Participation in Risk Sharing under Ambiguity

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Abstract: This paper is about (non) participation in efficient risk sharing among agents who have ambiguous beliefs about uncertain states of nature. The question we ask is whether and how can ambiguous beliefs give rise to some agents not participating in efficient risk sharing. Ambiguity of beliefs is described by the multiple-prior expected utility of Gilboa and Schmeidler (1989), or the variational preferences of Maccheroni et al. (2006). The main result says that if the aggregate risk is relatively small, then the agents whose beliefs are the most ambiguous do not participate in risk sharing. The higher the ambiguity of those agents' beliefs, the more likely is their non-participation. Another factor making non-participation more likely is low risk aversion of agents whose beliefs are less ambiguous. We discuss implications of our results on agents' participation in trade in equilibrium in assets markets.

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1. Introduction

Expected utility hypothesis together with (strict) risk aversion and common probabilities have strong implications on efficient risk sharing among multiple agents. First, agents' consumption plans are comonotone with aggregate resources. Second, every agent participates in risk sharing by holding at least a small fraction of the aggregate risk. These results are at odds with empirical observations. Individual consumption often deviates from positive correlation with the aggregate consumption.[1] A large fraction of population in the US is not participating in asset markets thereby abstaining from sharing the aggregate financial risk.

Ambiguity of beliefs has been suggested as a way to reconcile the differences between the observed patterns and the rules of efficient risk sharing. Two closely related, standard models of decision making with ambiguous beliefs are the multiple-prior expected utility of Gilboa and Schmeidler (1989) and the variational preferences of Maccheroni et al. (2006). Under the multiple-prior expected utility hypothesis, an agent has a set of probability measures (or priors) as her beliefs and evaluates an uncertain prospect by taking the minimum of expected utilities over the set of beliefs. One of the main implications of the multiple-prior model is the possibility of non-participation in trade. A simple illustration of this is the portfolio inertia of Dow and Werlang (1992). An agent with multiple-prior expected utility and risk-free initial wealth does not invest in a risky asset for a range prices. As long as the expected return on the risky asset under the most pessimistic belief is below the return on the risk-free asset and the expected return under the most optimistic belief is above the risk-free return, the agent will choose zero investment in the risky asset.

Mukerji and Tallon (2001, 2004) and Cao et al. (2005) have shown that non-participation in trade can occur in an equilibrium in asset markets with multiple-prior expected utilities. Cao et al. (2005) considered a CARA-normal model of asset markets where agents know the true variance of the payoff of a risky asset but have ambiguous beliefs about its mean. Those ambiguous beliefs are specified by intervals of values around the true mean. There is heterogeneity of ambiguous beliefs. Agents with high ambiguity have bigger intervals than those with low

[1] Positive correlation is implied by comonotonicity, see LeRoy and Werner (2014, pg. 158).

ambiguity. In equilibrium, agents with high ambiguity do not participate in trade of the asset. The threshold for non-participation depends on the variance of the payoff of the outstanding asset supply, the dispersion of ambiguous beliefs, and the (common) degree of risk aversion. Low variance of asset supply, low risk aversion, and high dispersion of beliefs all lead to greater non-participation. A related CARA-normal model has been considered by Easley and O'Hara (2009) - with similar results - in their study of financial regulation and its role in mitigating the effects of ambiguity on market participation. Ozsoylev and Werner (2011) consider asset markets with asymmetric information and show that ambiguous beliefs may lead to limited participation and market illiquidity in rational expectations equilibrium.

In this paper we focus on non-participation in efficient risk sharing. The question we ask is whether and how can ambiguity of beliefs give rise to some agents not participating in risk sharing, that is, having risk-free consumption in Pareto optimal allocations. First, we show that an agent whose set of priors is a strict superset of another agent's set of priors is more likely not to participate in risk sharing in the sense that she does not participate whenever the other agent chooses so. Our main result says that if the aggregate risk is small, then agents with the highest ambiguity - those whose sets of priors are supersets of some other agents' sets of priors - do not participate in risk sharing in interior Pareto optimal allocations. The bigger the set of priors of an agent with the highest ambiguity, the greater is the aggregate risk for which she does not participate in risk sharing. Another factor leading to non-participation of agents with the highest ambiguity is low risk aversion of agents with less ambiguous beliefs.

Because of the First Welfare Theorem, properties of Pareto optimal allocations hold for equilibrium allocations in assets markets if markets are complete. If the aggregate risk is small, agents with the highest ambiguity will have risk-free consumption in an equilibrium. Whether those agents will or will not trade the assets depends on their initial endowments. If the initial endowment is risk free, then the agent will not trade. Otherwise, if her initial endowment is risky, then she will trade so as to achieve a risk-free equilibrium consumption. Thus, she will purchase full insurance in asset markets. Our results are in concordance with the findings of Cao et al. (2005) in their specialized setting.

Properties of efficient allocations for multiple-prior expected utilities and other non-expected utilities have been extensively studied in the literature over the past two decades. Billot et al. (2001) show that if agents have at least one prior in common and there is no aggregate risk, then all interior Pareto optimal allocations are risk free. Rigotti et al. (2008) extend that result to other models of convex preferences under ambiguity such as variational preferences and the smooth ambiguity model of Klibanoff et al. (2005). Recent paper by Ghirardato and Siniscalchi (2018) provides further extensions to non-convex preferences under ambiguity. They identify a sufficient behavioral condition for risk-free optimal allocations with no aggregate risk. Gierlinger (2018) studies risk sharing and trade in complete markets with sunspot uncertainty.

Comonotonicity and measurability of individual consumption plans with respect to the aggregate endowment when there is aggregate risk have been studied in Chateauneuf et al. (2000) and Dana (2004) for non-additive (or Choquet) expected utilities of Schmeidler (1989), and, in greater generality, in Strzalecki and Werner (2011) for multiple-prior utilities, variational preferences, and the smooth ambiguity model. Kelsey and Chakravarty (2015) study efficient risk sharing for Choquet expected utilities with heterogeneous ambiguity. Relationship between ambiguity aversion and trade in complete markets is the subject of a paper by de Castro and Chateauneuf (2011). They show that if initial endowments are unambiguous, then the set of individually rational net trades gets smaller when agents become more ambiguity averse in the sense of Ghirardato and Marinacci (2002). Kajii and Ui (2006) study the existence of Pareto-improving “agreeable” bets between two agents with Choquet expected utilities when there is no aggregate risk. Dominiak et al. (2012) provide some extensions of the results to non-convex preferences. Kajii and Ui (2009) and Martins-da-Rocha (2010) study interim efficient allocations in an economy with asymmetric information and multiple-prior expected utilities. Araujo et al. (2017) consider uncertainty described by infinitely many states and study the existence of Pareto optimal allocations with convex and non-convex preferences under ambiguity.

The paper is organized as follows. In Section 2 we introduce the multiple-prior expected utility and define risk-adjusted beliefs that are the basic tool in the analysis of Pareto optimal allocations. In Section 3 we review properties of Pareto optimal allocations for multiple-prior expected utilities. Our main results about non-participation in risk sharing are presented in Section 4. In Section 5 we extend the results to variational preferences. We conclude in Section 6 with comments on the assumptions.

2. Ambiguity and Risk-Adjusted Beliefs

We consider a static (single-period) economy under uncertainty with I agents. Uncertainty is described by a finite set of states S. There is a single consumption good consumed in every state. State contingent consumption plans (or acts) are vectors $c \in R^S_+$. Agent i has a utility function $U_i : R^S_+ \to R$ on state-contingent consumption plans. Utility U_i is assumed to be a multiple-prior (or MinMax) expected utility. That is,

$$U_i(c) = \min_{P \in \mathcal{P}_i} \mathbf{E}_P[v_i(c)], \tag{1}$$

for some utility function $v_i : R_+ \to R$ and a closed and convex set $\mathcal{P}_i \subseteq \Delta$ of probability measures on S. We assume throughout that

(**A**) v_i is strictly increasing, concave and differentiable for every i.

The set of probability measures $\mathcal{P}_i$ represents agent ith ambiguous beliefs (or priors) about uncertain states of nature. The bigger that set, the higher the ambiguity. More specifically, we say that agent j has *higher ambiguity* than agent i if

$$\mathcal{P}_i \subset \mathcal{P}_j. \tag{2}$$

If $\mathcal{P}_i \subset \text{int}\mathcal{P}_j$, where $\text{int}\mathcal{P}_i$ is the interior of $\mathcal{P}_i$ relative to Δ, then agent j has *strictly higher ambiguity than agent* i. [2]

Multiple-prior utility functions are not differentiable. The natural generalization of the derivative, or the marginal utility, for concave non-differentiable utility function is the superdifferential. The superdifferential $\partial U_i(c)$ at $c \in R^S_+$ is the set of all vectors $\phi \in \mathcal{R}^S$ (supergradients) such that $U_i(c') \leq U_i(c) + \phi(c' - c)$ for every

[2] The relation of having higher ambiguity should not be confused with that of being more ambiguity averse introduced by Ghirardato and Marinacci (2002). The latter requires that in addition to (2) utility function v_j is an affine transformations of v_i.

$c' \in R^S_+$. For concave multiple-prior expected utility (1), the superdifferential at an interior consumption plan $c \in R^S_{++}$ is

$$\partial U_i(c) = \{\phi \in R^S : \phi(s) = v'(c(s))P(s)\ \forall s, \ \text{ for some } P \in \bar{\mathcal{P}}_i(c)\}, \tag{3}$$

where $\bar{\mathcal{P}}_i(c) \subset \mathcal{P}$ is the subset of priors for which the minimum expected utility is attained. That is,

$$\bar{\mathcal{P}}_i(c) = \arg\min_{P \in \mathcal{P}_i} \mathbf{E}_P[v_i(c)]. \tag{4}$$

It is convenient to normalize the supergradient vectors in $\partial U_i(c)$ so that they lie in the probability simplex Δ. That set is

$$Q_i(c) = \{\pi \in \Delta : \pi(s) = \frac{v'(c(s))P(s)}{\mathbf{E}_P[v'(c)]}, \ \forall s, \ \text{ for some } P \in \bar{\mathcal{P}}_i(c)\}, \tag{5}$$

for $c \in R^S_{++}$. Probability measures in $Q_i(c)$ will be called *risk-adjusted beliefs* in accordance with the terminology often used in asset pricing. If utility index v_i is linear, then $Q_i(c) = \bar{\mathcal{P}}_i(c)$ for every c. Note that if $\mathcal{P}_i \subset \mathring{\Delta}$, then $Q_i(c) \subset \mathring{\Delta}$ for every $c \in R^S_{++}$, where $\mathring{\Delta}$ is the strictly positive probability simplex.

Probability measure π is a risk-adjusted belief at $c \in R^S_{++}$ if and only if

$$\mathbf{E}_\pi(c') \geq \mathbf{E}_\pi(c) \text{ for every } c' \in R^S_+ \text{ such that } U_i(c') \geq U_i(c). \tag{6}$$

Probability measures satisfying (6) are sometimes called subjective beliefs at c, see Rigotti et al. (2008) where also a proof of the equivalence can be found.

We state the following result for the use later.[3]

Lemma 1: *For every $c \in R^S_{++}$, the following hold*

(i) If c is risk free, then $Q_i(c) = \mathcal{P}_i$.

(ii) If c is risky, then $Q_i(c) \cap \mathrm{int}\mathcal{P}_i = \emptyset$.

(iii) If c is risky, v_i is strictly concave and $\mathcal{P}_i \subset \mathring{\Delta}$, then

$Q_i(c) \cap \mathcal{P}_i = \emptyset$.

[3]We use the terms risk free and risky to mean, respectively, deterministic and non-deterministic consumption plans.

Proof: Part (i) is obvious. To prove (ii), suppose by contradiction that there exists π such that $\pi \in \text{int}\mathcal{P}_i$ and $\pi \in Q_i(c)$. Let $\hat{c} = E_\pi c$. Since c is risky and v_i is concave it follows that

$$\min_{P \in \mathcal{P}_i} \mathbf{E}_P[v_i(c)] < \mathbf{E}_\pi[v_i(c)] \leq v_i(\hat{c}). \tag{7}$$

That is, $U_i(\hat{c}) > U_i(c)$. Since U_i is continuous and c is interior, we obtain from (6) that $E_\pi \hat{c} > E_\pi c$. This contradicts $\hat{c} = E_\pi c$.

The proof of (iii) is the same as for (ii) except that, for $\pi \in \mathcal{P}_i$ the first inequality in (7) is weak but the second is strict because of strict concavity of v_i and $\pi \in \mathring{\Delta}$. We obtain $U_i(\hat{c}) > U_i(c)$, and the rest of the argument applies. □

Risk-adjusted beliefs provide a simple characterization of Pareto optimal allocations. We recall first some standard definitions. A feasible allocation is a collection of consumption plans $\{c_i\}_{i=1}^I$ such that $c_i \in R_+^S$ for every i and $\sum_{i=1}^I c_i = \omega$, where $\omega \in R_{++}^S$ the aggregate endowment of the economy assumed to be strictly positive. A feasible allocation $\{c_i\}$ is *Pareto optimal* if there is no other feasible allocation $\{\tilde{c}_i\}$, such that $U_i(\tilde{c}_i) \geq U_i(c_i)$ for all i and $U_j(\tilde{c}_j) > U_j(c_j)$ for some j. The following characterization of interior Pareto optimal allocations can be found in Rigotti et al. (2008) (see also Kaji and Ui (2009) and de Castro and Chateauneuf (2011)):

Proposition 1: *An interior allocation $\{c_i\}$ is Pareto optimal if and only if there exists a probability measure π such that $\pi \in Q_i(c_i)$ for all i.*

3. Efficient Risk Sharing

The most fundamental rule of efficient risk sharing for expected utility functions is comonotonicity of individual consumption plans with the aggregate endowment. Comonotonicity means that every agent's consumption in every state is a non-decreasing function of the aggregate endowment in that state. It holds if agents have common probabilities and are strictly risk averse. If in addition their utility functions are differentiable, then strict comonotonicity (i.e., individual consumption being a strictly increasing function of the aggregate endowment) holds for interior Pareto optimal allocations. Comonotonicity implies that if there is no aggregate risk (i.e., aggregate endowment is risk free), then every agent's consumption plan is risk free. Strict comonotonicity implies that if there is aggregate risk, then every agent's consumption plan is risky.

Properties of Pareto optimal allocations for multiple-prior expected utilities depend on agreement among agents' beliefs. The minimal agreement is that the sets of priors are overlapping,

$$\bigcap_{i=1}^{I} \mathcal{P}_i \neq \emptyset, \tag{8}$$

so that there exists at least one common belief. Strzalecki and Werner (2011) show by means of a counterexample that condition (8) is not sufficient for comonotonicity of Pareto optimal allocations for strictly concave multiple-prior expected utilities. Their sufficient condition for comonotonicity requires existence of common conditional beliefs and is quite stringent. Nevertheless, the condition of overlapping sets of priors guarantees that every Pareto optimal allocation is risk free if there is no aggregate risk. More precisely, Billot et al. (2001) show that if agents' utility functions are strictly concave and there is no aggregate risk, then (8) is sufficient (and necessary) for every Pareto optimal allocation to be risk free. The same holds for concave utility functions provided that the intersection of sets of priors has non-empty interior[4], that is, $int \bigcap_{i=1}^{I} \mathcal{P}_i \neq \emptyset$.

A stronger condition of belief agreement is that agents have a common set of ambiguous beliefs. Chateauneuf et al. (2000) (see also Dana (2004)) show that if the common set of probabilities is the core of a convex capacity[5], then Pareto optimal allocations are comonotone.

We say that an agent *participates in risk sharing* in a feasible allocation if there is aggregate risk and the agent's consumption plan is risky. If the consumption plan is strictly comonotone with the risky aggregate endowment, the the agent participates in risk sharing, but not vice versa. We show in Proposition 2 that for a common set of priors and strictly concave utility functions (satisfying assumption (A)) every agent participates in risk sharing if there is aggregate risk.

Proposition 2: *Suppose that v_i is strictly concave and $\mathcal{P}_i = \mathcal{P}$ for some $\mathcal{P} \subset \mathring{\Delta}$, for every i. If there is aggregate risk, then every agent participates in risk sharing in every interior Pareto optimal allocation.*

PROOF: Suppose by contradiction that there is an interior Pareto optimal allo-

[4]For completeness, we prove this result in Proposition 5 in the Appendix.

[5]Multiple-prior expected utility with core of convex capacity as a set of priors has an equivalent representation as Choquet expected utility of Schmeidler (1989).

cation $\{c_i\}$ and an agent j such that c_j is risk free. By Lemma 1 (i), we have $Q_j(c_j) = \mathcal{P}$. Since there is aggregate risk, there exists at least one other agent k such that c_k is risky. Lemma 1 (iii) implies that $Q_k(c_k) \cap \mathcal{P} = \emptyset$. This contradicts Proposition 1. □

4. Non-Participation in Risk Sharing

If agents have different but overlapping sets of beliefs, non-participation in risk sharing (i.e., some agents' consumption plans being risk free) can occur in Pareto optimal allocations. We start this section with an observation that agents who have high ambiguity are more likely not to participate in risk sharing than those with low ambiguity. This is a simple consequence of Lemma 1.

Proposition 3: *If $\{c_i\}$ is an interior Pareto optimal allocation and c_j is risk free for some agent j, then c_k is risk free for every agent k who has strictly higher ambiguity than j. The same holds if k has higher ambiguity than j, v_k is strictly concave, and $\mathcal{P}_k \subset \mathring{\Delta}$.*

Proof: Let $\pi \in \bigcap_{i=1}^{I} Q_i(c_i)$. If c_j is risk free, then, by Lemma 1 (i), $\pi \in \mathcal{P}_j$. If k has strictly higher ambiguity than j, then $\pi \in \mathrm{int}\mathcal{P}_k$. Lemma 1 (ii) implies that c_k is risk free. Applying Lemma 1 (iii) instead of (ii) gives the same conclusion if k has higher ambiguity than j and the additional assumptions hold. □

The main question we ask in this section is under what conditions agents with high ambiguity do not participate in risk sharing. There are three separate conditions: (1) when their ambiguity is sufficiently high; (2) when the aggregate risk is small, and (3) when risk aversion of agents with low ambiguity is low. We explain each of these conditions below.

First, an agent whose ambiguity is the highest possible does not participate in risk sharing in every interior Pareto optimal allocation.

Proposition 4: *If agent k has strictly higher ambiguity than some agent j and $\mathcal{P}_k = \Delta$, then agent's k consumption plan is risk free in every interior Pareto optimal allocation.*

Proof: Consider an interior Pareto optimal allocation $\{c_i\}$. By Proposition 1, there exist $\pi \in \bigcap_{i=1}^{I} Q_i(c_i)$. Since $\mathcal{P}_j \subset \mathring{\Delta}$ and therefore $Q_j(c_j) \subset \mathring{\Delta}$, it follows that $\pi \in \mathring{\Delta}$. Lemma 1 (ii) implies that c_k is risk free. □

Second, an agent with high ambiguity does not participate in risk sharing if the aggregate risk is sufficiently small. We say that there is *small aggregate risk* if the aggregate endowment ω lies in an ϵ-neighborhood $B_\epsilon(D)$ of risk-free consumption plans for some small $\epsilon > 0$. Here $D = \{\lambda e : \lambda \geq 0\}$ is the set of risk-free consumption plans where $e = (1, \ldots, 1)$.

We have

Theorem 1: *Suppose that sets of priors are overlapping (8) and every utility function v_i is strictly concave. If agent k has strictly higher ambiguity than some agent j, then there exists $\epsilon > 0$ such that if $\omega \in B_\epsilon(D)$, then agent's k consumption plan is risk free in every interior Pareto optimal allocation.*

PROOF: See Appendix □

Theorem 1 is the main result of this paper. Condition (8) guarantees that individual risk in Pareto optimal allocations is small when the aggregate risk is small. The assumption of strict concavity of utility functions can be weakened to concavity provided that the intersection of the sets of priors has non-empty interior.

Third, low risk aversion of agents with low ambiguity makes them likely to take all the risk so that agents with high ambiguity do not participate in risk sharing. Suppose that agent k has higher ambiguity than agent j whose utility function v_j is linear. If v_k is strictly concave and $\mathcal{P}_k \subset \mathring{\Delta}$, then it follows from Lemma 1 (iii) and Proposition 1 that agent k does not participate in risk sharing in every interior Pareto optimal allocation. The agent with linear utility function provides full insurance to agents who have higher ambiguity and strictly concave utility functions. This extends the well-known result for expected utility functions with common probabilities which says that a risk-neutral agent provides full insurance to strictly risk-averse agents in every interior Pareto optimal allocation.

More generally, we show that if agent k has strictly higher ambiguity than agent j and agent's j risk aversion is sufficiently small, then agent k does not participate in risk sharing. We assume that each utility function v_i is twice continuously differentiable. The Arrow-Pratt measure of risk aversion is $A_i(x) = -\frac{v_i''(x)}{v_i'(x)}$. The supremum of $A_i(x)$ on the interval $[0, \hat{\omega}]$ where $\hat{\omega} = \max_s \omega_s$ is the *global measure* *of risk aversion* of utility function v_i.[6] Of course, $\hat{A}_i = 0$ if and only if v_i is linear.

Theorem 2: *If agent k has strictly higher ambiguity than some agent j, then there exists $\epsilon > 0$ such that if the global measure of risk aversion $\hat{A}_j$ of agent j is less than ϵ, then agent's k consumption plan is risk free in every interior Pareto optimal allocation.*

PROOF: See Appendix □

The following example illustrates the results of this section.

Example 1: There are two states of nature and two agents. Agents have CARA utility functions $v_i(x) = -e^{-\rho_i x}$ with $\rho_i > 0$. Note that $\hat{A}_i = \rho_i$ because of constant risk aversion. Agents' sets of prior beliefs are intervals $\mathcal{P}_i = \{(q, 1-q) : q_l^i \leq q \leq q_h^i\}$, where $0 \leq q_l^i < q_h^i \leq 1$. Let $q_l^2 < q_l^1$ and $q_h^1 < q_h^2$, so that agent 2 has strictly higher ambiguity. The aggregate endowment is $\omega = (d, d + \Delta d)$, for some $d > 0$ and $\Delta d > 0$, so that there is aggregate risk.

Consider an allocation $\{c_1, c_2\}$ where agent 2 does not participate in risk sharing, that is, $c_1 = (a, a+\Delta d)$ and $c_2 = (b, b)$ for some $a > 0, b > 0$ such that $a+b = d$. The risk-adjusted probability of agent 1 at c_1 is $\frac{1}{q_h^1+(1-q_h^1)e^{-\rho_1 \Delta d}}$ $(q_h^1, (1-q_h^1)e^{-\rho_1 \Delta d})$. The risk-adjusted probabilities of agent 2 at risk-free c_2 are the set of priors $\mathcal{P}_2$. The risk-adjusted probability of agent 1 is contained in $\mathcal{P}_2$, if and only if $\frac{1}{q_h^1+(1-q_h^1)e^{-\rho_1 \Delta d}} q_h^1 \leq q_2^h$, or equivalently

$$e^{\rho_1 \Delta d} \frac{q_h^1}{1-q_h^1} \leq \frac{q_h^2}{1-q_h^2}, \tag{9}$$

provided that $q_h^1 < 1$ and $q_h^2 < 1$. Inequality (9) is a necessary and sufficient condition for Pareto optimality of allocations where agent's 2 consumption is risk free. Suppose that $q_1^h = \frac{5}{8}$ and $q_2^h = \frac{7}{8}$. We have $e^{\rho_1 \Delta d} \leq 4\frac{1}{5}$, or $\rho_1 \Delta d \leq 1.43$. For given Δd, it follows that if the risk aversion of agent 1 satisfies $\rho_1 \leq \frac{1.43}{\Delta D}$, then agent 2 does not participate in risk sharing in every Pareto optimal allocation.

[6] Our use of the standard concepts of the theory of aversion to risk should be taken with care. Those concepts have been developed in the setting of expected utility and their meaning for the multiple-prior expected utility may not be the same. For instance, linear utility v exhibits risk neutrality for expected utility, but this does not mean that the agent with multiple-prior expected utility is indifferent between the expectation of a consumption plan delivered with certainty and the consumption plan itself.

This illustrates Theorem 2. The same holds for given ρ_1, if the aggregate risk is small so that $\Delta d \leq \frac{1.43}{\rho_1}$, as per Theorem 1. If $q_h^2 = 1$, then agent 2 does not participate in risk sharing regardless of the aggregate risk or the risk aversion of agent 1. This illustrates Proposition 4. □

5. Risk Sharing with Variational Preferences

Results of Sections 2-4 can be extended to variational preferences. Variational preferences are closely related to multiple-prior expected utility and have been extensively studied in the literature (see Maccheroni et al. (2006) and Strzalecki (2011)). We provide an outline of an extension.

Variational preferences have a utility representation of the form

$$\min_{P \in \Delta} \left\{ E_P[v_i(c)] + \psi_i(P) \right\} \tag{10}$$

for some strictly increasing and continuous utility function $v_i : \mathcal{R}_+ \to \mathcal{R}$ and some convex and lower semicontinuous function $\psi_i : \Delta \to [0, \infty]$ such $\psi_i(Q) = 0$ for some $Q \in \Delta$. Function ψ_i is a "cost" function of beliefs. We assume that v_i is concave and differentiable, and that ψ_i is continuous.

The superdifferential of variational utility function has the same representation (3) as for multiple-prior expected utility, with the set of minimizing probabilities of (10), see Maccheroni et al. (2006). The normalized supergradients at any risk-free consumption plan are the zero-cost probabilities, that is, probability measures $Q \in \Delta$ such that $\psi_i(Q) = 0$. We denote the set of such measures by $\mathcal{P}_i^0$. Lemma 1 holds with the set of zero-cost probabilities $\mathcal{P}_i^0$ in place of set of beliefs $\mathcal{P}_i$ for multiple-prior expected utility. Proposition 1 has been extended to variational preferences in Rigotti et al. (2008). Our new Proposition 2 holds for variational preferences with the set of zero-cost probabilities $\mathcal{P}_i^0$ in place of $\mathcal{P}_i$. In particular, if all agents' cost functions are scale-multiples of the same function and their utility functions are strictly concave, then all agents participate in efficient risk sharing.

Non-participation in efficient risk-sharing can occur with variational preferences if sets of zero-cost probabilities are different across agents. Results of Section 4 hold for variational preferences with sets of zero-cost probabilities in place of sets of beliefs for multiple-prior expected utilities. Agents whose sets of zero-cost beliefs are strict supersets of other agents' sets of zero-cost beliefs are more likely not to participate in risk sharing. If the aggregate risk is small, then those agents do not participate in risk sharing at in any interior Pareto optimal allocation.

6. Concluding Remarks

We conclude with a discussion of some assumptions we made in Sections 2-4. The assumption of differentiability of utility functions in (A) is not essential for the results of Sections 2 and 3, and for Proposition 3. Of course, representation (5) of the superdifferential cannot be used but, for instance, Lemma 1 can be extended to any concave multiple-prior expected utility using normalized superdifferentials and its properties found in Rockafellar (1970). Theorems 1 and 2 require that utility functions be twice continuously differentiable. We restricted our attention to interior consumption plans and interior Pareto optimal allocations in most of Sections 2-4. Again Lemma 1, the results of Section 3, and Proposition 3 can be extended to hold for boundary allocations using normalized superdifferentials. Hypotheses of Theorems 1 and 2 may not be true for boundary allocations as it can be easily seen in an Edgeworth-box illustration of an economy with 2 states and two agents.

Appendix

For two probability measures $P, Q \in \Delta$, let $|P - Q|$ denote the total-variation distance between them. That is,

$$|P - Q| = \sum_{s=1}^{S} |P(s) - Q(s)|. \tag{11}$$

Further, let $V_\delta(\mathcal{P}) \subset \Delta$ denote the δ-neighborhood of the set $\mathcal{P} \subset \Delta$ in the variational distance for $\delta > 0$. Let $\hat{\omega} = \max_s \omega_s$ and let $\hat{A}_i = \sup\{A_i(x) : x \in [0, \hat{\omega}]\}$ where $A_i(x) = -\frac{v_i''(x)}{v_i'(x)}$ is the Arrow-Pratt measure of risk aversion.

Proof of Theorem 1:

First we show the following lemma.

Lemma 2: *If $c \in B_\epsilon(D)$ and $c \in R^S_{++}$, then $Q_i(c) \subset V_\delta(\mathcal{P}_i)$ for $\delta = e^{2\epsilon\hat{A}_i} - 1$.*

Proof: Take any $Q \in Q_i(c)$. Let $P \in \mathcal{P}_i$ be such that $Q(s) = \frac{v_i'(c(s))P(s)}{\mathbf{E}_P[v_i'(c)]}$ for every s. Further, let $\underline{c} = \min_s c_s$ and $\bar{c} = \max_s c_s$. We have

$$|P - Q| = \sum_{s=1}^{S} P(s) \frac{|\mathbf{E}_P[v_i'(c)] - v_i'(c(s))|}{\mathbf{E}_P[v_i'(c)]} \leq \frac{v_i'(\underline{c}) - v_i'(\bar{c})}{v_i'(\bar{c})} = \frac{v_i'(\underline{c})}{v_i'(\bar{c})} - 1. \tag{12}$$

Further,

$$\ln v_i'(\underline{c}) - \ln v_i'(\bar{c}) = \int_{\underline{c}}^{\bar{c}} A_i(x)dx \leq 2\epsilon\hat{A}_i, \tag{13}$$

where we used the fact that $\bar{c} - \underline{c} \leq 2\epsilon$ for $c \in B_\epsilon(D)$. Combining, we obtain

$$|P - Q| \leq e^{2\epsilon\hat{A}_i} - 1. \tag{14}$$

Therefore $Q \in V_\delta(\mathcal{P}_i)$. $\square$

We proceed now with the proof of Theorem 1. Since $\mathcal{P}_j \subset \text{int}\mathcal{P}_k$, there exists δ be such that $V_\delta(\mathcal{P}_j) \subset \text{int}\mathcal{P}_k$. Let $\tilde{\epsilon}$ be such that $e^{2\tilde{\epsilon}\hat{A}_j} - 1 = \delta$. By Lemma 2, $Q_j(c_j) \subset \text{int}\mathcal{P}_k$ for every $c_j \in B_{\tilde{\epsilon}}(D)$. If c_j is part of an interior Pareto optimal allocation and $Q_j(c_j) \subset \text{int}\mathcal{P}_k$, then it follows from Proposition 1 and Lemma 1 (ii) that c_k is risk free. Hence, it suffices to show that there exists $\epsilon > 0$, such that if $\omega \in B_\epsilon(D)$, then $c_j \in B_{\tilde{\epsilon}}(D)$ for every consumption plan c_j that is part of an interior Pareto optimal allocation.

Let $\mathcal{E}_j(\omega)$ be the set of Pareto optimal consumption plans of agent j. Let $b \in D$ be such that $b >> \omega$. Mapping $\mathcal{E}_j(\cdot)$ is an upper hemi-continuous correspondence on the compact set $[0, b]$. Let $\hat{D} = D \cap [0, b]$. By assumption (8), $\mathcal{E}_j(\omega) \subset \hat{D}$ if $\omega \in \hat{D}$. Therefore there exists ϵ be such that if $\omega \in B_\epsilon(\hat{D})$, then $\mathcal{E}_j(\omega) \subset B_{\tilde{\epsilon}}(\hat{D})$. This concludes the proof of Theorem 1. $\Box$

Proof of Theorem 2:

We start with a lemma.

Lemma 3: *If $\hat{A}_i < \epsilon$, then $Q_i(c) \subset V_\delta(\mathcal{P}_i)$ for $\delta = e^{\epsilon\hat{\omega}} - 1$ and every $c \in R^S_{++}$ such that $c \leq \omega$.*

Proof: The proof is similar to Lemma 2. For any $Q \in Q_i(c)$ let $P \in \mathcal{P}_i$ be such that $Q(s) = \frac{v_i'(c(s))P(s)}{\mathbf{E}_P[v_i'(c)]}$ for every s. We have

$$|P - Q| = \sum_{s=1}^{S} P(s) \frac{|\mathbf{E}_P[v_i'(c)] - v_i'(c(s))|}{\mathbf{E}_P[v_i'(c)]} \leq \frac{v_i'(0) - v_i'(\hat{\omega})}{v_i'(\hat{\omega})}. \tag{15}$$

Since

$$\ln v_i'(0) - \ln v_i'(\hat{\omega}) = \int_0^{\hat{\omega}} A_i(x)dx \leq \epsilon\hat{\omega}, \tag{16}$$

it follows that

$$|P - Q| \leq e^{\epsilon\hat{\omega}} - 1, \tag{17}$$

Therefore $Q \in V_\delta(\mathcal{P}_i)$ for $\delta = e^{\epsilon\hat{\omega}} - 1$. $\Box$

We proceed with the proof of Theorem 2. Since $\mathcal{P}_j \subset \text{int}\mathcal{P}_k$, there exists δ such that $V_\delta(\mathcal{P}_j) \subset \text{int}\mathcal{P}_k$. Let ϵ be such that $e^{\epsilon\hat{\omega}} - 1 = \delta$. If $\hat{A}_j < \epsilon$ and c_j is part of an interior Pareto optimal allocation $\{c_i\}$, then $Q_j(c_j) \subset \text{int}\mathcal{P}_k$, where we used Lemma 3. Using Proposition 1 and Lemma 1 (ii) we obtain that c_k is risk free. $\Box$

Proof of footnote 4 in Section 3:

Proposition 5: *If int$\bigcap_{i=1}^I \mathcal{P}_i \neq \emptyset$ and there is no aggregate risk, then every Pareto optimal allocation is risk free.*

Proof: Let $\{c_i\}$ be a feasible allocation. Let $\pi \in \text{int}\bigcap_{i=1}^I \mathcal{P}_i$. For each i, let $\hat{c}_i = E_\pi(c_i)$. Since $\bar{\omega}$ is risk free, it follows that allocation $\{\hat{c}_i\}$ is feasible. Inequality (7) implies that $U_i(\hat{c}_i) \geq U_i(c_i)$, with strict inequality if c_i is risky. It follows that every Pareto optimal allocation is risk free. $\Box$

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