

Risk Aversion for Variational and Multiple-Prior Preferences*

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Abstract: The objective of this paper is to identify variational preferences and multiple-prior (maxmin) expected utility functions that exhibit aversion to risk under some probability measure from among the priors. Risk aversion has profound implications on agents' choices and on market prices and allocations. Our approach to risk aversion relies on the theory of mean-independent risk of Werner (2009). We identify necessary and sufficient conditions for risk aversion of convex variational preferences and concave multiple-prior expected utilities. The conditions are stability of the cost function and of the set of probability priors, respectively, with respect to a probability measure. The two stability properties are new concepts. We show that cost functions defined by the relative entropy distance or other divergence distances have that property. Set of priors defined as cores of convex distortions of probability measures or neighborhoods in divergence distances have that property, too.

Keywords: Risk Aversion, Mean-Independent Risk, Multiple-Prior Expected Utility, Variational Preferences.

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1. Introduction

Risk aversion has important implications on agents' choices and on prices and allocations in security markets under expected utility. Good examples are results on optimal risk sharing among many risk-averse agents (see LeRoy and Werner (2001)). The standard theory of risk aversion due to Arrow (1965) and Pratt (1964) is limited to expected utility. In Werner (2009) we developed a new theory of risk aversion aimed primarily at applications to preference that have been put forward as alternatives to expected utility; the so called non-expected utilities. The theory applies to non-expected utilities that may not be distribution invariant (or probabilistically sophisticated). That is, when there is no probability measure on uncertain states such that acts or contingent claims are ranked only according to probability distributions of claims that they induce. Many non-expected utilities are not distribution invariant.

The basic concept of Werner (2009) is the mean-independent risk: for a probability measure π on a finite state space, contingent claim ϵ is a mean-independent risk at another contingent claim z if the conditional expectation $E_\pi(\epsilon|z)$ of ϵ on z equals zero. Utility function U exhibits aversion to mean-independent risk if there exists a probability measure π such that $U(y) \geq U(x)$ whenever x differs from y by a mean-independent risk at π , that is, if $x = z + \epsilon$ and $y = z + \lambda\epsilon$ for some ϵ and z such that ϵ is a mean-independent risk at z and $0 \leq \lambda \leq 1$. Aversion to mean-independent risk under π implies, in particular, that $U(z) \geq U(z + \epsilon)$ whenever ϵ is a mean-independent risk at z . Thus, the agent whose initial position is a contingent claim z rejects a gamble given by the mean-independent risk ϵ . If utility function U is concave, this last condition is equivalent to aversion to mean-independent risk (see Werner (2009)). Arrow (1965) and Pratt (1964) defined risk aversion under the expected utility hypothesis by this condition restricted to risk-free initial claims. Under expected utility, risk aversion in the Arrow-Pratt sense implies rejection of gambles with mean-independent risk. Every utility function that is monotone decreasing with respect to the standard Rothschild-Stiglitz (or stochastic dominance) order of more risky is averse to mean-independent risk. The converse does not hold.

The objective of this paper is to provide characterizations of aversion to mean-independent risk for two popular classes of non-expected utilities: variational pref-

ferences, and multiple-prior (maxmin) expected utilities. Under the variational preferences model, the decision maker has a primary probability belief but considers alternative probability beliefs as well. These other probability beliefs involve cost (in terms of utility) specified by a cost function. The decision criterion is the minimum over all priors of the sum of the expected utility of an act and the cost of the prior. A motivation for variational preferences comes from the literature on robust control and model misspecification, see Hansen and Sargent (2001) and Maccheroni, Marinacci and Rustichini (2006).

Under the multiple-prior expected utility model, the decision maker has a set of probability measures over uncertain states as probability beliefs instead of a single measure under expected utility. This multiplicity of probability measures reflects her ambiguous information about the states, or uncertainty of her expectations. The decision criterion is the minimum over the set of multiple priors of the expected utility of a contingent claim. Taking the minimum reflects the decision maker's concern with the "worst case" scenario. An axiomatic derivation of multiple-prior expected utility has been given by Gilboa and Schmeidler (1989).

The standard motivation for multiple-prior expected utility comes from the Ellsberg paradox. The pattern of preference over bets on balls drawn from an urn in the Ellsberg experiment is incompatible with expected utility, but can be explained by multiple-prior expected utility. For a single urn with 30 red balls and 60 green or yellow balls with unknown proportions of the two colors, a multiple-prior expected utility with the set of all probability measures that assign probability $1/3$ to drawing red ball and arbitrary probabilities (summing up to $2/3$) to drawing yellow or green ball leads to the desired pattern of preferences.

We identify in this paper conditions on the cost function and the von Neumann-Morgenstern utility function that guarantee that variational preferences exhibit aversion to mean-independent risk for some probability measure π . The condition on the cost function is called *stability* with respect to probability measure π . It requires that, for every probability measure P and every partition of states F , the cost of a probability measure that coincides with P on elements of F and has conditional probabilities of π within each element of F has lower cost than the cost of P . Our Theorem 1 states that variational preferences with concave utility function exhibits aversion to mean-independent risk under π if and only if the cost

function is π -stable.

We show that cost functions defined by any divergence distance from primary probability belief π are π -stable. It follows that the variational preferences proposed by Hansen and Sargent (2001) with concave utility function and relative entropy as the cost function - these are called multiplier preferences - exhibit aversion to mean-independent risk. Cost functions defined by the Euclidean distance fail to be π -stable unless π is the uniform probability measure.

For multiple-prior expected utility, we identify conditions on the set of probability priors and the von Neumann-Morgenstern utility function that guarantee aversion to mean-independent risk for some probability measure from the set of priors. The condition on the set of priors is also called *stability* with respect to probability measure π and requires that, for every probability measure in the set of priors and every partition of states, the probability measure obtained by the same operation as in the definition of a stable cost function lies in the set of priors. Theorem 2 states that concave multiple-prior expected utility exhibits aversion to mean-independent risk under π if and only if the set of priors is π -stable.

The most important cases of sets of priors that are π -stable are cores of convex distortions of probability measure π and neighborhoods of π in the relative entropy or other divergence distances. Euclidean neighborhoods of π are not π -stable unless π is the uniform probability measure. The set of priors in the Ellsberg experiment is not π -stable for any probability π in the set. The reason is the existence of an unambiguous non-trivial event – red ball drawn – to which all priors assign the same probability. We show that, in general, the existence of non-trivial unambiguous event precludes mean-independent risk aversion under any probability measure.

The paper is organized as follows: In Section 2 we introduce the setting and briefly review definitions and some results about mean-independent risk and aversion to mean-independent risk. In Section 3 we present a characterization of variational preferences that exhibit aversion to mean-independent risk. Section 4 contains the respective results for multiple-prior expected utilities. In Section 5 we study divergence distances and neighborhoods while Section 6 is devoted to other examples such as distortions of probability measures and the Euclidean distance. Section 7 is about risk aversion and existence of unambiguous events.

2. Mean-Independent Risk Aversion

There is a finite set $S = \{1, \dots, S\}$ of states of nature (with $S > 1$.) The set of all (additive) probability measures on S is Δ , the unit simplex in \mathcal{R}^S . The subset of strictly positive probability measures is denoted by $\mathring{\Delta}$. Any S -dimensional vector $x = (x_1, \dots, x_S) \in \mathcal{R}^S$ is called *contingent claim*. The expected value $\sum_{s=1}^S P(s)x_s$ of x under a probability measure $P \in \Delta$ is denoted by $E_P(x)$. For partition F of states, $E(x|F)$ denotes conditional expectation of x on F . For contingent claim z , $E(x|z)$ denotes conditional expectation of x on z , that is, conditional expectation of x on the partition generated by z . All these conditional expectations are contingent claims.

Our study of risk aversion for variational preferences and multiple-prior expected utilities builds on the concepts of mean-independent risk and aversion to mean-independent risk of Werner (2009). These concepts have been developed primarily for applications to non-expected utilities that may not be distribution invariant. That is, when there is no probability measure on states such that contingent claims are ranked only according to probability distributions of claims that they induce.¹ The standard Rothschild-Stiglitz (or stochastic dominance) concept of more risky is a ranking based on induced probability distributions and hence can not be used for preferences that are not distribution invariant. Variational preferences and multiple-prior expected utilities are often not distribution invariant. (See Sections 3 and 4 for further discussion.)

We present now the basic concepts and some results of Werner (2009). Let $\pi \in \mathring{\Delta}$ be a strictly positive probability measure. Contingent claim $\epsilon \in \mathcal{R}^S$ is a *mean-independent risk at $z \in \mathcal{R}^S$* if $E_\pi(\epsilon|z) = 0$. For two contingent claims $x, y \in \mathcal{R}^S$ with the same expectation, $E_\pi(x) = E_\pi(y)$, x *differs from y by mean-independent risk* if there exist $z, \epsilon \in \mathcal{R}^S$ and $0 \leq \lambda \leq 1$ such that ϵ is a mean-independent risk at z , and $x = z + \epsilon$ and $y = z + \lambda\epsilon$.

Definition 1: Utility function U on \mathcal{R}^S is *averse to mean-independent risk* if there exists a probability measure π such that $U(y) \geq U(x)$ whenever x differs from y by mean-independent risk.

Every utility function that is decreasing with respect to the Rothschild-Stiglitz

¹Distribution invariance is often called probabilistic sophistication in decision theory, following Machina and Schmeidler (1992).

order of more risky is averse to mean-independent risk (Werner (2009), Theorem 2.1). The converse is not true. Every concave expected utility is averse to mean-independent risk. Concave expected utility takes the form $\sum_{s=1}^S \pi(s)v(x_s)$ for a concave utility function $v : \mathcal{R} \rightarrow \mathcal{R}$, and is denoted by $E_\pi[v(x)]$.

Aversion to mean-independent risk is closely related to preference for conditional expectations.

Definition 2: Utility function U on \mathcal{R}^S exhibits *preference for conditional expectations* under probability measure π if $U(E_\pi(x|F)) \geq U(x)$ for every $x \in \mathcal{R}^S$ and every partition of states F .

Equivalently, U exhibits preference for conditional expectations if the agent rejects any mean-independent risk, that is, $U(z) \geq U(z + \epsilon)$ for every $\epsilon, z \in \mathcal{R}^S$ such that ϵ is mean-independent risk at z . It follows from Theorem 5.1 in Werner (2009) that a quasi-concave utility function U is averse to mean-independent risk under π if and only if it exhibits preference for conditional expectations under π . Werner (2009) provides a characterization of concave utility functions that are mean-independent risk averse in terms of superdifferentials. Recall that the superdifferential of a concave function U at $x \in \mathcal{R}^S$ is the set $\partial U(x)$ consisting of all vectors $\phi \in \mathcal{R}^S$ that satisfy $U(y) \leq U(x) + \phi(y - x)$ for every $y \in \mathcal{R}^S$. Theorem 6.1 in Werner (2009) says that concave utility function U on \mathcal{R}^S is averse to mean-independent risk if and only if for every x there exists $\phi \in \partial U(x)$ such that if $x_s = x_{s'}$, then $\frac{\phi_s}{\pi(s)} = \frac{\phi_{s'}}{\pi(s')}$.

We conclude this section by introducing an operation on probability measures that will be repeatedly used in the rest of the paper. For every partition F of states and every probability measure $P \in \Delta$, we define another probability measure $P_F^\pi \in \Delta$ by

$$P_F^\pi(A) = \sum_{i=1}^k \pi(A|A_i)P(A_i) \quad (1)$$

for every $A \subset S$, where sets A_i 's are elements of the partition F . Probability measure P_F^π coincides with P on elements of partition F and has the same conditional probabilities within each element of partition F as π . For the trivial partition $F = \{S\}$, probability measure P_F^π equals π ; for the full partition $F = \{\{s\}_{s \in S}\}$, P_F^π equals P . Two elementary properties of probability measure P_F^π will be used:

Lemma 1: For every $\pi \in \mathring{\Delta}$, $P \in \Delta$, $x \in \mathcal{R}^S$, and every partition F ,

$$(i) \quad E_{P_F^\pi}(x|F) = E_\pi(x|F),$$

$$(ii) \quad \text{if } x \text{ is } F\text{-measurable, then } E_{P_F^\pi}(x) = E_P(x).$$

Lemma 1 implies that $E_{P_F^\pi}(x) = E_P[E_\pi(x|F)]$ for every x , and every F .

3. Variational Preferences.

Variational preferences (or variational utility) take the form

$$\min_{P \in \Delta} \{E_P[v(x)] + c(P)\}, \quad (2)$$

for some strictly increasing and continuous utility function $v : \mathcal{R} \rightarrow \mathcal{R}$ and some convex and lower semicontinuous function $c : \Delta \rightarrow [0, \infty]$ such that $c(\pi) = 0$ for some $\pi \in \Delta$. We shall maintain these assumptions on c and v throughout the paper. For probability measure P , $c(P)$ is the cost (in terms of utility) of considering belief P . We shall call c the cost function. Cost function c is finite, if $c(P) < \infty$ for every $P \in \Delta$.

An axiomatization of variational preferences is due to Maccheroni, Marinacci and Rustichini (2006). Variational preferences are often not distribution invariant. An exception is when the cost function is a divergence distance, see Maccheroni et al (2006) and Strzalecki (2008). Strzalecki (2009) provides a comprehensive study of distribution invariance (i.e., probabilistic sophistication) of variational preferences.

We introduce a property of cost function being decreasing with respect to taking π -conditional probabilities as in (1). We call this property π -stability.

Definition 3: Cost function c is π -stable for $\pi \in \mathring{\Delta}$ if

$$c(P_F^\pi) \leq c(P) \quad (3)$$

for every $P \in \Delta$ and every partition F .

If c is π -stable, then probability measure π has the lowest cost, that is, $c(\pi) = 0$.

Theorem 1: If cost function c is π -stable for $\pi \in \mathring{\Delta}$, then variational preferences with cost c and arbitrary concave utility v are averse to mean-independent risk under π . Conversely, if c is finite and variational preferences with cost c and concave utility v are averse to mean-independent risk under $\pi \in \mathring{\Delta}$, then c is π -stable.

PROOF: Suppose that c is π -stable. Since variational utility with concave utility v is concave, it follows from Theorem 5.1 in Werner (2009) that it suffices to show that it exhibits preference for conditional expectation under π . Consider an arbitrary partition F . From Lemma 1, we have that

$$E_P[v(E_\pi(x|F))] = E_{P_F^\pi}[v(E_{P_F^\pi}(x|F))]. \quad (4)$$

Using conditional Jensen's inequality on the right-hand side of (4), we obtain

$$E_P[v(E_\pi(x|F))] \geq E_{P_F^\pi}[v(x)]. \quad (5)$$

Adding (5) and (3) side by side and taking minimum on both sides gives

$$\min_{P \in \Delta} \{E_P[v(E_\pi(x|F))] + c(P)\} \geq \min_{P \in \Delta} \{E_{P_F^\pi}[v(x)] + c(P_F^\pi)\} \quad (6)$$

The right-hand side of (6) is greater than or equal to $\min_{P \in \Delta} \{E_P[v(x)] + c(P)\}$. This shows preference for conditional expectations for variational preferences with cost c and utility v .

The proof of the converse implication can be found in the Appendix. \square

Theorem 1 implies that variational preferences with finite cost c and concave utility v are averse to mean-independent risk if and only if the same holds for variational preferences with cost c and linear utility. Aversion to mean-independent risk for variational preferences with linear utility has a simple characterization that can be derived from Theorem 6.1 in Werner (2009). The superdifferential of variational preferences with cost c and linear utility coincides with the set of minimizing probability measures (see Theorem 18 of Maccheroni et al (2006)). That is, the superdifferential at x is

$$\mathcal{M}_c(x) = \arg \min_{P \in \Delta} \{E_P(x) + c(P)\}. \quad (7)$$

It follows from Theorem 1 and Theorem 6.1 that

Proposition 1: *Variational preferences with finite cost c and any concave utility v are averse to mean-independent risk under π if and only if, for every x , there exists $P \in \mathcal{M}_c(x)$ such that*

$$\text{if } x_s = x_{s'}, \text{ then } \frac{P(s)}{\pi(s)} = \frac{P(s')}{\pi(s')}. \quad (8)$$

For finite cost function c , condition (8) is equivalent to π -stability.

4. Multiple-Prior Expected Utilities

Multiple-prior (or maxmin) expected utility takes the form

$$\min_{P \in \mathcal{P}} E_P[v(x)], \tag{9}$$

for some strictly increasing and continuous utility function $v : \mathcal{R} \rightarrow \mathcal{R}$ and some convex and closed set $\mathcal{P} \subset \Delta$ of probability measures. We shall maintain these assumptions on \mathcal{P} and v throughout the paper. If the set \mathcal{P} consists of a single probability measure, then multiple-prior expected utility reduces to the standard expected utility. If \mathcal{P} is the set of all probability measures Δ , then it reduces to the Wald's criterion $\min_{s \in S} v(x_s)$.

An axiomatization of multiple-prior expected utility (9) with an arbitrary closed and convex set of priors has been first given by Gilboa and Schmeidler (1989). Multiple-prior expected utilities are often not distribution invariant under any probability measure on states. An exception is when the set of priors is a convex distortion of a probability measure (see Section 6).² The multiple-prior expected utility with linear utility $\min_{P \in \mathcal{P}} E_P(x)$ has been extensively studied in the context of coherent measures of risk (see Föllmer and Schied (2002)).

Multiple-prior expected utility is a special case of variational preferences. Indeed, for the cost function $c_{\mathcal{P}}$ defined by

$$c_{\mathcal{P}}(P) = \begin{cases} 0 & \text{if } P \in \mathcal{P} \\ +\infty & \text{if } P \notin \mathcal{P} \end{cases} \tag{10}$$

variational preferences (2) take the form of multiple-prior expected utility (9) with set of priors \mathcal{P} .

We introduce the concept of stability of a set of priors with respect to probability measure π .

Definition 4: Set of probability measures $\mathcal{P} \subset \Delta$ is π -stable for $\pi \in \mathring{\Delta}$ if $P_F^\pi \in \mathcal{P}$ for every $P \in \mathcal{P}$ and every partition F .

²Marinacci (2002) studies probabilistic sophistication of multiple-prior expected utilities on a continuum state space. For further results and examples, see Grant and Kaji (2005) and Strzalecki (2009).

Note that set of priors \mathcal{P} is π -stable if and only if cost function $c_{\mathcal{P}}$ is π -stable. Further, if \mathcal{P} is π -stable, then $\pi \in \mathcal{P}$. We have the following theorem.

Theorem 2: *Multiple-prior expected utility with set of priors \mathcal{P} and concave utility v is averse to mean-independent risk under π if and only if \mathcal{P} is π -stable.*

PROOF: The first implication, that the multiple-prior expected utility is averse to mean-independent risk, follows from Theorem 1. The converse implication does not follow from Theorem 1 since cost function $c_{\mathcal{P}}$ is not finite (unless $\mathcal{P} = \Delta$). The proof can be found in the Appendix. \square

It follows from Theorem 2 that multiple-prior expected utility with set of priors \mathcal{P} and concave utility v is averse to mean-independent risk if and only if the same holds for multiple-prior expected utility with set of priors \mathcal{P} and linear utility. The latter has a simple characterization derived from Theorem 6.1 in Werner (2009).

The superdifferential of multiple-prior expected utility with set of priors \mathcal{P} and linear utility coincides with the set of minimizing probability measures (see Aubin (1998)). That is, the superdifferential at x is

$$\mathcal{M}_{\mathcal{P}}(x) = \arg \min_{P \in \mathcal{P}} E_P(x). \quad (11)$$

It follows from Theorem 2 and Theorem 6.1 that

Proposition 2: *Multiple-prior expected utility with set of priors \mathcal{P} and any concave utility v is averse to mean-independent risk under π if and only if, for every x , there exists $P \in \mathcal{M}_{\mathcal{P}}(x)$ such that*

$$\text{if } x_s = x_{s'}, \text{ then } \frac{P(s)}{\pi(s)} = \frac{P(s')}{\pi(s')}. \quad (12)$$

Condition (12) is equivalent to the set of priors \mathcal{P} being π -stable.

5. Divergence Distances and Neighborhoods

For a convex function $\Phi : \mathcal{R}_+ \rightarrow \mathcal{R}_+$ such that $\Phi(1) = 0$, the divergence distance (or statistical measure of distance) between probability measures $P \in \Delta$ and $\pi \in \mathring{\Delta}$ is

$$d_{\Phi}(P, \pi) = \sum_{s=1}^S \pi(s) \Phi\left(\frac{P(s)}{\pi(s)}\right). \quad (13)$$

It can be shown that the distance function d_Φ is non-negative and a convex function of P . If Φ is strictly convex, then $d_\Phi(P, \pi)$ is zero if and only if P equals π . In general d_Φ is not a metric for it is asymmetric and violates the triangle inequality.

Important examples of divergence distances are the following:

Example 1. (KULLBACK-LEIBLER RELATIVE ENTROPY) For $\Phi(t) = t \ln(t) - t + 1$, the divergence distance is the *relative entropy*

$$d_{kl}(P, \pi) = \sum_{s=1}^S P(s) \ln \left(\frac{P(s)}{\pi(s)} \right). \quad (14)$$

Example 2. (GINI INDEX) For $\Phi(t) = (t - 1)^2$, the divergence distance is the *Gini index*

$$\sum_{s=1}^S \frac{(P(s) - \pi(s))^2}{\pi(s)}. \quad (15)$$

Example 3. (TOTAL VARIATION) For $\Phi(t) = |t - 1|$, the divergence distance is the *total variation*

$$\sum_{s=1}^S |P(s) - \pi(s)|. \quad (16)$$

A neighborhood of probability measure $\pi \in \mathring{\Delta}$ in divergence distance is the set

$$\mathcal{N}_\Phi(\pi, \epsilon) = \{P \in \Delta : d_\Phi(P, \pi) \leq \epsilon\} \quad (17)$$

for $\epsilon > 0$. We restrict our attention to neighborhoods that are contained in the interior of the probability simplex.

We have the following

Proposition 3: *The divergence distance $d_\Phi(\cdot, \pi)$ is π -stable for every convex Φ and $\pi \in \mathring{\Delta}$. Further, the neighborhood $\mathcal{N}_\Phi(\pi, \epsilon)$ is π -stable.*

PROOF: It suffices to show that

$$d_\Phi(P_F^\pi, \pi) \leq d_\Phi(P, \pi) \quad (18)$$

for every $P \in \Delta$ and every partition F . The proof of (18) can be found in Strzalecki and Werner (2009, Proposition 7). We present it here for completeness. From (1) and (13), we obtain

$$d_\Phi(P_F^\pi, \pi) = \sum_{i=1}^k \pi(A_i) \phi \left(\frac{P(A_i)}{\pi(A_i)} \right) \quad (19)$$

From Jensen's inequality and convexity of Φ , it follows that

$$\pi(A_i)\phi\left(\frac{P(A_i)}{\pi(A_i)}\right) \leq \sum_{s \in A_i} \pi(s)\Phi\left(\frac{P(s)}{\pi(s)}\right). \quad (20)$$

Using (19) and (20) we obtain

$$d_{\Phi}(P_F^{\pi}, \pi) \leq \sum_{i=1}^k \sum_{s \in A_i} \pi(s)\Phi\left(\frac{P(s)}{\pi(s)}\right) = d_{\Phi}(P, \pi). \quad (21)$$

□

Variational preferences with cost function c given as a scale-multiple of a divergence distance, that is, $c(P) = \theta d_{\Phi}(P, \pi)$ where $\theta > 0$ is a scale parameter, are called divergence preferences. It follows from Theorem 1 and Proposition 3 that divergence preferences exhibit aversion to mean-independent risk under π . Hansen and Sargent(2001) and Strzalecki (2008) consider divergence preferences with cost function c given by $c(P) = \theta d_{kl}(P, \pi)$, where d_{kl} is the relative entropy of (14) and π is the agent's reference belief. Such variational preferences are called multiplier preferences. They exhibit aversion to mean-independent risk under π .

Multiple-prior expected utilities with neighborhoods in divergence distances as the sets of priors have been used in asset pricing models. For example, Cao, Wang and Zhang (2005) consider relative entropy neighborhoods; Epstein and Wang (1995) consider total variation neighborhoods.

6. Cores of Convex Distortions and Other Examples

An important class of stable sets of priors are cores of convex distortions. Let $f : [0, 1] \rightarrow [0, 1]$ be an increasing and convex function satisfying $f(0) = 0$ and $f(1) = 1$. Set function $f \circ \pi$ is the distortion of probability π by f . The *core* of $f \circ \pi$ is

$$\text{core}(f \circ \pi) = \{P \in \Delta : P(A) \geq f(\pi(A)), \forall A\} \quad (22)$$

Multiple-prior expected utility with set of priors $\text{core}(f \circ \pi)$ for convex distortion function f can be written as

$$\sum_{i=1}^S v(x_{(i)})[f(\pi\{s : x_s \geq x_{(i)}\}) - f(\pi\{s : x_s \geq x_{(i-1)}\})], \quad (23)$$

where we used $x_{(i)}$ to denote the i -th highest value from among all x_s , so that $x_{(1)} \geq x_{(2)} \geq \dots \geq x_{(S)}$. Utility function (23) is the rank-dependent expected utility axiomatized by Quiggin (1982) and, in the case of linear v , by Yaari (1987). Convex distortion of a probability measure is a convex capacity. Multiple-prior expected utility with the set of priors $\text{core}(f \circ \pi)$ is the Choquet (non-additive) expected utility of Schmeidler (1989) with the capacity $f \circ \pi$.

Proposition 4: *If f is convex, then $\text{core}(f \circ \pi)$ is π -stable.*

PROOF: Every contingent claim x when regarded as random variable on probability space $(S, 2^S, \pi)$ dominates $E_\pi(x|F)$ in the sense of second order stochastic dominance, for every partition F . It is well known (see Yaari (1987)) that rank-dependent expected utility (23) with linear v and convex f is monotone decreasing with respect to the second order stochastic dominance. Therefore

$$\min_{P \in \text{core}(f \circ \pi)} E_P(x) \leq \min_{P \in \text{core}(f \circ \pi)} E_P(E_\pi(x|F)). \quad (24)$$

We apply (24) to $x = \chi_A$ for $A \subset S$. The left-hand side equals $f(\pi(A))$ while the right-hand side is $\min P_F^\pi(A)$ over all P in $\text{core}(f \circ \pi)$. Thus

$$f(\pi(A)) \leq P_F^\pi(A) \quad (25)$$

for every $P \in \text{core}(f \circ \pi)$ and every A and F . This shows that $\text{core}(f \circ \pi)$ is π -stable. \square

Proposition 4 and Theorem 2 imply that every rank-dependent expected utility with concave utility and convex distortion function is averse to mean-independent risk. Yaari (1987) proved (using an inequality of Hardy, Littlewood and Polya) that rank-dependent expected utility (23) with linear utility v and convex distortion function f is decreasing with respect to the Rothschild-Stiglitz order of more risky. Therefore it is also averse to mean-independent risk. Chew, Karni and Safra (1987) proved that rank-dependent expected utility on the set of all distributions on a real interval is decreasing with respect to the relation of R-S more risky if and only if utility v is concave and distortion function f is convex. The proof in Chew, Karni and Safra (1987) (see also Chew and Mao (1995)) relies on Gateaux differentiability of the utility function on the space of distributions.

Examples of cores are intervals of probabilities. The set

$$\mathcal{P}_l = \{P \in \Delta : P(s) \geq \gamma_s, \forall s\}, \quad (26)$$

where $\gamma_s \in (0, 1)$ are lower bounds on probabilities such that $\sum_s \gamma_s \leq 1$, is the core of convex distortion of probability measure π defined by $\pi(s) = \frac{\gamma_s}{\sum_s \gamma_s}$, and is π -stable. The distortion function is a convex function f_l such that $f_l(t) = \lambda t$ for every $t \leq \max_{A \subset S, A \neq S} \pi(A)$ and $f_l(1) = 1$. The set

$$\mathcal{P}_u = \{P \in \Delta : P(s) \leq \lambda_s, \forall s\}, \quad (27)$$

where $\lambda_s \in (0, 1)$ are upper bounds on probabilities such that $\sum_s \lambda_s \geq 1$, is the core of convex distortion of probability measure π given by $\pi(s) = \frac{\gamma_s}{\sum_s \gamma_s}$ and is π -stable. The distortion function is f_u given by $f_u(t) = \max\{\lambda(t - 1) + 1, 0\}$ for every $t \in [0, 1]$. Since the intersection (and the union) of any two π -stable sets is π -stable, it follows that the order interval of probabilities

$$[\gamma\pi, \lambda\pi] = \{P \in \Delta : \gamma\pi \leq P \leq \lambda\pi\}, \quad (28)$$

where $\gamma \leq 1 \leq \lambda$, is π -stable, too. It is the core of $f \circ \pi$ for convex function $f = \max\{f_l, f_u\}$. The set of all possible priors Δ is π -stable for every $\pi \in \mathring{\Delta}$.

Proposition 4 does not extend to the more general class of cores of convex capacities. The set of priors in our discussion of the Ellsberg paradox in Section 1 is the core of a convex capacity but it is not π -stable for any probability measure π (see Section 7).

The Euclidean distance from a reference probability measure π is not π -stable unless π is the uniform probability measure. Similarly, Euclidean neighborhoods of π are not π -stable unless π is the uniform probability measure. This can be demonstrated using conditions Propositions 1 and 3, respectively. We present here an argument for π -stability of an Euclidean neighborhood of uniform probability measure π and lack thereof for non-uniform π . Let $\mathcal{P} = \{P \in \Delta : \|P - \pi\| \leq \epsilon\}$, and assume that $\mathcal{P} \subset \mathring{\Delta}$. For every non-deterministic x , the set $\mathcal{M}_{\mathcal{P}}(x)$ consists of a unique probability measure P_x^* . The first-order conditions for P_x^* as a solution to the minimization in (11) imply that if $x_s = x_{s'}$, then $P_x^*(s) - \pi(s) = P_x^*(s') - \pi(s')$. If π is uniform, then this implies $P_x^*(s) = P_x^*(s')$ which is condition (12) in this case.

If π is not uniform, say $\pi(1) \neq \pi(2)$, and $S \geq 3$, then one can show that for x such that $x_1 = x_2 \neq x_3$ the first-order conditions for P_x^* contradict (12). The Euclidean distance from uniform probability measure is a monotone transformation of the Gini-index (15) and the Euclidean neighborhood is the Gini-index neighborhood.

7. Risk Aversion and Unambiguous Events.

Under multiple-prior expected utility, the agent's probabilistic beliefs about events are described by the set of probability measures \mathcal{P} . A natural definition (see Nehring (1999)) of an *unambiguous event* for \mathcal{P} is as an event $A \subset S$ such that $P(A) = P'(A)$ for all $P, P' \in \mathcal{P}$. Of course, the *trivial* events, \emptyset and S , are always unambiguous.

It turns out that, if a set of priors - other than a singleton set - permits non-trivial unambiguous events, then it cannot be π -stable for any π . Thus, the existence of a non-trivial unambiguous event precludes mean-independent risk aversion.

Proposition 5: *Let \mathcal{P} be π -stable. There exists a non-trivial unambiguous event for \mathcal{P} if and only if $\mathcal{P} = \{\pi\}$.*

PROOF: Suppose that \mathcal{P} is π -stable, has a non-trivial unambiguous event A , and, by contradiction, there exists $P \in \mathcal{P}$ such that $P \neq \pi$. Let s be such that $\pi(s) \neq P(s)$. Suppose first that $s \notin A$. Consider a partition F of S into two sets: $A \cup \{s\}$, and its complement. Note that the complement of $A \cup \{s\}$ is non-empty, for it cannot be that $P(A) = \pi(A)$, $\pi(s) \neq P(s)$, and $A \cup \{s\} = S$. For the probability measure P_F^π defined by (1), we have

$$P_F^\pi(A) = \pi(A) \frac{\pi(A) + P(s)}{\pi(A) + \pi(s)} \neq \pi(A). \quad (29)$$

Since $P_F^\pi \in \mathcal{P}$, this contradicts the assumption that A is unambiguous. If $s \in A$, then we consider the complement event A^c instead of A . Event A^c is unambiguous and $s \notin A^c$, so that the above arguments apply. This concludes the proof of the non-trivial part of the proposition. \square

For the set of priors in our discussion of the Ellsberg paradox in Section 1, the event of red ball drawn from the urn is unambiguous. It has probability 1/3. Proposition 5 implies that there is no measure π in the set of priors such that

π -stability holds. Thus, the form of ambiguity of beliefs in the Ellsberg paradox precludes risk aversion.

8. Concluding Remarks.

The main contribution of the paper is identifying the property of stability of a cost function and a set of priors as necessary and sufficient for aversion to mean-independent risk of variational preferences and multiple-prior expected utilities. It should be noted that the condition of stability of a set of priors has some similarity to the condition of rectangularity introduced by Epstein and Schneider (2003) in their study of dynamic consistency of multiple-prior preferences. The similarity appears superficial though. In the setting of this paper, a set of priors \mathcal{P} is called rectangular with respect to a fixed partition of states F if $P_F^\pi \in \mathcal{P}$ for all probability measures P and π in \mathcal{P} .

In Strzalecki and Werner (2009) we extend the standard results on optimal risk sharing among risk averse agents from expected utilities to multiple-prior utilities and variational preferences using some results from this paper.

Appendix

Proof of Theorem 1. The proof of the second part proceeds in two steps. First we show that if variational preferences with cost c and concave utility v are mean-independent risk averse under π , then so are the variational preferences with cost c and linear utility function. Second, we show that if variational preferences with finite cost c and linear utility are mean-independent risk averse under π , then c is π -stable.

The first step relies on the characterization of aversion to mean-independent risk in terms of superdifferentials established in Theorem 6.1 in Werner (2009). It says that a concave utility function U is averse to mean-independent risk under π if and only if for every x there exists $\phi \in \partial U(x)$ such that

$$\text{if } x_s = x_{s'}, \text{ then } \frac{\phi_s}{\pi(s)} = \frac{\phi_{s'}}{\pi(s')}. \quad (30)$$

The superdifferential of variational preferences with cost c and concave utility v at a point x at which v is differentiable is (see Theorem 18, of Maccheroni et al (2006))

$$\{\phi \in \mathcal{R}^S : \phi_s = v'(x_s)P(s), \forall s \in S, \text{ for some } P \in \mathcal{M}_c^v(x)\}, \quad (31)$$

where $\mathcal{M}_c^v(x)$ denotes the set of minimizing probability measures. That is

$$\mathcal{M}_c^v(x) = \arg \min_{P \in \Delta} \{E_P[v(x)] + c(P)\} \quad (32)$$

For linear utility, the superdifferential is simply the set of minimizing probabilities $\mathcal{M}_c(x)$ of (7).

We need to show that (30) holds for some $P \in \mathcal{M}_c(x)$ in place of ϕ . For every x there exist $y \in \mathfrak{R}^S$ and a constant $k \in \mathfrak{R}$ such that $v(y) = x + k$ and v is differentiable at y . The differentiability of v at y can be guaranteed by a suitable choice of constant k because concave function v has at most a countable set of points of nondifferentiability. Since $\mathcal{M}_c^v(y) = \mathcal{M}_c(x + k)$ and $\mathcal{M}_c(x + k) = \mathcal{M}_c(x)$, the sought out condition follows from the assumption of mean-independent risk aversion of variational preference with cost c and utility v via condition (30).

For the second step we have the following

Lemma 2: For every $\bar{P} \in \Delta$ such that $c(\bar{P}) < \infty$, there exists $\bar{x} \in \mathfrak{R}^S$ such that

$$\bar{P} = \arg \min_{P \in \Delta} \{E_P(\bar{x}) + c(P)\} \quad (33)$$

PROOF: One can take \bar{x} so that $-\bar{x} \in \partial c(\bar{P})$, where ∂c denotes the subdifferential of convex function c . By the definition of the subdifferential it holds $c(P) \geq c(\bar{P}) + E_P(-\bar{x}) - E_{\bar{P}}(-\bar{x})$ for every $P \in \Delta$. \square

Suppose that, for some $\hat{P} \in \Delta$,

$$c(\hat{P}_F^\pi) > c(\hat{P}) \quad (34)$$

Since c is finite, we have $c(\hat{P}_F^\pi) < \infty$. By Lemma 2, there exists \hat{x} be such that (33) holds for \hat{P}_F^π . Using Lemma 1 and (34), we obtain

$$\min_{P \in \Delta} \{E_P[\hat{x}] + c(P)\} = E_{\hat{P}}[E_\pi[\hat{x}|F]] + c(\hat{P}_F^\pi) > E_{\hat{P}}[E_\pi[\hat{x}|F]] + c(\hat{P}). \quad (35)$$

The right-hand side of (35) exceeds $\min_{P \in \Delta} \{E_P[E_\pi[\hat{x}|F]] + c(P)\}$. This contradicts preference for F -conditional expectation of variational preferences with cost c and linear utility. This concludes the proof. \square

Proof of Theorem 2. The first step of the Proof of Theorem 1 in this Appendix implies that if multiple-prior expected utility with set of priors \mathcal{P} and concave utility v is averse to mean-independent risk under π then so is the multiple-prior expected utility with set of priors \mathcal{P} and linear utility.

We show now that if multiple-prior expected utility with set of priors \mathcal{P} and linear utility is mean-independent risk averse under π , then \mathcal{P} is π -stable. Suppose by contradiction that \mathcal{P} is not π -stable. Then there exists probability measure $\bar{P} \in \mathcal{P}$ such that $\bar{P}_F^\pi \notin \mathcal{P}$ for some partition F . By the separation theorem, there exists $y \in \mathfrak{R}^S$ such that

$$E_{\bar{P}_F^\pi}(y) < \min_{P \in \mathcal{P}} E_P(y) \quad (36)$$

Using Lemma 1 and the fact that $\bar{P} \in \mathcal{P}$, we obtain from (36) that

$$\min_{P \in \mathcal{P}} E_P[E_\pi(y|F)] < \min_{P \in \mathcal{P}} E_P(y) \quad (37)$$

This contradict preference for conditional expectations under π , which is equivalent to mean-independent risk aversion. This contradiction concludes the proof. \square

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