Risk Aversion as a Foundation of Expected Utility*

Jan Werner
Department of Economics
University of Minnesota
Minneapolis, MN 55455

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Abstract: An agent's preferences exhibit risk aversion with respect to some probabilities of states of the world if she prefers deterministic outcome equal to the expected value of a state-contingent outcome under these probabilities to the state-contingent outcome itself. We show that if preferences exhibit risk aversion with respect to some probabilities and satisfy the independence axiom (sure thing principle), then they have an expected utility representation with respect to these probabilities. This gives a simple axiomatization of risk-averse expected utility with a finite set of states of the world. We also provide an axiomatization of general expected utility based on a condition on agent's attitude toward risk.

1. Introduction

An agent's preferences exhibit risk aversion if she prefers deterministic outcome equal to the expectation of a state-contingent outcome to the state-contingent outcome itself. Risk aversion gives rise to many important results in economics of uncertainty. The most frequently used specification of risk-averse preferences is expected utility with concave von Neumann-Morgenstern utility function. The question we ask in this paper is when does a risk-averse preference have an expected utility representation? The answer turns out to be surprisingly simple. It is the independence axiom (sure-thing principle) that, when added to risk aversion, implies expected utility representation (with concave von Neumann-Morgenstern utility function). This gives a simple axiomatization of expected utility.

We consider a setting of uncertainty described by a finite set of states of the world. Outcomes take values in real numbers or an Euclidean space. The intended interpretation is that of consumption of a single or multiple goods under uncertainty. Of course, our definition of risk aversion refers to some probabilities of states. It could be that these probabilities are "objective" probabilities, but it does not have to be so. The agent may not know any objective probabilities and yet her preference may exhibit risk aversion with respect to some probabilities. This simply means that, when faced with a binary choice between state-contingent consumption plan c and a deterministic consumption plan $E(c)$, where $E(c)$ is calculated using these probabilities, she will choose $E(c)$, for every c. Our result says that if preferences exhibit risk aversion with respect to some probabilities and satisfy the independence axiom, then they have expected utility representation with respect to these probabilities. These probabilities become agent's "subjective" probabilities of states, and they are revealed by agent's risk aversion.

The setting of finite-state uncertainty and a continuum of outcomes plays an important role in economic theory. It is the basic setting for equilibrium theory under uncertainty and most of financial economics. Yet, the setting proved to be problematic for an axiomatization to expected utility. The famous axiomatization of expected utility due to Savage [11] requires that there be a continuum of states. The axiomatization of Anscombe and Aumann [1] requires an extension of the domain of preferences to include arbitrary lotteries on outcomes. Alternatives to our axiomatization of expected utility under finite-state uncertainty are due to Wakker [13] (see LeRoy and Werner [9] for a recent exposition), Gul [4], Hens [5] and Stigum [12]. Wakker's axiomatization involves strengthening of the independence axiom to cardinal coordinate independence.

Results analogous to our main theorem hold for risk proclivity and risk neutrality in place of risk aversion. For instance, risk proclivity with respect to some probabilities along with the independence axiom implies expected utility representation with respect to these probabilities, with convex von Neumann-Morgenstern utility function. Again, it is the agent's attitude toward risk that reveals her subjective probabilities. We also provide an axiomatization of general, neither concave nor convex, expected utility. The condition on risk attitude in this axiomatization concerns risk compensation for small risks.

The paper is organized as follows: In section 2 we present and prove our main axiomatization of risk-averse expected utility. Section 3 contains the axiomatization of general expected utility. Extensions of our two axiomatizations to arbitrary Euclidean spaces of outcomes are discussed in section 4. Most of the proofs can be found in the Appendix.

2. Risk Aversion and Concave Expected Utility

Uncertainty is represented by a finite set of states $S = \{1, \dots, S\}$ with $S > 1$. At first, we assume that there is a single consumption good. A typical consumption plan is a S-vector $c = (c_1, \dots, c_S)$, where c_s represents consumption conditional on state s. For $x \in \mathcal{R}$, we write $\mathbf{x} = (x, \dots, x)$ to denote the deterministic consumption plan equal to x in every state.

An agent's preferences over state-contingent consumption plans are indicated by a strictly increasing and continuous utility function $U : \mathcal{R}^S \to \mathcal{R}$. Let $\mathcal{U}$ denote the set of all such utility functions.

Let Δ be the (open) simplex of strictly positive vectors of probabilities of states. For $\pi \in \Delta$ and any S-vector such as, say, $c \in \mathcal{R}^S$, we use $E_\pi(c) \in \mathcal{R}$ to denote the expectation with respect to π, and $\mathbf{E}_\pi(c) \in \mathcal{R}^S$ to denote the vector with $E_\pi(c)$ in every coordinate.

Utility function U exhibits *risk aversion* with respect to probabilities π if the agent prefers deterministic consumption equal to the expected value of a consumption plan to the consumption plan itself, that is,

$$U(c) \leq U(\mathbf{E}_\pi(c)), \tag{1}$$

for every $c \in \mathcal{R}^S$.

Utility function U has an *expected utility representation* with respect to π, if there exists function $v : \mathcal{R} \to \mathcal{R}$ (a *von Neumann-Morgenstern utility*) such that

$$U(c) \geq U(d) \;\text{ iff }\; E_\pi[v(c)] \geq E_\pi[v(d)] \tag{2}$$

for all $c, d \in \mathcal{R}^S$. As usual, the von Neumann-Morgenstern utility, if it exists, is unique up to a positive affine transformation. Utility function U that has expected utility representation exhibits risk aversion if and only if the von Neumann-Morgenstern utility function v is concave. In such case we say that U has *concave expected utility representation.*

A condition that has long been recognized as crucial for an axiomatization of expected utility under uncertainty is the *independence axiom* (also called *sure-thing principle*). It says that

$$U(c_{-s}x) \geq U(d_{-s}x) \qquad \text{iff} \quad U(c_{-s}y) \geq U(d_{-s}y) \tag{3}$$

for all $c, d \in \mathcal{R}^S$, $x, y \in \mathcal{R}$, and $s \in S$. Here $c_{-s}x$ denotes the consumption plan c with consumption c_s in state s replaced by x.

We have the following

Theorem 2.1 *Assume that $S \geq 3$. Utility function $U \in \mathcal{U}$ obeys the independence axiom and exhibits risk aversion with respect to probabilities $\pi \in \Delta$ if and only if it has a concave expected utility representation with respect to π.*

PROOF: To prove the only non-trivial sufficiency part of the theorem, suppose that U obeys the independence axiom. It follows from Debreu [2] that U has state-separable representation

$$U(c) \geq U(d) \ \text{ iff } \ \sum_{s=1}^{S} v_s(c_s) \geq \sum_{s=1}^{S} v_s(d_s) \tag{4}$$

for all $c, d \in \mathcal{R}^S$, for some functions $v_s : \mathcal{R} \to \mathcal{R}$, $\ s = 1, \ldots S$. We shall assume that each function v_s in (4) is differentiable. A considerably more difficult proof without this extra assumption can be found in the Appendix.

For each $x \in \mathcal{R}$, consider the following constrained maximization problem

$$\max_c \sum_s v_s(c_s) \tag{5}$$

subject to

$$E_\pi(c) = x. \tag{6}$$

Risk aversion implies that $\mathbf{x}$ is a solution to (5). The first-order conditions are

$$v_s'(x) = \lambda \pi_s, \qquad s = 1, \ldots, S, \tag{7}$$

where λ is a strictly positive Lagrange multiplier.

Eq. (7) can be rewritten as

$$v_s'(x) = \frac{\pi_s}{\pi_1} v_1'(x). \tag{8}$$

Since (8) holds for every x, we obtain

$$v_s(x) = \frac{\pi_s}{\pi_1} v_1(x) + \psi_s, \tag{9}$$

for every x and some $\psi_s \in \mathcal{R}$. It follows that $\sum_s v_s(c_s) = \frac{1}{\pi_1} \sum_s \pi_s v_1(c_s) + \psi$ where $\psi = \sum_{s=2}^{S} \psi_s$. Therefore, expected utility $E_\pi[v(c)]$ with $v \equiv v_1$ represents utility function U. Risk aversion implies that function v is concave. □

If there are only two states, then the sufficiency part of Theorem 2.1 does not hold. Our proof does not apply since the independence axiom does not imply state-separable representation with only two states. In fact, the independence axiom is satisfied by every strictly increasing utility on $\mathcal{R}^2$. Theorem 2.1 can be extended to the case of $S = 2$ by replacing the independence axiom by some other condition that guarantees a state-separable representation. For such condition, see Debreu [2].

It should be clear that an analogous result holds under *risk proclivity* defined by $U(c) \geq U(\mathbf{E}_\pi(c))$ for every c. Risk proclivity and the independence axiom are equivalent to convex expected utility representation. *Risk neutrality*, defined by $U(c) = U(\mathbf{E}_\pi(c))$ for every c, is equivalent to (linear) representation of $U(c)$ by $E_\pi(c)$ even without the independence axiom.

3. Axiomatization of Expected Utility

In this section we provide an axiomatization of general, not necessarily concave, expected utility. In the proof of Theorem 2.1 risk aversion with respect to π implies that the marginal rate of substitution between consumption in two different states at any deterministic consumption plan equals the ratio of probabilities. This combined with state-separability of utility function leads to expected utility representation. Consequently, an axiomatization of expected utility can be obtain by imposing the independence axiom along with the requirement that the aforementioned marginal rates of substitution be the same for all deterministic consumption plans. This is the approach taken by Hens [5]. Here we take a different approach. We provide a condition that pertains directly to agent's attitude toward risks.

We say that $z \in \mathcal{R}^S$ is a π-*risk* for probabilities of states $\pi \in \Delta$ if $E_\pi(z) = 0$. *Risk compensation* for π-risk z at a deterministic consumption level $x \in \mathcal{R}$ is $\rho(x, z) \in \mathcal{R}$ defined by

$$U(\mathbf{x} - \rho(x, z)) = U(\mathbf{x} + z) \tag{10}$$

We introduce the following condition on the asymptotic order of risk compensation:

(C) $\lim_{\lambda \to 0} \frac{\rho(x, \lambda z)}{\lambda} = 0$, for every $x \in \mathcal{R}$ and every π-risk $z \in \mathcal{R}^S$.

If utility function U exhibits risk neutrality with respect to π, then $\rho(x,z)=0$ for every x and every π-risk z, and (C) holds. More important, if U has expected utility representation with respect to π with differentiable von Neumann-Morgenstern utility, then condition (C) holds. A simple proof can be given if the von Neumann-Morgenstern utility is twice continuously-differentiable while the more general case follows from Theorem 3.1 below. The second-order approximation of risk compensation under expected utility (see, for instance, LeRoy and Werner [9]) is

$$\rho(x,\lambda z) \cong \frac{1}{2} r_a(x)\sigma^2(z)\lambda^2, \tag{11}$$

for small λ, where $r_a(x)$ is the Arrow-Pratt measure of absolute risk aversion and $\sigma^2(z)$ is the variance of z. Thus, $\frac{\rho(x,\lambda z)}{\lambda}$ is approximately proportional to λ and converges to zero as λ goes to zero. Note also that if U has differentiable expected utility representation with respect to probabilities different from π, then condition (C) does not hold for π-risks. The reason is that for an expected utility with respect to, say, π' approximation (11) of risk compensation for π-risk has an additional first-order term $\lambda E_{\pi'}(z)$.

A condition similar to (C) has been found by Nielsen [10] to be necessary and sufficient for differentiability of a concave von Neumann-Morgenstern utility.

We have

Theorem 3.1 *Assume that $S \geq 3$. Differentiable utility function $U \in \mathcal{U}$ obeys the independence axiom and satisfies condition (C) for some $\pi \in \Delta$ if and only if it has an expected utility representation with respect to π.*

The proof can be found in the Appendix.

Expected Utility with Multiple Goods

If there are L goods, then state-contingent consumption plans are vectors in $\mathcal{R}^{SL}$ and von Neumann-Morgenstern utility function v is a multivariable function on $\mathcal{R}^L$. Theorem 2.1 extends without any change to the case of L goods. In its proof, equation (8) becomes $Dv_s(x) = \frac{\pi_s}{\pi_1} Dv_1(x)$, where $Dv_s(x)$ is the gradient vector, and it implies (9).

An extension of Theorem 3.1 is more problematic since equation (10) does not lead to a well-defined concept of risk compensation (see Kihlstrom and Mirman [6]) with multiple goods. Therefore we introduce risk compensations measured in individual goods. *Risk compensation* for π-risk z in consumption of good l at a deterministic consumption level $x \in \mathcal{R}^L$ is $\rho_l(x, z) \in \mathcal{R}$ defined by

$$U(\mathbf{x} - \rho_l(x, z)\mathbf{e}_l) = U(\mathbf{x} + ze_l), \tag{12}$$

where e_l is the lth unit vector in $\mathcal{R}^L$, $l = 1, \ldots, L$. We extend condition (C) to

(C.1) $\lim_{\lambda \to 0} \frac{\rho_l(x, \lambda z)}{\lambda} = 0$, for every $x \in \mathcal{R}$, every π-risk $z \in \mathcal{R}^S$, and $l = 1, \ldots, L$.

If condition (C) is replaced by $(C.1)$, then Theorem 3.1 and its proof extend to the case of L goods.

Appendix.

Lemma A.1: *Let $v : \mathcal{R} \to \mathcal{R}$ be a continuous function and let $0 < \lambda < 1$. If*

$$v(y+h) + v(y-h) \leq v(y+\lambda h) + v(y-\lambda h) \tag{13}$$

for every y and h, then v is concave.

PROOF: Using (13) repeatedly n times we obtain

$$v(y+h) + v(y-h) \leq v(y+\lambda^n h) + v(y-\lambda^n h) \tag{14}$$

Taking the limit in (14) as n goes to infinity, there results

$$\frac{1}{2}v(y+h) + \frac{1}{2}v(y-h) \leq v(y), \tag{15}$$

for every y and h. This implies that v is concave. □

PROOF OF THEOREM 2.1: The independence axiom implies that U has representation $\sum_s v_s(c_s)$. We prove that function v_s is concave for each s. For $\epsilon \in \mathcal{R}$ and states s, t with $s \neq t$, we define $\tilde{\epsilon} \in \mathcal{R}^S$ by $\tilde{\epsilon}_t = \epsilon$, $\tilde{\epsilon}_s = -(\pi_t/\pi_s)\epsilon$ and $\tilde{\epsilon}_k = 0$ for all $k \neq t, s$. For every $x \in \mathcal{R}$, $E_\pi(\mathbf{x} + \tilde{\epsilon}) = x$ and, by risk aversion, $U(\mathbf{x} + \tilde{\epsilon}) \leq U(\mathbf{x})$. That is,

$$v_t(x+\epsilon) + v_s(x - \frac{\pi_t}{\pi_s}\epsilon) \leq v_t(x) + v_s(x) \tag{16}$$

Next, consider deterministic consumption plan $\mathbf{x} + \epsilon$ equal to $x + \epsilon$ in every state. Again, by risk aversion, we have $U(\mathbf{x} + \epsilon - \tilde{\epsilon}) \leq U(\mathbf{x} + \epsilon)$. That is,

$$v_t(x) + v_s(x + (1 + \frac{\pi_t}{\pi_s})\epsilon) \leq v_t(x+\epsilon) + v_s(x+\epsilon) \tag{17}$$

Adding (16) and (17) side-by-side and rearranging terms, we obtain

$$v_s(x + (1 + \frac{\pi_t}{\pi_s})\epsilon) + v_s(x - \frac{\pi_t}{\pi_s}\epsilon) \leq v_s(x) + v_s(x+\epsilon). \tag{18}$$

If we change variables in (18) by setting

$$y \equiv x + \frac{\epsilon}{2}, \quad h \equiv (1 + 2\frac{\pi_t}{\pi_s})\frac{\epsilon}{2}, \quad \lambda \equiv \frac{1}{1 + 2\frac{\pi_t}{\pi_s}}, \tag{19}$$

we obtain (13). Since x and ϵ were chosen arbitrarily, y and h are arbitrary while λ is fixed with $0 < \lambda < 1$. Thus Lemma A.1 can be applied implying that function v_s is concave.

Since each function v_s is concave and continuous, it is differentiable except for at most countably many points. Therefore, function $\sum_s v_s(\cdot)$ is differentiable except for at most countably many points in $\mathcal{R}^S$. By the argument of the proof in Section 2,

$$v_s'(x) = \frac{\pi_s}{\pi_1} v_1'(x), \tag{20}$$

for $x \in \mathcal{R}$, with exception of at most countably many points. This implies (see Kuczma [7, pg. 74]) that

$$v_s(x) = \frac{\pi_s}{\pi_1} v_1(x) + \psi_s \tag{21}$$

for every x. □

Proof of Theorem 3.1: We first prove sufficiency of the independence axiom and condition (C) for expected utility representation. Since U is differentiable and $S \geq 3$, the independence axiom is equivalent to the following condition (see Leontief [8]) on marginal rates of substitution between consumption in any two states:

$$\frac{\partial_s U(c_{-k}x)}{\partial_t U(c_{-k}x)} = \frac{\partial_s U(c_{-k}y)}{\partial_t U(c_{-k}y)} \tag{22}$$

for every $k \neq s, t$ and every c, x, y, where $\partial_s U$ denotes the partial derivative of U with respect to c_s. Condition (22) implies the existence of state-separable representation (4) with differentiable functions v_s (see the proof of Green [3]). Without loss of generality we can assume that $U(c) = \sum_s v_s(c_s)$.

For $\epsilon \in \mathcal{R}$ and state $s \neq 1$, define $\tilde{\epsilon} \in \mathcal{R}^S$ by $\tilde{\epsilon}_s = \epsilon$, $\tilde{\epsilon}_1 = -(\pi_s/\pi_1)\epsilon$ and $\tilde{\epsilon}_t = 0$ for every $t \neq s, 1$. Of course, $\tilde{\epsilon}$ is a π-risk. For every $x \in \mathcal{R}$, we have

$$\frac{1}{\epsilon}[U(\mathbf{x} + \tilde{\epsilon}) - U(\mathbf{x})] = \frac{1}{\epsilon}[v_s(x + \epsilon) - v_s(x) + v_1(x - \frac{\pi_s}{\pi_1}\epsilon) - v_1(x)] = \tag{23}$$

$$= \frac{v_s(x+\epsilon) - v_s(x)}{\epsilon} + \frac{v_1(x - \frac{\pi_s}{\pi_1}\epsilon) - v_1(x)}{\epsilon} \tag{24}$$

Since v_s and v_1 are differentiable, we obtain

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}[U(\mathbf{x} + \tilde{\epsilon}) - U(\mathbf{x})] = v_s'(x) - \frac{\pi_s}{\pi_1} v_1'(x) \tag{25}$$

On the other hand, we have

$$\frac{1}{\epsilon}[U(\mathbf{x} + \tilde{\epsilon}) - U(\mathbf{x})] = \frac{1}{\epsilon}[U(\mathbf{x} - \rho(x,\tilde{\epsilon})) - U(\mathbf{x})] = -\frac{\rho(x,\tilde{\epsilon})}{\epsilon}\ [\frac{U(\mathbf{x} - \rho(x,\tilde{\epsilon})) - U(\mathbf{x})}{-\rho(x,\tilde{\epsilon})}] \tag{26}$$

If ϵ converges to zero so does the risk compensation $\rho(x,\tilde{\epsilon})$ and the limit of the right-most term in (26) equals the derivative of U at $\mathbf{x}$ in the direction of deterministic consumption. It follows from condition (C) that

$$\lim_{\epsilon \to 0} \frac{\rho(x,\tilde{\epsilon})}{\epsilon} = 0. \tag{27}$$

Consequently,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}[U(\mathbf{x} + \tilde{\epsilon}) - U(\mathbf{x})] = 0. \tag{28}$$

Combining (25) with (28), we obtain that

$$v_s'(x) = \frac{\pi_s}{\pi_1} v_1'(x). \tag{29}$$

holds for every x and every s. The rest of the proof is the same as of Theorem 2.1.

Next we prove necessity of the independence axiom and condition (C) for expected utility representation. Of course, the independence axiom holds for expected utility, and so only condition (C) is of concern. For every π-risk z we have

$$\lim_{\lambda \to 0} \frac{1}{\lambda}(E_\pi[v(\mathbf{x} + \lambda z)] - v(x)) = v'(x) E_\pi(z) = 0. \tag{30}$$

Note that differentiability of v follows from differentiability of U and the uniqueness, up to affine transformation, of expected utility representation. As in (26), we can write

$$\lim_{\lambda \to 0} \frac{1}{\lambda}(E_\pi[v(\mathbf{x} + \lambda z)] - v(x)) = -\frac{\rho(x,\lambda z)}{\lambda}\ [\frac{E_\pi[v(\mathbf{x} - \rho(x,\lambda z))] - v(x)}{-\rho(x,\lambda z)}]. \tag{31}$$

Since the right-most term in (31) converges to $v'(x)$, we obtain from (30) and (31) that

$$\lim_{\lambda \to 0} \frac{\rho(x, \lambda z)}{\lambda} = 0. \tag{32}$$

This completes the proof.

□

References

1. F. Anscombe and R. Aumann, A definition of subjective probability, *Annals of Mathematical Statistics*, 34: 199-205, (1963).
2. G. Debreu, Topological methods in cardinal utility theory. In K. Arrow, S. Karlin and P. Suppes, eds., "Mathematical Methods in Social Sciences," Stanford University Press, (1959).
3. H.A. Green, "Aggregation in Economic Analysis," Princeton University Press, Princeton, (1964).
4. F. Gul, Savage's theorem with a finite number of states. *Journal of Economic Theory*, 57:99–110, (1992).
5. T. Hens, A note on Savage's theorem with a finite number of states, *Journal of Risk and Uncertainty*, 4:63-71, (1992).
6. R.E. Kihlstrom and L.J. Mirman, Risk aversion with many commodities. *Journal of Economic Theory*, 8:361–88, (1974).
7. M. Kuczma, "An Introduction to the Theory of Functional Equations and Inequalities," PWN, Kraków, (1985).
8. W. Leontief, A note on interrelation of subsets of independent variables of a ontinuous function with continuous first derivatives. *Bulletin of the American Mathematical Society*, 53:343–350, (1947).
9. S. LeRoy and J. Werner, "Principles of Financial Economics," Cambridge University Press, (2001).
10. L.T. Nielsen, Differentiable von Neumann-Morgenstern utility. *Economic Theory*, 14:285–97, (1999).
11. L. Savage, "The Foundations of Statistics," Wiley, New York, (1954).
12. B.P. Stigum, Finite state space and expected utility maximization, *Econometrica* 40, 253-259, (1972).
13. P. Wakker, "Additive Representations of Preferences," Kluwer, (1989).