Economics 8706 **Exercises/Projects**

Spring 2008, S4

1. Consider security markets with infinite time-horizon. There are J securities with dividends $x_{jt} \geq 0$ for every $t = 1, 2, \ldots$. Agents' portfolio holdings are restricted by the transversality constraint:

$$\liminf_{T \to \infty} \sum_{\xi_T \subset \xi_t} q(\xi_T) p(\xi_T) h(\xi_T) \geq 0 \quad \forall \xi_t, \tag{1}$$

where q are strictly positive event prices (assumed to exist).

(i) Is the existence of strictly positive event prices sufficient for there being no no unlimited arbitrage under the transversality constraint? Prove it (or disprove it).

(ii) Prove the following result: If security prices p are such that there exist strictly positive event prices q and markets are complete, then the set of budget feasible consumption plans under the wealth constraint at p is the same as the set of budget feasible consumption plans at p under the transversality constraint.

2. Consider an Arrow-Debreu equilbrium $(P, \{c^i\})$ for an infinite-horizon economy under uncertainty (see page 13 of Course Handouts). The pricing functional is

$$P(c) = \sum_{t=0}^{\infty} \sum_{\xi_t \in F_t} q(\xi_t) c(\xi_t)$$

where $q(\xi_t) > 0$. Define security prices by

$$p_j(\xi_t) = \frac{1}{q(\xi_t)} \sum_{\tau > t} \sum_{\xi_\tau \subset \xi_t} q(\xi_\tau) x_j(\xi_\tau)$$

Assume that security markets are complete at p.

Prove that there exist short sales restrictions b^i such that $p, \{c^i\}$ and a suitable chosen portfolio allocation are an equilibrium in security markets under short sales constraints

$$h(\xi_t) \geq -b^i(\xi_t), \qquad \forall\ \xi_t. \tag{2}$$

3. (Continuation of Question 2, Problem Set 1.) First, consider two-date security markets with short sales constraints on all securities as in (2). Say that security price vector $p \in \mathcal{R}^J$ is *viable* if there does not exist portfolio h such that $ph \leq 0$ and $x_s h \geq 0$, $\forall\, s$, with at least one strict inequality, and $h_j \geq 0$ for all securities j with exception of at most one security.

(i) Suppose that agents' utility functions are strictly increasing. Prove that if p is an equilibrium price vector under short sales constraints and the constraints are such that $b^i >> 0$, for every agent i, then p is viable. [The supply of securities is assumed to be 0.]

(ii) Prove that p is viable if and only if for every j there exists a vector of strictly positive state prices $q^j \in \mathcal{R}^S_{++}$ such that

$$p_j = \sum_s q^j_s x_{js} \quad \text{and} \quad p_k \geq \sum_s q^j_s x_{ks}, \ \forall\, k \neq j. \tag{3}$$

Consider now security markets with infinite time-horizon and short sales constraints (2).

(iii) Suppose that agents' utility functions are strictly increasing. Prove that if p is an equilibrium price vector under short sales constraints in infinite-time markets and the constraints are such that $b^i >> 0$ for every agent i, then p satisfies a dynamic version of (3) for every ξ_t.

(iv) Define the present value of security j at date 0 as

$$\sum_{t=1}^{\infty} \sum_{\xi_t \in F_t} q^j(\xi_t) x_j(\xi_t) \tag{4}$$

and the price bubble as the difference between p_{j0} and present value (4). Can there be a price bubbles in equilibrium under short-sales constraints?

4. Analyze the possibility of price bubbles in equilibrium under debt constraint.