsymbol{\pi}_{\mathcal{G}} = \{(q_1, q_2, q_3) \in \Delta^3 : q_1 = q_2\}$, that is all measures with equal probabilities of states 1 and 2.

If $\mathcal{G}$ is the trivial partition, then the $\mathcal{G}$-measurable acts are simply the constant acts. The set of minimizing probabilities for every constant act is the whole set of priors $\mathcal{P}$. Conditional probabilities for trivial partition coincide with unconditional probabilities. Proposition 2 implies that conditional beliefs for the trivial partition (i.e., unconditional beliefs) are consistent and the set of consistent unconditional beliefs is the whole set $\mathcal{P}$. They are the subjective beliefs for constant acts, see RSS [28]. Consistency of conditional beliefs for other partitions is not always guaranteed. This is illustrated by the following.

Example 3. The set of priors arising in the context of the Ellsberg Paradox (with one urn and balls of 3 colors) is $\mathcal{P} = \{(p_1, p_2, p_3) \in \Delta^3 : p_1 \geq b, p_2 \geq b, p_3 = \frac{1}{3}\}$, where b is a lower bound such that $0 < b < \frac{1}{3}$. Consider the partition $\mathcal{G} = \{\{1\}, \{2, 3\}\}$. Subjective beliefs at $\mathcal{G}$-measurable acts are $\mathcal{P}(f) = \{(b, \frac{2}{3} - b, \frac{1}{3})\}$ if $f(1) < f(3)$, $\mathcal{P}(f) = \{(b, \frac{2}{3} - b, \frac{1}{3})\}$ if $f(1) > f(3)$, and $\mathcal{P}(f) = \mathcal{P}$ if $f(1) = f(3)$. The former two sets consist of single probability measures that are not $\mathcal{G}$-concordant with each other. Therefore, the set of consistent $\mathcal{G}$-conditional beliefs is empty.

We now present a characterization of consistent conditional beliefs for the multiple-priors model. For any probability measures $Q \in \mathring{\Delta}_{\mathcal{G}}$ and $P \in \Delta$, we define another probability measure $P_{\mathcal{G}}^Q$ by

$$P_{\mathcal{G}}^Q(A) = \sum_{i=1}^k Q(A|G_i)P(G_i) \tag{13}$$

for every $A \subseteq S$. The probability measure $P_{\mathcal{G}}^Q$ coincides with P on elements of partition $\mathcal{G}$ and has conditional probabilities of Q within each element of the partition; in other words, it takes the marginals from P and conditionals from Q. Note that $P_{\mathcal{G}}^Q = Q$ if $\mathcal{G}$ is the trivial partition.

Theorem 4. *For every multiple-priors utility with concave utility and set of priors $\mathcal{P} \subseteq \mathring{\Delta}_{\mathcal{G}}$, the probability measure Q is a consistent $\mathcal{G}$-conditional belief if and only if*

$$P_{\mathcal{G}}^Q \in \mathcal{P} \quad \text{for every } P \in \mathcal{P}. \tag{14}$$

PROOF: See Appendix.

An important class of sets of priors that give rise to consistent conditional beliefs for multiple-priors utility are stable sets introduced by Werner ([35]). A set probability measures $\mathcal{P}$ is called *Q-stable* for $Q \in \mathring{\Delta}$ if (14) holds for every partition $\mathcal{G}$. If $\mathcal{P}$ is Q-stable, then Q is a consistent $\mathcal{G}$-conditional belief for every partition $\mathcal{G}$. Examples of Q-stable sets of priors include cores of convex distortions of Q. For an increasing and convex function $\varphi : [0,1] \to [0,1]$ that satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$ the core of the distortion φ of Q is

$$\{P \in \Delta : P(A) \geq \varphi(Q(A)) \text{ for every } A \subseteq S\}. \tag{15}$$

Sets of priors with lower bound $\{P \in \Delta : P \geq \gamma Q\}$ or upper bound $\{P \in \Delta : P \leq \lambda Q\}$, for $\gamma, \lambda \in [0,1]$ are cores of convex distortions of Q. The set $\mathcal{P}$ in Example 2 is a core of a convex distortion of $Q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Another important class of Q-stable sets are neighborhoods of Q in a divergence distance.

Divergence distance between probability measures $P \in \Delta$ and $Q \in \mathring{\Delta}$ is defined by

$$d(P,Q) = \sum_{s \in S} \psi\left(\frac{P(s)}{Q(s)}\right) Q(s) \qquad (16)$$

for a convex function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\psi(1) = 0$ and $\lim_{t \to \infty} \psi(t)/t = \infty$. A special case of (16) is the Kullback-Leibler divergence or relative entropy that obtains by taking $\psi(t) = t \ln(t) - t + 1$. Other examples are relative Gini index and total variation, see [23]. A neighborhood of Q in divergence distance is the set $\{P \in \Delta : d(P,Q) \leq \epsilon\}$ for some $\epsilon > 0$.

By Definition 4, $\mathcal{G}$-conditional beliefs are strongly consistent if they are the same for every $\mathcal{G}$-measurable strictly positive act. Since constant acts are $\mathcal{G}$-measurable for every $\mathcal{G}$ and subjective beliefs at any strictly positive constant act are the whole set of priors $\mathcal{P}$ (assuming differentiable utility function), it follows that, if $\mathcal{G}$-conditional beliefs are strongly consistent, then they must equal the set all probability measures that are $\mathcal{G}$-concordant with some probability measure in $\mathcal{P}$, denoted $\mathcal{P}_\mathcal{G}$. The necessary and sufficient conditions for this are:

Proposition 3. *For every multiple-priors utility with concave and differentiable utility and set of priors $\mathcal{P} \subseteq \mathring{\Delta}_\mathcal{G}$, the $\mathcal{G}$-conditional beliefs are strongly consistent if and only if*

$$P_\mathcal{G}^Q \in \mathcal{P} \text{ for every } P, Q \in \mathcal{P}. \qquad (17)$$

Then the strongly consistent $\mathcal{G}$-conditional beliefs consist of all probability measures in $\mathcal{P}_\mathcal{G}$.

PROOF: See Appendix.

Property (17) is the requirement of rectangularity of $\mathcal{P}$ with respect to $\mathcal{G}$ that has been introduced by Epstein and Schneider [13] in their analysis of dynamic consistency of multiple-priors utility. Examples of rectangular sets of priors can be found there.

Proposition 3 implies that unconditional beliefs are strongly consistent for every differentiable utility function and every set of priors $\mathcal{P}$, and they are equal to the whole set $\mathcal{P}$. This observation can be extended to any partition that consists of unambiguous events.

Definition 5. An event $E \subseteq S$ is *unambiguous* if $P(E) = Q(E)$ for every $P, Q \in \mathcal{P}$. A partition $\mathcal{G}$ is *unambiguous* if it consists of unambiguous events.

If there exists an unambiguous event other than S or $\emptyset$, then there exits a non-trivial unambiguous partition. Ghirardato and Marinacci [14] and Nehring [25] provide axiomatic characterizations of the multiple-priors preferences with sets of priors that have unambiguous events. If $\mathcal{G}$ is an unambiguous partition, then $P_{\mathcal{G}}^Q = Q$ for every $P, Q \in \mathcal{P}$ and property (17) holds. Proposition 3 implies the following

Corollary 3. *For every multiple-priors utility with concave and differentiable utility and set of priors $\mathcal{P} \subseteq \mathring{\Delta}_{\mathcal{G}}$, if the partition $\mathcal{G}$ is unambiguous, then the $\mathcal{G}$-conditional beliefs are strongly consistent, and the strongly consistent $\mathcal{G}$-conditional beliefs consist of all probability measures in $\mathcal{P}_{\mathcal{G}}$.*

We shall review now the results on optimal risk sharing (Section 3) for the case when agents have multiple-priors utilities with concave utility functions. If the set of priors $\mathcal{P}_i$ of agent i is Q_i-stable for every i and probability measures Q_i are $\mathcal{E}$-concordant, then Theorem 1 implies that every interior Pareto optimal allocation is essentially $\mathcal{E}$-measurable. If utility functions are strictly concave and each set of priors $\mathcal{P}_i$ is Q-stable with respect to the common probability measure Q, then Theorem 3 implies that every interior Pareto optimal allocation is comonotone. The condition of stability is much stronger than what is required in Theorem 1 in that it implies consistency of conditional beliefs for every partition $\mathcal{G}$, not just for the partition $\mathcal{E}$ induced by the aggregate endowment. A good illustration of the difference is when the aggregate endowment is constant. For arbitrary sets of priors, if those sets have nonempty intersection, then, by Theorem 1, every interior Pareto optimal allocation is essentially constant (i.e., essentially measurable with respect to the trivial partition).

The most interesting case of strongly consistent beliefs arise for multiple-priors utilities and unambiguous partitions (see Corollary 3). We say that the aggregate endowment is unambiguous if partition $\mathcal{E}$ is unambiguous for every agent. Corollary 2 implies the following

Corollary 4. *Suppose that every agent has multiple-priors utility with strictly concave and differentiable utility. If the aggregate endowment is unambiguous, then the following conditions are equivalent:*

(i) There exists an interior $\mathcal{E}$-measurable Pareto optimal allocation.

(ii) All interior Pareto optimal allocations are $\mathcal{E}$-measurable.

(iii) $\bigcap_{i=1}^{I} \mathcal{P}_{i\mathcal{E}} \neq \emptyset$

A related result can be found in de Castro and Chateauneuf ([10], Theorem 5.3).

4.2 Variational Preferences

Variational preferences have a utility representation of the form

$$\min_{P \in \Delta} \left\{ \mathbb{E}_P[v(f)] + c(P) \right\}, \tag{18}$$

for some strictly increasing and continuous utility function $v : \mathbb{R}_+ \to \mathbb{R}$, such that $v(\mathbb{R}_+) = \mathbb{R}_+$,[8] and some convex and lower semicontinuous function $c : \Delta \to [0, \infty]$ such that there exists $Q \in \Delta$ with $c(Q) = 0$. In this specification c can be interpreted as the cost (in terms of utility) of considering every belief. Observe that the preference is $\mathcal{G}$-monotone if and only if $\mathcal{P}^{\text{fin}} \subseteq \mathring{\Delta}_{\mathcal{G}}$ where $\mathcal{P}^{\text{fin}} = \{P \in \Delta : c(P) < \infty\}$. The axiomatization of variational preferences is due to Maccheroni, Marinacci, and Rustichini [23]. Hansen and Sargent [16] considered variational preferences with a cost function c given by $c(P) = \theta R(P, Q)$, where Q is the agent's reference belief, R is the relative entropy measure and $\theta > 0$ is a scale parameter. Such variational preferences are called multiplier preferences; their axiomatization is due to Strzalecki [33]. A more general subclass of variational preferences are divergence preferences with cost function $c(P) = \theta d(P, Q)$ for a divergence distance d given by (16).

Let $\mathcal{P}^v(f)$ denote the set of priors for which the minimum in (18) is attained. That is,

$$\mathcal{P}^v(f) = \arg\min_{P \in \Delta} \left\{ \mathbb{E}_P[v(f)] + c(P) \right\} \tag{19}$$

Let $\mathcal{P}^v_{\mathcal{G}}(f)$ denote the set of probability measures that are $\mathcal{G}$-concordant with some probability in $\mathcal{P}^v(f)$. Further, let $\mathcal{P}(f)$ and $\mathcal{P}_{\mathcal{G}}(f)$ be the sets of minimizing probabilities and the induced $\mathcal{G}$-conditional probabilities, respectively, for linear utility function.

If the function v is differentiable at a strictly positive act f, then the superdifferential of utility function (18) at f is (by Theorem 18 of [23])

$$\{\phi \in \mathbb{R}^S : \phi_s = v'(f(s))P(s), \ \forall s \in S, \ \text{ for some } P \in \mathcal{P}^v(f)\} \tag{20}$$

---

[8]The unboundedness of v is guaranteed by Axiom A7 of [23].

If f is $\mathcal{G}$-measurable, then the marginal utility $v'(f(s))$ is the same within each cell of the partition $\mathcal{G}$ and every normalized vector in the superdifferential (20) is $\mathcal{G}$-concordant with some measure in $\mathcal{P}^v(f)$. Therefore

$$\boldsymbol{\pi}_{\mathcal{G}}(f) = \mathcal{P}^v_{\mathcal{G}}(f) \tag{21}$$

for every $f \in \mathcal{F}_{\mathcal{G}}$. If v is not differentiable at f, then only one inclusion holds: $\pi_{\mathcal{G}}(f) \supset \mathcal{P}^v_{\mathcal{G}}(f)$.

Just like for the multiple-priors model, if Q is a consistent $\mathcal{G}$-conditional belief for variational preferences with some concave utility function v, then Q is a consistent $\mathcal{G}$-conditional belief for every concave utility, in particular, for the linear utility. We have the following proposition.

Proposition 4. *For every $\mathcal{G}$-monotone variational preference with concave utility, the set of consistent $\mathcal{G}$-conditional beliefs is*

$$\boldsymbol{\pi}_{\mathcal{G}} = \cap_{f \in \mathcal{F}_{\mathcal{G}}} \mathcal{P}_{\mathcal{G}}(f). \tag{22}$$

PROOF: See Appendix.

For a constant act f the set of minimizing probabilities (19) consists of all probability measures with zero cost, $\mathcal{P}^0 = \{Q \in \Delta : c(Q) = 0\}$. This implies that unconditional beliefs for variational preferences are consistent for every concave utility, and the set of consistent unconditional beliefs is $\mathcal{P}^0$ (see RSS [28]).

The following result is an analog of Theorem 4 for variational preferences.

Theorem 5. *For every variational preferences with concave utility, if*

$$c(P^Q_{\mathcal{G}}) \leq c(P) \quad \text{for every } P \in \Delta, \tag{23}$$

then probability measure Q a consistent $\mathcal{G}$-conditional belief.[9]

PROOF: See Appendix.

---

[9]Werner [35] shows that condition (23) is also necessary for Q to be consistent $\mathcal{G}$-conditional belief if cost function c is finite.

An important class of cost functions satisfying condition (23) are rescaled divergence measures of the form $c(P) = \theta d(P, Q)$, where d is given by (16). The relative entropy cost function of Hansen and Sargent [16] belongs to that class. This follows from the following

Proposition 5. *If $d(P, Q)$ is a divergence measure with $Q \in \mathring{\Delta}$, then*

$$d(P_{\mathcal{G}}^Q, Q) \leq d(P, Q) \tag{24}$$

for every $P \in \Delta$ and every partition $\mathcal{G}$.

PROOF: See Appendix.

In the discussion of strongly consistent conditional beliefs we focus on unambiguous partitions. An event is unambiguous for variational preferences if it has the same probability for every probability measure considered possible by the agent, that is, probability measure with finite cost. Formally

Definition 6. *An event $E \subseteq S$ is unambiguous if $P(E) = Q(E)$ for every $P, Q \in \mathcal{P}^{\mathrm{fin}}$. A partition $\mathcal{G}$ is unambiguous if it consists of unambiguous events.*

If there exists an unambiguous event other than S or $\emptyset$, then there exits a non-trivial unambiguous partition. Strzalecki [34] provides a characterization of cost functions that give rise to unambiguous events. If $\mathcal{G}$ is an unambiguous partition and the utility function is differentiable, then the set of subjective beliefs at every strictly positive $\mathcal{G}$-measurable act is equal to the set $\mathcal{P}^0$. $\mathcal{G}$-conditional beliefs are the set $\mathcal{P}_{\mathcal{G}}^0$, i.e., $\mathcal{G}$-conditional probabilities induced by probability measures from $\mathcal{P}^0$. It follows that

Proposition 6. *For every $\mathcal{G}$-monotone variational preference with concave utility, if partition $\mathcal{G}$ is unambiguous, then the $\mathcal{G}$-conditional beliefs are strongly consistent, and the set of strongly consistent $\mathcal{G}$-conditional beliefs consists of all probability measures in $\mathcal{P}_{\mathcal{G}}^0$.*

We review now the results on optimal risk sharing for the case when agents have variational preferences with concave utility functions. If the cost function c_i for agent i is a rescaled divergence measure from Q_i and probability measures Q_i are $\mathcal{E}$-concordant, then every interior Pareto optimal allocation is essentially $\mathcal{E}$-measurable (Theorem 1). If utility

functions are strictly concave and each cost function c_i is a rescaled divergence measure from the common probability measure Q, then every interior Pareto optimal allocation is comonotone (Theorem 3). For the case of no aggregate risk, Theorem 1 implies that, for arbitrary cost functions, if the sets of zero-cost probability measures have nonempty intersection, then every interior Pareto optimal allocation is essentially constant.

For the case of unambiguous aggregate endowment, we obtain from Corollary 2 the following

Corollary 5. *Suppose that every agent has variational preferences with strictly concave and differentiable utility. If the aggregate endowment is unambiguous, then the following conditions are equivalent:*

(i) *There exists an interior $\mathcal{E}$-measurable Pareto optimal allocation*

(ii) *All interior Pareto optimal allocations are $\mathcal{E}$-measurable*

(iii) $\bigcap_{i=1}^{I} \mathcal{P}_{i\mathcal{E}}^{0} \neq \emptyset$

4.3 Smooth Model of Ambiguity Aversion

The utility representation in the smooth model of Klibanoff, Marinacci and Mukherji ([19]; henceforth KMM) takes the form

$$\mathbb{E}_{\mu}[\phi(\mathbb{E}_P v(f))], \tag{25}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R}_+ \to \mathbb{R}$ are strictly increasing and concave functions that are differentiable in the interior of their domains. The probability measure μ is the second-order prior, that is, a probability distribution on the set of probability measures Δ. Observe that the preference is $\mathcal{G}$-monotone if and only if supp $(\mu) \subseteq \mathring{\Delta}_{\mathcal{G}}$.

A measure that plays an important role in the analysis is the "average measure" $P^{\mu} \in \Delta$ defined as $P^{\mu}(s) = \mathbb{E}_{\mu}[P(s)]$ for every $s \in S$. As RSS [28] show, the measure P^{μ} is a subjective belief at every strictly positive constant act. More generally, Proposition 5 of RSS [28] implies that the utility representation (25) is differentiable at every strictly positive act f with the gradient being a vector whose sth coordinate for $s \in S$ is

$$v'(f(s))\mathbb{E}_{\mu}\big[\phi'(\mathbb{E}_P v(f))P(s)\big]. \tag{26}$$

23

The subjective belief at f is the gradient vector (26) normalized to be a probability measure. The set of $\mathcal{G}$-conditional beliefs at f consists of all probability measures in $\mathring{\Delta}_{\mathcal{G}}$ that are $\mathcal{G}$-concordant with the subjective belief at f. In general, conditional beliefs in the smooth model need not be consistent.

Example 4. Let there be 3 states and let the second-order prior μ assign equal probabilities to two probability vectors in Δ^3: $(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$. Suppose that v is the linear utility and ϕ is strictly concave. Consider the partition $\mathcal{G} = \{\{1,2\}, \{3\}\}$ and two $\mathcal{G}$-measurable acts $f = (7,7,1)$ and $g = (2,2,8)$. The unique subjective belief at f is $\big(2\phi'(4) + 3\phi'(5), \phi'(4) + \phi'(5), 3\phi'(4) + 2\phi'(5)\big)$ normalized to be a probability vector. The subjective belief at g is $\big(2\phi'(5) + 3\phi'(4), \phi'(5) + \phi'(4), 3\phi'(5) + 2\phi'(4)\big)$, normalized. These two subjective beliefs are not $\mathcal{G}$-concordant since ratios of probabilities of states 1 and 2 are different. Therefore, the set of consistent $\mathcal{G}$-conditional beliefs is empty.

A sufficient condition for strong consistency of $\mathcal{G}$-conditional beliefs in the smooth model is the concordancy of all measures in the support of the second-order prior μ.

Proposition 7. *If all probability measures in the support of μ are $\mathcal{G}$-concordant, then $\mathcal{G}$-conditional beliefs are strongly consistent and the set of strongly consistent $\mathcal{G}$-conditional beliefs consists of all measures that are $\mathcal{G}$-concordant with P^μ.*

PROOF: See Appendix.

Unambiguous events and partitions can be defined for the smooth model (see an axiomatic derivation in KMM [19]) and they lead to strongly consistent conditional beliefs.

Definition 7. An event $E \subseteq S$ is *unambiguous* if there exists a $\gamma \in [0,1]$ such that $P(E) = \gamma$, μ-almost-everywhere. A partition $\mathcal{G}$ is *unambiguous* if it consists of unambiguous events.

Similarly to the multiple-priors and variational preferences, conditional beliefs in the smooth model are strongly consistent if the partition $\mathcal{G}$ is unambiguous.

Proposition 8. *For every $\mathcal{G}$-monotone smooth ambiguity preference and every unambiguous partition $\mathcal{G}$, the $\mathcal{G}$-conditional beliefs are strongly consistent and the set of strongly consistent $\mathcal{G}$-conditional beliefs consists of all measures that are $\mathcal{G}$-concordant with P^μ.*

24

PROOF: See Appendix.

For smooth ambiguity preferences, Theorem 1 implies that interior Pareto optimal allocations are $\mathcal{E}$-measurable if all probability measures in the support of the second-order priors μ_i are $\mathcal{E}$-concordant with the average measure $P_i^\mu \in \mathring{\Delta}_{\mathcal{G}}$, and measures P_i^μ are $\mathcal{E}$-concordant. For the case of unambiguous aggregate endowment, we have

Corollary 6. *Suppose that every agent has smooth KMM preferences with strictly concave and differentiable utility. If the aggregate endowment is unambiguous, then the following conditions are equivalent:*

  (i) There exists an interior unambiguous Pareto optimal allocation

  (ii) All interior Pareto optimal allocations are $\mathcal{E}$-measurable

  (iii) The measures P^{μ_i} are identical

5   Relation to the Literature and Remarks

Chateauneuf, Dana and Tallon [8] study properties of Pareto optimal allocations when agents have Choquet expected utilities, that is, expected utilities with nonadditive probabilities, or capacities, introduced be Schmeidler [32]. Proposition 3.1 in Chateauneuf, Dana and Tallon [8] says that, if agents have Choquet expected utilities with the same convex capacity, then Pareto optimal allocations are comonotone. This result is not implied by our Theorem 3, the reason being that conditional beliefs need not be consistent for such preferences. In Example 3 the set of priors is the core of a convex capacity and there is no consistent conditional belief.

Every Choquet expected utility with convex capacity has the property of comonotonic independence, that is, for each subset of comonotone acts, there exists a probability measure such that the preferences coincide with the expected utility with this probability measure. Chateauneuf, Dana and Tallon [8] show that if agents have Choquet expected utilities with the same convex capacity then Pareto optimal allocations are the same as if agents had expected utilities with a common probability measure identified in the subset of acts that are comonotone with the aggregate endowment.

Our observation from Section 4.1 that Pareto optimal allocations are comonotone when sets of priors are cores of convex distortions of a common probability measure can be found in Chateauneuf, Dana and Tallon ([8], Proposition 4.1). Cores of convex distortions of probability measures do generate consistent conditional beliefs.

Kajii and Ui [17] study an economy with no aggregate risk and derive necessary and sufficient conditions for the existence of an agreeable *bet*, i.e., a special kind of trade involving only two payoffs, as opposed to an arbitrary trade as considered in [6] and [28] that can be an arbitrary vector of payoffs. In another paper, Kajii and Ui [18] study interim efficient allocations in an economy with asymmetric information. They provide a characterization of interim efficiency for Bewley preferences (Bewley's [**?**]) and a sufficient condition for interim efficiency for multiple-priors utilities. Martins-da-Rocha [22] provides a full characterization of interim efficiency for multiple-priors utilities and for a general class of preferences. The condition of nonempty intersection of the sets of "compatible priors" for posterior beliefs in [18] and [22] is similar to our condition of nonempty intersection of consistent conditional beliefs.

The papers by Kajii and Ui [18] and Martins-da-Rocha [22], as well as the papers by Bewley [5], and Rigotti and Shannon [27] study Bewley's [**?**] incomplete preferences. We suspect that our methods extend to such preferences and even more general incomplete preferences after appropriately modifying the completeness and continuity axioms; however we leave this extension to the interested reader.

A   Proofs

Proof of Proposition 1: (i) $\Rightarrow$ (ii) Suppose that Q is a $\mathcal{G}$-conditional belief at a $\mathcal{G}$-measurable act f and that (4) does not hold. Then there exists an act g such that $\mathbb{E}_Q[g|\mathcal{G}] = f$ and $g \succ f$. Since Q is $\mathcal{G}$-concordant with a subjective belief P at f, it follows that $\mathbb{E}_P[g|\mathcal{G}] = f$. This implies $\mathbb{E}_P[g] = \mathbb{E}_P(f)$ which together with $g \succ f$ contradicts P being a subjective belief at f upon recalling that $\succsim$ is continuous.

(i) $\Rightarrow$ (ii) Suppose that (4) holds. Let $A = \{h \in \mathcal{F} : h \succsim f\}$ and $B = \{g \in \mathcal{F} : \mathbb{E}_Q[g|\mathcal{G}] = f\}$. Note that A is a convex set and $\operatorname{ri} A = \operatorname{int} A \subseteq \{h \in \mathcal{F} : h \succ f\}$. Also B is a convex set

and by (4), we have that $B \subseteq \{g \in \mathcal{F} : f \succsim g\}$. Hence, $\operatorname{ri} A \cap \operatorname{ri} B = \emptyset$. By Theorem 11.3 of Rockafellar [30], there exists a probability measure $P \in \Delta$ such that $\mathbb{E}_P(h) \geq \mathbb{E}_P(f)$ for every $h \succsim f$ and $\mathbb{E}_P(f) \geq \mathbb{E}_P(g)$ for every g such that $\mathbb{E}_Q[g|\mathcal{G}] = f$. It follows that P is a subjective belief at f. Suppose that P is not concordant with Q. Then there exists $g \in \mathcal{F}$ such that $\mathbb{E}_Q[g|\mathcal{G}] = f$ but $\mathbb{E}_P[g|\mathcal{G}](s) \geq f(s)$ for all $s \in S$ and there exists $G_j \in \mathcal{G}$ with $P(G_j) > 0$ and $\mathbb{E}_P[g|\mathcal{G}](s) > f(s)$ for all $s \in G_j$. By the law of iterated expectations $\mathbb{E}_P(g) > \mathbb{E}_p(f)$. Contradiction. $\square$

Proof of Corollary 1: Condition (4) of Proposition 1 can be written as $\mathbb{E}_Q[g|\mathcal{G}] \succsim g$ for every $g \in \mathcal{F}$ such that $\mathbb{E}_Q[g|\mathcal{G}] = f$. It follows that $Q \in \mathring{\Delta}_\mathcal{G}$ is a consistent $\mathcal{G}$-conditional belief for $\mathcal{G}$-monotone and convex $\succsim$ if and only if (5) holds for every g such that $\mathbb{E}_Q[g|\mathcal{G}]$ is strictly positive. An inspection of the proof of Proposition 1 reveals that the equivalence remains true with (5) required to hold only for strictly positive acts g. Part (ii) follows from Proposition 1 (i). $\square$

Proof of Theorem 3: Let $\{f_i\}$ be a Pareto optimal allocation such that f_i is strictly positive for every i. Theorem 1 and strict convexity of preferences imply that f_i is $\mathcal{E}$-measurable for every i. Suppose that there are i and i' such that f_i and $f_{i'}$ are not comonotone. Then there exist events E_j and E_k in the partition $\mathcal{E}$ such that $f_i(E_j) < f_i(E_k)$ and $f_{i'}(E_j) > f_{i'}(E_k)$.

Let $\mathcal{E}_{jk}$ denote the partition obtained from partition $\mathcal{E}$ by replacing two cells E_j and E_k by their union $E_j \cup E_k$. Since $\mathcal{E}_{jk} \in \Sigma_c$, there exists $Q \in \bigcap_{i=1}^I \pi^i_{\mathcal{E}_{jk}}$. Let $\tilde{f}_i = \mathbb{E}_Q[f_i|\mathcal{E}_{jk}]$. Act $\tilde{f}_i$ differs from f_i in that consumptions states belonging to event $E_j \cup E_k$ are replaced by their expectation under Q conditional on $E_j \cup E_k$. Further, let $\tilde{f}_{i'} = \mathbb{E}_Q[f_{i'}|\mathcal{E}_{jk}]$ and $\varepsilon_i = f_i - \tilde{f}_i$ and $\varepsilon_{i'} = f_{i'} - \tilde{f}_{i'}$.

Since $\mathbb{E}_Q(\varepsilon_i) = \mathbb{E}_Q(\varepsilon_{i'}) = 0$ and acts ε_i and $\varepsilon_{i'}$ differ only in two cells, it holds $\varepsilon_{i'} = -\lambda \varepsilon_i$, for some $\lambda > 0$. Suppose that $\lambda \geq 1$. We will show that transferring ε_i from agent i to agent i' makes both of them strictly better off. Transferring ε_i from agent i leaves him with $\tilde{f}_i$. Corollary 1 (ii) implies that $\tilde{f}_i \succsim_i f_i$. Since $\succ_i$ is strictly convex and $\tilde{f}_i \neq f_i$, we actually have that $\tilde{f}_i \succ_i f_i$. Transferring ε_i to agent i' leaves him with $f_{i'} + \varepsilon_i = \left(\frac{1}{\lambda}\right)\tilde{f}_{i'} + \left(\frac{\lambda-1}{\lambda}\right)f_{i'}$. As for agent i, it holds that $\tilde{f}_{i'} \succ_{i'} f_{i'}$. Using strict convexity we obtain $f_{i'} + \varepsilon_{i'} \succ_{i'} f_{i'}$.

If $\lambda < 1$, then transferring $\varepsilon_{i'}$ from agent i' to agent i makes both agents better off. Thus

we obtained a contradiction to Pareto optimality of allocation $\{f_i\}$. $\square$

Proof of Proposition 2: The proof is straightforward if the function v is differentiable. We have from (11) that $\boldsymbol{\pi}_{\mathcal{G}} = \cap_{f \in \mathcal{F}_{\mathcal{G}}} \mathcal{P}_{\mathcal{G}}^v(f)$. Upon observing that $\mathcal{P}^v(f) = \mathcal{P}(v(f))$, we obtain (12).

For an arbitrary concave v and $f \in \mathcal{F}_{\mathcal{G}}$, it holds $\pi_{\mathcal{G}}(f) \supseteq \mathcal{P}_{\mathcal{G}}^v(f)$, with equality if v is differentiable at f. Since v has at most a countable set of points of non-differentiability, one can show that for every $\mathcal{G}$-measurable act f there exists a $\mathcal{G}$-measurable act g such that v is differentiable at g and $v(f) = \lambda v(g)$ for some scalar $\lambda > 0$. It holds $\mathcal{P}^v(f) = \mathcal{P}^v(g)$ and $\boldsymbol{\pi}_{\mathcal{G}}(g) = \mathcal{P}^v(g)$. Consequently, $\boldsymbol{\pi}_{\mathcal{G}}(f) \cap \boldsymbol{\pi}_{\mathcal{G}}(g) = \mathcal{P}^v(f)$. Using the same argument as in the case of differentiable v, we obtain (12). $\square$

Proof of Theorem 4: It suffices to show equivalence for linear utility since the set $\boldsymbol{\pi}_{\mathcal{G}}$ does not depend on the utility function by Proposition 2. We first prove that (14) implies that Q is a consistent $\mathcal{G}$-consistent belief. By Corollary 1 (i), it suffices to show that the multiple-priors utility with linear utility and set of priors $\mathcal{P}$ satisfying (14) exhibits preference for $\mathcal{G}$-conditional expectations (5).

For every $f \in \mathcal{F}$, we have $\mathbb{E}_P[\mathbb{E}_Q[f|\mathcal{G}]] = \mathbb{E}_{P_{\mathcal{G}}^Q}[f]$. Therefore

$$\min_{P \in \mathcal{P}} \mathbb{E}_P[[\mathbb{E}_Q[f|\mathcal{G}]] = \min_{P \in \mathcal{P}} \mathbb{E}_{P_{\mathcal{G}}^Q}[f] \geq \min_{P \in \mathcal{P}} \mathbb{E}_P[f] \qquad (27)$$

where we used (14). Inequality (27) implies that multiple-priors utility with set of priors $\mathcal{P}$ and linear utility exhibits preference for conditional expectation under Q.

To show the converse, suppose by contradiction that $\bar{P}_{\mathcal{G}}^Q \notin \mathcal{P}$ for some $\bar{P} \in \mathcal{P}$. By the separation theorem, there exists $\hat{f} \in \mathbb{R}^S$ such that

$$\mathbb{E}_{\bar{P}_{\mathcal{G}}^Q}(\hat{f}) < \min_{P \in \mathcal{P}} \mathbb{E}_P(\hat{f}). \qquad (28)$$

Since adding any constant act to $\hat{f}$ would not change inequality (28) we can assume that

$\hat{f} \in \mathcal{F}$. Using $\mathbb{E}_P[\mathbb{E}_Q[\hat{f}|\mathcal{G}]] = \mathbb{E}_{P_{\mathcal{G}}^Q}[\hat{f}]$ and (28) we obtain

$$\min_{P \in \mathcal{P}} \mathbb{E}_P[\mathbb{E}_Q[\hat{f}|\mathcal{G}]] < \min_{P \in \mathcal{P}} \mathbb{E}_P[\hat{f}] \tag{29}$$

This contradicts the preference for $\mathcal{G}$-conditional expectations under Q for linear multiple-priors utility with $\mathcal{P}$, and hence implies that Q is not a consistent $\mathcal{G}$-conditional belief. $\square$

Proof of Proposition 3: If $\mathcal{G}$-conditional beliefs are strongly consistent, then, as already noted, it holds $\boldsymbol{\pi}_{\mathcal{G}} = \mathcal{P}_{\mathcal{G}}$. Condition (17) follows then from Theorem 4.

For the converse implication, suppose that condition (17) is satisfied. Using Theorem 4, we have $\boldsymbol{\pi}_{\mathcal{G}} = \mathcal{P}_{\mathcal{G}}$. We have to show that $\pi_{\mathcal{G}}(f) = \mathcal{P}_{\mathcal{G}}$ for every $f \in \mathcal{F}_{\mathcal{G}}$. Since $\pi_{\mathcal{G}} \subset \pi_{\mathcal{G}}(f)$, it follows $\mathcal{P}_{\mathcal{G}} \subset \pi_{\mathcal{G}}(f)$. Further, since v is differentiable, it holds $\pi_{\mathcal{G}}(f) = \mathcal{P}_{\mathcal{G}}^v(f)$. Since $\mathcal{P}^v(f) \subset \mathcal{P}$, it follows that $\mathcal{P}_{\mathcal{G}}^v(f) \subset \mathcal{P}_{\mathcal{G}}$, and hence the conclusion. $\square$

Proof of Proposition 4: The proof is straightforward if the function v is differentiable. We have from (11) that $\boldsymbol{\pi}_{\mathcal{G}} = \cap_{f \in \mathcal{F}_{\mathcal{G}}} \mathcal{P}_{\mathcal{G}}^v(f)$. Upon observing that $\mathcal{P}^v(f) = \mathcal{P}(v(f))$, we obtain the conclusion.

For an arbitrary concave v and $f \in \mathcal{F}_{\mathcal{G}}$, it holds $\pi_{\mathcal{G}}(f) \supseteq \mathcal{P}_{\mathcal{G}}^v(f)$, with equality if v is differentiable at f. Since v has at most a countable set of points of nondifferentiability, one can show that for every $f \in \mathcal{F}_{\mathcal{G}}$ there exists $g \in \mathcal{F}_{\mathcal{G}}$ such that v is differentiable at g and $v(f) = v(g) + k$ for some $k \in \mathbb{R}$. Because the set of minimizing probabilities is invariant to additive shifts, it holds $\mathcal{P}^v(f) = \mathcal{P}^v(g)$ and $\boldsymbol{\pi}_E(g) = \mathcal{P}^v(g)$. Consequently $\boldsymbol{\pi}_{\mathcal{G}}(f) \cap \boldsymbol{\pi}_{\mathcal{G}}(g) = \mathcal{P}^v(f)$. Using the same argument as in the case of differentiable v, we obtain (22). $\square$

Proof of Theorem 5: It suffices to show that variational preferences with linear utility and cost function c satisfying (23) exhibits preference for $\mathcal{G}$-conditional expectations (5). The result follows then from Proposition 1 and the observation that the set $\boldsymbol{\pi}_{\mathcal{G}}$ does not depend on the utility function as long as the utility function is concave (Proposition 4).

If c satisfies (23), then, for every $f \in \mathcal{F}$,

$$\min_{P \in \Delta} \left\{ \mathbb{E}_P[\mathbb{E}_Q[f|\mathcal{G}]] + c(P) \right\} \geq \min_{P \in \Delta} \left\{ \mathbb{E}_{P_{\mathcal{G}}^Q}[f] + c(P_{\mathcal{G}}^Q) \right\} \geq \min_{P \in \Delta} \left\{ \mathbb{E}_P[f] + c(P) \right\}, \qquad (30)$$

where we used the fact that $\mathbb{E}_P[\mathbb{E}_Q[f|\mathcal{G}]] = \mathbb{E}_{P_{\mathcal{G}}^Q}[f]$. Inequality (30) shows preference for $\mathcal{G}$-conditional expectation under Q for variational preferences with cost function c and linear utility. $\square$

Proof of Proposition 5: We have

$$d(P_{\mathcal{G}}^Q, Q) = \sum_{j=1}^{K} \sum_{s \in G_j} \phi\left(\frac{Q(s)P(G_j)}{Q(s)Q(G_j)} \right) Q(s) =$$

$$= \sum_{j=1}^{K} \phi\left(\frac{P(G_j)}{Q(G_j)} \right) Q(G_j) \leq \sum_{j=1}^{K} \sum_{s \in G_j} \phi\left(\frac{P(s)}{Q(s)} \right) Q(s) = d(P, Q)$$

where the last inequality follows from Jensen's inequality. $\square$

Proof of Proposition 7: Because all $P \in \text{supp}\,(\mu)$ are $\mathcal{G}$-concordant, for any $G_j \in \mathcal{G}$ for any $s, s' \in G_j$ there exists $\alpha > 0$ such that $P(s) = \alpha P(s')$ for all $P \in \text{supp}\,(\mu)$. The ratio of subjective probabilities of states s and s' at any $\mathcal{G}$-measurable strictly positive act f is
$$\frac{\mathbb{E}_\mu \left\{ (\phi'(\mathbb{E}_P v(f)) \cdot v'(f(s)) \cdot P(s)) \right\}}{\mathbb{E}_\mu \left\{ (\phi'(\mathbb{E}_P v(f)) \cdot v'(f(s)) \cdot P(s')) \right\}} = \frac{\mathbb{E}_\mu \left\{ (\phi'(\mathbb{E}_P v(f)) \cdot \alpha P(s')) \right\}}{\mathbb{E}_\mu \left\{ (\phi'(\mathbb{E}_P v(f)) \cdot P(s')) \right\}} = \alpha = \frac{P^\mu(s)}{P^\mu(s')}. \ \square$$

Proof of Proposition 8: Under the assumptions of Proposition 8, it holds that $\mu\{P \in \Delta : P(G) = P^\mu(G) \text{ for all } G \in \mathcal{G}\} = 1$. Then, for any strictly positive $\mathcal{G}$-measurable act f, the expression $\phi'(\mathbb{E}_P v(f))$ does not depend on P. Thus, the gradient (26) is proportional to the vector with an sth coordinate equal to $v'(f(s))\mathbb{E}_\mu[P(s)]$. This implies that the subjective belief at any strictly positive $\mathcal{G}$-measurable act f is $\mathcal{G}$-concordant with P^μ. $\square$

References

[1] R. Aiyagari. Uninsured Idiosyncratic Risk and Aggregate Saving. *Quarterly Journal of Economics*, **109**, 659-684, (1994)

[2] J-P. Aubin. *Optima and Equilibria, An Introduction to Nonlinear Analysis*, Springer. 1998.

[3] Bewley, T. Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers, in *Contributions to Mathematical Economics in Honor of Gerard Debreu*, W. Hildenbrand and A. MasColell eds., North Holland, (1986)

[4] Bewley, T. Knightian Decision Theory: Part I *Discussion Paper, Cowles Foundation* (1986)

[5] Bewley, T. Market Innovation and Entrepreneurship: A Knightian View *Discussion Paper, Cowles Foundation* (1989)

[6] A. Billot, A. Chateauneuf, I. Gilboa and J-M. Tallon. Sharing Beliefs: Between Agreeing and Disagreeing. *Econometrica*, **68**, No. 4, 685-694, (2000)

[7] D. Cass and K. Shell, Do Sunspots Matter? *Journal of Political Economy*, 91(2), (1983), pp. 193-227.

[8] A. Chateauneuf, R.A. Dana and J.M. Tallon. Risk sharing rules and Equilibria with non-additive expected utilities, *Journal of Mathematical Economics*, **34**, pp. 191–214, (2000).

[9] A. Chateauneuf and J. H. Faro. Ambiguity through Confidence Functions, *Journal of Mathematical Economics*, forthcoming.

[10] L. de Castro and A. Chateauneuf. Ambiguity Aversion and Trade, mimeographed, March 2008.

[11] Chew, S.H, E. Karni and Z. Safra. Risk aversion in the theory of expected utility with rank dependent preferences. *Journal of Economic Theory*, (1987), pp. 370-381.

[12] Cogley, T. and T. Sargent Diverse Beliefs, Survival and the Market Price of Risk, Discussion Paper, New York University,

[13] Epstein, L.G. and M. Schneider. Recursive multiple-priors, *Journal of Economic Theory*, 113, (2003), pp. 1-31.

[14] Ghirardato, P. and M. Marinacci. Ambiguity Made Precise: A Comparative Foundation, *Journal of Economic Theory*, 102, (2002), 251-289.

[15] I. Gilboa and D. Schmeidler. Maxmin expected utility with non unique prior, *Journal of Mathematical Economics*, 18, (1989), 141-153.

[16] Hansen, L. and T. Sargent. "Robust Control and Model Uncertainty," *American Economic Review*, **91**, pp 60-66, (2001).

[17] Kajii, A. and Ui, T. Agreeable Bets with Multiple Priors *Journal of Economic Theory*, 128:299–305, 2006.

[18] Kajii, A. and Ui, T. Trade with Heterogeneous Multiple Priors *Journal of Economic Theory*, 144:337–353, 2009.

[19] P. Klibanoff, M. Marinacci and S. Mukherji. A Smooth Model of Decision Making under Ambiguity *Econometrica*, **73**, No. 6, 1849-92, (2005)

[20] D. Krueger and F. Perri Public versus Private Risk Sharing Working Paper, Federal Reserve Bank of Minneapolis, December 2009.

[21] LeRoy, S. and J. Werner. *Principles of Financial Economics*, Cambridge University Press, 2001.

[22] Martins-da-Rocha, V. F. Interim efficiency with MEU-preferences *Journal of Economic Theory*, forthcoming

[23] Maccheroni, F, M. Marinacci and A. Rustichini. Ambiguity Aversion, Robustness, and the Variational Representation of Preferences, *Econometrica*, **74**, No. 6, 1447-1498, (2006)

[24] Milgrom, P. and N. Stokey. Information, Trade and Common Knowledge. *Journal of Economic Theory*, 26, 1982, 17–27.

[25] Nehring, K. Capacities and Probabilistic Beliefs: A Precarious Coexistence, *Mathematical Social Sciences*, **38**, 197-213, (1999)

[26] Quiggin, J. A theory of anticipated utility, *Journal of Economic Behavior and Organization*, 3, pp. 323-343, (1982)

[27] Rigotti, L. and Shannon, C. Uncertainty and Risk in Financial Markets, *Econometrica*, 73, pp. 203–243, (2005)

[28] Rigotti, L., Ch. Shannon and T. Strzalecki. Subjective Beliefs and Ex-Ante Trade, *Econometrica*, 76, (2008), 1176-1190

[29] Rockafellar, T. *The Theory of Subgradients and Its Applications to Problems of Optimization.* Helderman, 1981.

[30] Rockafellar, T. *Convex analysis.* Princeton university press, 1997.

[31] Rothschild, M. and J.E. Stiglitz, Increasing Risk. I: A Definition, *Journal of Economic Theory*, 2, (1970), 225-243.

[32] Schmeidler, D. Subjective Probability and Expected Utility Without Additivity, *Econometrica*, 57, (1989), 571-587.

[33] Strzalecki, T. Axiomatic Foundations of Multiplier Preferences, Harvard Institute of Economic Research Discussion Paper No. 2181, (2009)

[34] Strzalecki, T. Probabilistic Sophistication and Variational Preferences, mimeographed, (2009)

[35] Werner J. Risk Aversion for Variational and Multiple-Prior Expected Utility, *Journal of Mathematical Economics*, forthcoming.

[36] Yaari, M. Some remarks on measures of risk aversion and on their uses. *Journal of Economic Theory*, 55:95–115, 1969.