Rational Asset Pricing Bubbles
and Debt Constraints[*]

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Abstract: Rational price bubble arises when the price of an asset exceeds the asset's fundamental value, that is, the present value of future dividend payments. The important result of Santos and Woodford (1997) says that price bubbles cannot exist in equilibrium in the standard dynamic asset pricing model with rational agents facing borrowing constraints as long as assets are in strictly positive supply and the present value of total future resources is finite. This paper explores the possibility of asset price bubbles under endogenous debt constraints induced by limited enforcement of debt repayment. Equilibria with endogenous debt constraints are prone to have infinite present value of total resources. We show that asset price bubble are likely to exist in such equilibria. Further, we demonstrate that there always exist equilibria with price bubbles on assets in zero supply.

1. Introduction

In times of instability in financial markets, news of price bubbles are frequent. The Global Financial Crisis of 2007-2009 provides several examples: real-estate market in the U.S, mortgage-backed securities, etc. Price bubble arises when the price of an asset exceeds the asset's fundamental value. As the notion of fundamental value invites many different interpretations, news of price bubbles can often be ascribed to misjudgement of the fundamental value. Economists generally agree though that the "dot-com" bubble of 1998-2000 and the Japanese stock market bubble of the 1980s were genuine price bubbles. Internet stock prices reached staggering levels in the U.S. in 2000. Nasdaq composite index rose by 200 % between 1998 and 2000. Returns on some internet stocks over this period were nearly 1000 %. Then, the market crashed down to virtually no value. Asset prices in Japan rose sharply in the 1980s. The Nikkei index soared by more than 400 % over the decade. Market capitalization of Japanese stocks exceeded the value of U.S. stock market in 1990. By 1991 prices plummeted down to levels from the early 1980s.

Financial economic theory does not provide a satisfactory answer to the question of whether and how price bubble can arise in asset markets. The theory of rational asset pricing bubbles defines the fundamental value of an asset as the present value of future dividend payments. The main result of the seminal paper by Manuel Santos and Michael Woodford (1997) says that price bubbles cannot exists in equilibrium in the standard dynamic asset pricing model as long as assets are in strictly positive supply and the present value of total future resources over the infinite time is finite. As most assets are believed to be in strictly positive supply and equilibria with infinite present value of aggregate resources are usually considered exotic, the Santos-Woodford result is interpreted as saying that price bubbles are unlikely to happen.

The no-bubble theorem of Santos and Woodford (1997) has been established for a particular type of constraint on agents' portfolio holdings - the borrowing constraint which restricts the amount of wealth an agent can borrow on a portfolio at any date. The primary role of borrowing constraints in dynamic asset markets is to prevent agents from engaging in Ponzi schemes. There are many alternative portfolio constraints that could be considered: debt (or solvency) constraints,

collateral constraints, short sales constraints, etc. This paper explores the possibility of asset price bubbles in dynamic asset markets under debt constraints. Debt constraints restrict the amount of debt an agent can carry on a portfolio at every date. Our main focus is on endogenous (or self-enforcing) debt constraints induced by limited commitment to market transactions (see Alvarez and Jermann (2000), Hellwig and Lorenzoni (2009), Martins-da-Rocha and Vailakis (2012), and others). If default on payoff of a portfolio is permitted and has precisely specified consequences, self-enforcing debt constraints are defined by a sequence of debt bounds such that an agent is unwilling to default even if his indebtedness is at the maximum allowed level.

We show that the no-bubble theorem extends to debt constraints including endogenous debt constraints. The peculiar feature of equilibria with endogenous debt constraints is that they often give rise to infinite present value of the aggregate endowment, or "low interest rates." This is exactly when the sufficient condition for non-existence of equilibrium price bubbles on assets in strictly positive supply is violated. Our main result, Theorem 2, shows that if there is an equilibrium with endogenous debt constraints in which debt bounds are different from zero to an extent we specify, then price bubbles may be "injected" on assets in strictly positive supply and present value of the aggregate endowment must be infinite. The term injecting means that a suitably chosen sequence of positive price bubbles can be added to equilibrium asset prices so that equilibrium be preserved with unchanged consumption plans. The idea of injecting bubbles is due to Kocherlakota (2008) whose results we extend and clarify. Bejan and Bidian (2013) proved a closely related result in a setting where debt bounds can be negative (so that agents are forced to hold some savings). We provide two examples of equilibria in asset markets under endogenous debt constraints with price bubbles on assets in strictly positive supply. One is the classical example of price bubble on zero-dividend asset (i.e., fiat money) in strictly positive supply due to Bewley (1980) (see also Kocherlakota (1992)). We show that debt bounds in this example are self enforcing when punishment for default is exclusion from further borrowing. The second is a variation of the leading example in Hellwig and Lorenzoni (2009).

We show in Theorem 3 that there always exist equilibria with price bubbles on infinitely-lived assets that are in zero supply. There is multiplicity of equilibria,

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and some equilibrium prices have bubbles.

The literature on asset price bubbles is vast and diverse. Gilles and LeRoy (1992) define price bubbles as manifestation of lack of countable additivity of pricing. Huang and Werner (2000) show that this notion is different from rational price bubbles in dynamic asset markets. Harrison and Kreps (1978) explore the possibility of speculative bubbles under short sales constraints and heterogeneous beliefs.[1] Speculative bubbles differ from rational bubbles in the way the fundamental value of dividends is defined. Instead of the present value of future dividends, it is the marginal valuation of dividends by an agent for whom such valuation is the highest. In presence of portfolio constraints and infinite-time horizon, these two notions may be different. Ofek and Richardson (2005) attribute the dot-com bubble to heterogeneous beliefs and short-sale constraints in a way that resembles speculative bubbles. Araujo at al (2011) explore the possibility of rational price bubbles when asset trades are collateralized by durable goods. Loewenstein and Willard (2000) extend the theory of rational bubbles to continuous-time asset market models. Miao and Wang (2012, 2014) develop a theory of price bubbles on productive assets in markets with credit constraints.

2. Dynamic Asset Markets with Debt Constraints

Time is discrete with infinite horizon and indexed by $t = 0, 1 \ldots$. Uncertainty is described by a set $\mathcal{S}$ of states of the world and an increasing sequence of finite partitions $\{\mathcal{F}_t\}_{t=0}^{\infty}$ of $\mathcal{S}$. The partition $\mathcal{F}_t$ specifies sets of states that can be verified by the information available at date t. An element $s_t \in \mathcal{F}_t$ is called a date-t event. The subset relation $s_\tau \subset s_t$ for $\tau \geq t$ indicates that event s_τ is a successor of s_t We use S^t to denote the set of all successors of event s_t from t to infinity, and S^{t+} the set of all successors of s_t excluding s_t. The set of one-period (date-$(t+1)$) successors of s_t is denoted by s_t^{+}. The unique one-period predecessor of s_t is denoted by s_t^{-}.

There is a single consumption good. A consumption plan is a scalar-valued process $c = \{c(s_t)\}_{s_t \in \mathcal{E}}$ adapted to $\{\mathcal{F}_t\}_{t=0}^{\infty}$. The consumption space is the space C of all adapted processes and it can be identified with $\mathcal{R}^{\infty}$. There are I consumers. Each consumer i has a consumption set C^i which is a convex subset of C_+, a

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[1]The term speculative bubble has only recently been introduced in Scheinkman and Xiong (2003).

strictly increasing utility function u^i on C^i, and an initial endowment $w^i \in C^i$. The aggregate endowment $\bar{w} \equiv \sum_i w^i$ is assumed positive, i.e., $\bar{w} \geq 0$.

Asset markets consist of J infinitely-lived assets traded at every date. The dividend process x_j of asset j is adapted to $\{\mathcal{F}_t\}_{t=0}^\infty$ and positive, i.e., $x(s_t) \geq 0$ for every s_t for $t \geq 1$. There are no dividends at date 0. The ex-dividend price of asset j in event s_t is denoted by $p_j(s_t)$. A portfolio of J assets held after trade at s_t is $h(s_t)$. Each agent has an initial portfolio α_0^i at date 0. The supply of assets is $\bar{\alpha}_0 \equiv \sum_i \alpha_0^i$.

Agent i faces the following budget constraints when trading in asset markets

$$c(s_0) + p(s_0)h(s_0) \leq w^i(s_0) + p(s_0)\alpha_0^i, \tag{1}$$

$$c(s_t) + p(s_t)h(s_t) \leq w^i(s_t) + [p(s_t) + x(s_t)]h(s_t^-) \quad \forall s_t \neq s_0. \tag{2}$$

In addition to these budget constraints some restriction on portfolio holdings has to be imposed for otherwise agents could engage in a Ponzi scheme, that is, borrow any amount of wealth at any date and roll-over the debt forever. In this paper we focus on debt constraints which impose limits on debt carried on a portfolio strategy at every date and in every event. Formally, the *debt constraints* on portfolio strategy h are

$$[p(s_{t+1}) + x(s_{t+1})]h(s_t) \geq -D(s_{t+1}), \quad \forall s_{t+1} \subset s_t \tag{3}$$

for every s_t. Bounds D are assumed to be positive in every event.[2] They may depend on prices and be different for different agents. Debt constraints should be distinguished from *borrowing constraints* seen in Santos and Woodford (1997) which are

$$p(s_t)h(s_t) \geq -B(s_t) \tag{4}$$

for some positive bounds B.

The set of consumption plans c satisfying budget constraints (1 - 2) and debt constraints (3) is denoted by $B_0^i(p, D^i, p_0\alpha_0^i)$. We included date-0 financial wealth $p_0\alpha_0^i$ among the determinants of the budget set for the use later when variations of financial wealth will be considered.

An *equilibrium under debt constraints* is a price process p and consumption-portfolio allocation $\{c^i, h^i\}_{i=1}^I$ such that (i) for each i, consumption plan c^i and

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[2]Debt constraints with negative bounds force agents to hold minimum savings. We exclude this from consideration.

portfolio strategy h^i maximize $u^i(c)$ over $(c, h) \in B_0^i(p, D^i, p_0 \alpha_0^i)$, and (ii) markets clear, that is

$$\sum_i h^i(s_t) = \bar{\alpha}_0, \qquad \sum_i c^i(s_t) = \bar{w}(s_t) + x(s_t)\bar{\alpha}_0,$$

for all s_t. We restrict our attention throughout to equilibria with positive prices.

3. Arbitrage, Event Prices, and Bubbles.

Portfolio $\hat{h}(s_t)$ is a *one-period arbitrage* in event s_t if

$$[p(s_{t+1}) + x(s_{t+1})]\hat{h}(s_t) \geq 0, \qquad \forall s_{t+1} \subset s_t \tag{5}$$

and

$$p(s_t)\hat{h}(s_t) \leq 0, \tag{6}$$

with at least one strict inequality. One-period arbitrage cannot exist in an equilibrium under debt constraints in any event. The reason is that one-period arbitrage portfolio could be added to an agent's equilibrium portfolio without violating debt constraints. This would result in higher consumption contradicting optimality of the equilibrium portfolio. It follows from Stiemke's Lemma that there is no one-period arbitrage if and only if there exist strictly positive numbers $q(s_t)$ for all s_t such that

$$q(s_t)p_j(s_t) = \sum_{s_{t+1} \subset s_t} q(s_{t+1})[p_j(s_{t+1}) + x_j(s_{t+1})] \tag{7}$$

for every s_t and every j. Strictly positive numbers $q(s_t)$ satisfying (7) and normalized so that $q(s_0) = 1$ are called *event prices*.

If asset prices admit event prices $q >> 0^3$ satisfying (7), then the present value of an asset and the bubble can be defined using any of those event prices. The *present value* of asset j in s_t under event prices q is

$$\frac{1}{q(s_t)} \sum_{\tau=t+1}^{\infty} \sum_{s_\tau \in s_t} q(s_\tau)x_j(s_\tau) \tag{8}$$

This intuitive definition of present value can be given more solid foundations from the view point of the theory of valuation of contingent claims, see Huang (2002).

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[3]The notation $q >> 0$ means that $q(s_t) > 0$ for all s_t.

Price bubble is the difference between the price and the present value of an asset. *Price bubble* on asset j at s_t is

$$\sigma_j(s_t) \equiv p_j(s_t) - \frac{1}{q(s_t)} \sum_{\tau=t+1}^{\infty} \sum_{s_\tau \in s_t} q(s_\tau) x_j(s_\tau) \tag{9}$$

If there exist multiple event prices, the price bubble may depend on the choice of event prices (see Huang (2002). Our notation does not reflect that possibility, but we shall keep this mind. Basic properties of price bubbles are stated in the following proposition, the proof of which is standard and therefore omitted.

Proposition 1: *Suppose that p admits strictly positive event prices q. Then*

(i) *Price bubbles are non-negative,*

$$0 \leq \sigma_j(s_t), \quad \forall s_t \; \forall j. \tag{10}$$

(ii) *If asset j is of finite maturity (that is, $x_{jt} = 0$ for $t \geq \tau$ for some τ, and that asset is not traded after date τ), then $\sigma_j(s_t) = 0$ for all s_t.*

(iii) *It holds*

$$q(s_t)\sigma_j(s_t) = \sum_{s_{t+1} \subset s_t} q(s_{t+1})\sigma_j(s_{t+1}) \tag{11}$$

for every s_t and every j.

Property (11) is referred to as the *discounted martingale property* of bubble σ_j with respect to event prices q. The reason for this terminology is that if a risk-free payoff lies in the one-period asset span for every event s_t, then the discount factor $\rho(s_t)$ can be defined as the product of one-period risk-free returns along the path of events from s_0 to s_t so that event prices rescaled by the discount factor $q(s_t)/\rho(s_t)$ become probabilities.[4] Property (11) says that discounted bubble $\rho(s_t)\sigma_j(s_t)$ is a martingale with respect to these probabilities.

The discounted martingale property of price bubbles together with their non-negativity have strong implications on the dynamics of bubbles. It follows that price bubble can be zero in an arbitrary event if and only if it is zero in every

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[4]For details, see LeRoy and Werner (2014, Chapter 26).

immediate successor of that event. Thus, non-zero price bubble can exist on an asset only if date-0 bubble is non-zero. Further, if price bubble is non-zero at date 0 (the issuance date of the asset), then there must exist a sequence of events throughout the event tree such that the bubble is strictly positive in every event of that sequence.

4. No-Bubble Theorem

In this section we establish sufficient conditions for non-existence of price bubbles in equilibrium and discuss their necessity.

We first introduce a restriction on agents' utility functions. For a consumption plan c and an event s_t, let $c(S^{t+})$ denote the consumption plan for all events in S^{t+}, that is, for events that are successors of event s_t. Let $c(-S^t)$ denote the consumption plan for all events not lying in S^t, that is, for events that are not successors of an event s_t, excluding s_t. It holds $c = (c(-S^t), c(s_t), c(S^{t+}))$. Further, let $\hat{w}(s_t) \equiv \bar{w}(s_t) + \bar{\alpha}_0 x(s_t)$.

We say that utility function u^i of agent i exhibits *uniform impatience* with respect to $\hat{w}$ if there exists γ^i satisfying $0 \leq \gamma^i < 1$ such that

$$u^i(c^i(-S^t), c^i(s_t) + \hat{w}(s_t), \gamma^i c^i(S^{t+})) > u^i(c^i), \tag{12}$$

for every s_t and every c^i such that $c^i \leq \hat{w}$.

Condition (12) concerns the trade-off in terms of utility between current consumption and consumption over the infinite future. It says that adding the aggregate (cum dividend) endowment to an agent's consumption in event s_t and scaling down her future consumption by scale-factor γ^i leaves the agent strictly better off. Restrictiveness of condition (12) lies in the requirement that factor γ^i is uniform over all feasible consumption plans and all events. Assumption of uniform impatience is usually imposed in existence theorems of an equilibrium in asset markets with infinite time, see Levine and Zame (1996) and Magill and Quinzii (1994). Any discounted time-separable expected utility with strictly increasing and continuous period-utility function exhibits uniform impatience with respect to the aggregate endowment $\hat{w}$ as long as $\hat{w}$ is bounded above and bounded away from zero (see LeRoy and Werner (2014, Chapter 30) for a proof).[5]

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[5]Other examples of uniform impatient utility functions are discussed in Santos and Woodford

Theorem 1: *Assume that agents' utility functions exhibit uniform impatience with respect to the aggregate endowment $\hat{w}$. Let p be equilibrium asset prices under debt constraints and q be strictly positive event prices associated with p. If present value of the aggregate endowment is finite,*

$$\sum_{t=1}^{\infty} \sum_{s_t \in F_t} q(s_t)\bar{w}(s_t) < \infty, \tag{13}$$

and assets are in strictly positive supply,

$$\bar{\alpha}_0 >> 0, \tag{14}$$

then price bubbles are zero.

Theorem 1 is an extension of the main result of Santos and Woodford (1997) from borrowing to debt constraints. A proof of Theorem 1 can be found in Appendix B (see also LeRoy and Werner (2014, Chapter 30). The assumption of uniform impatience can be dispensed with if markets are complete[6] and debt limits are the "natural" bounds equal to the present value of current and future initial endowments (see Remark 1 with proof in Appendix B).

If either one of conditions (13) or (14) is violated, there may exist price bubbles. We show in Section 6 that there always exists equilibria with price bubbles on assets in zero supply. Here, we present an example of an asset price bubble in an equilibrium with infinite present value of the aggregate endowment. The condition of infinite present value of the aggregate endowment is often referred to as *low interest rates*. The example is due to Bewley (1980) (see also Kocherlakota (1992) and Huang and Werner (2000)) and it shows an equilibrium with price bubble on a zero-dividend asset, that is, fiat money with strictly positive price.

Example 1: There is no uncertainty. There are two agents with utility functions

$$u^i(c) = \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

(1997). Yet, uniform impatience is not an innocuous restriction on utility functions. An example of utility function that does not satisfy (12) in the setting with no uncertainty is $\inf_t c_t + \sum_{t=0}^{\infty} \beta^t c_t$. See Werner (1997) and Araujo, Novinski and Pascoa (2011) for discussions of the possibility of price bubbles for utility functions that may not satisfy uniform impatience.

[6]That is, the one-period payoff matrix $[p_j(s_t^+) + x_j(s_t^+)]_{j \in J}$, where p are equilibrium prices, has rank equal to the number of one-period successors of s_t, for every event s_t.

where $0 < \beta < 1$. Their endowments are $w_t^1 = B$ and $w_t^2 = A$ for even dates $t \geq 2$, and $w_t^1 = A$ and $w_t^2 = B$ for odd dates $t \geq 1$, where $A > B$. Date-0 endowments will be specified later.

There is one asset that pays zero dividend at every date, that is, fiat money. Initial asset holdings are $\alpha_0^1 = 1$ and $\alpha_0^2 = 0$ so that the total supply is 1. Debt bounds are $D_t = p_t$ so that agents can short sell at most one share of the asset. It will be seen in Section 5 that these debt limits are self enforcing.

There exists a stationary equilibrium with consumption plans that depend only on current endowment, strictly positive prices p_t, and debt constraint binding the agent with low endowment at every date t. Such equilibrium has consumption plans $c_t^1 = B + \eta$ and $w_t^2 = A - \eta$ for even dates $t \geq 0$, and $c_t^1 = A - \eta$ and $w_t^2 = B + \eta$ for odd dates $t \geq 1$, asset holdings $h_t^1 = -1$ and $h_t^2 = 2$ for even dates and $h_t^1 = 2$ and $h_t^2 = -1$ for odd dates, and constant prices

$$p_t = \frac{1}{3}\eta. \tag{15}$$

The first-order condition for the unconstrained agent,

$$\frac{\beta^t}{c_t^1}p_t = \frac{\beta^{t+1}}{c_{t+1}^1}p_{t+1}, \tag{16}$$

holds provided that $\eta = \frac{\beta A - B}{(1+\beta)}$ and $\beta A > B$. For the constrained agent, the first-order condition requires that the left-hand side in (16) is greater than the right-hand side, and it holds. Transversality condition (see (41) in Appendix A) holds, too. If date-0 endowments are $w_0^1 = B + \frac{1}{3}\eta$ and $w_0^2 = A - \frac{1}{3}\eta$, then this is an equilibrium.

Event prices associated with equilibrium prices (15) are $q_t = 1$ for every t. The present value of the aggregate endowment $\sum_{t=0}^{\infty} q_t \bar{w}_t$ is infinite. $\square$

A sufficient condition for finite present value of the aggregate endowment is that there exist portfolio $\theta \in \Re_+^J$ and date T such that

$$\bar{w}(s_t) \leq \theta x(s_t), \tag{17}$$

for every $s_t \in F_t$ and every $t \geq T$. This follows from the fact that present value of dividend stream x_t^j is finite for every asset j, see Proposition 1. Another sufficient condition - for standard utility functions - is that the equilibrium allocation

be Pareto optimal and interior. Needless to say, neither one of those sufficient conditions holds in Example 1.

5. Bubbles under Endogenous Debt Constraints.

An important class of debt constraints that may lead to equilibria with infinite present value of the aggregate endowment and price bubbles on assets in strictly positive supply are endogenous debt constraints.

Endogenous debt constraints are induced by limited commitment to market transactions. We have assumed thus far that agents are fully committed to repay any debt incurred by their portfolio decisions. We shall relax this assumption now. Agents may consider defaulting on the payoff of a portfolio at any date. Whether an agent would want to default or not depends on gains and losses that such action would present to him. Once those gains and losses of default are precisely specified, endogenous (or self-enforcing) debt constraints can be defined as a sequence of debt bounds such that the agent is unwilling to default even if his indebtedness is at the maximum allowed level.

We assume throughout this section that each agent's utility function induces *continuation utility* for every event s_t. That is, for every s_t there exists a utility function $u^i_{s_t}$ on consumption plans for event s_t and all its successor events, that is, for events in S^t, such that

$$u^i_{s_t}(c(S^t)) \leq u^i_{s_t}(c'(S^t)) \quad \text{iff} \quad u^i(\bar{c}(-S^t), c(S^t)) \leq u^i(\bar{c}(-S^t), c'(S^t)), \qquad (18)$$

for every $c, c', \bar{c}$. To simplify notation, we write $u^i_{s_t}(c)$ instead of $u^i_{s_t}(c(S^t))$.

Condition (18) requires that agent's preferences be independent of unrealized events and past consumption. It holds for time-separable expected utility and for recursive utility of Epstein and Zin (1989).

We proceed now to a formal definition of endogenous debt constraints. Let $B^i_{s_t}(p, D^i, \Phi^i(s_t))$ denote the budget set in event s_t at date t under debt constraints with bounds D^i when initial financial wealth (or debt) at s_t is $\Phi^i(s_t)$. Specifically, this set consists of all consumption plans and portfolio holdings for events in S^t satisfying budget constraints (2) and debt constraints (3) for $s_\tau \in S^t$. Further, let $U^{*i}_{s_t}(p, D^i, \Phi^i(s_t))$ be the maximum event-s_t continuation utility over all consumption plans in the budget set $B^i_{s_t}(p, D^i, \Phi^i(s_t))$.

Gains and losses of default are described by a sequence of reservation utility levels that agent i can obtain if she defaults. We denote this sequence by $\bar{V}_d^i = \{\bar{V}_d^i(s_t)\}$ and call it default utilities. We focus on two specifications of default utilities that have been proposed in recent literature. They are

$$\bar{V}_d^i(s_t) = u_{s_t}^i(w^i), \tag{19}$$

and

$$\bar{V}_d^i(s_t) = U_{s_t}^{*i}(p, 0, 0). \tag{20}$$

Under the first specification (19), default results in permanent exclusion from the markets so that the agent is forced to consume her endowment from s_t on (see Alvarez and Jermann (2000)). Under the second specification (20), default results in prohibition from taking any debt from s_t on, but the agent continues to participate in the markets under zero-debt constraints (see Hellwig and Lorenzoni (2009)). Note that default utilities (20) depend on asset prices p.

Debt bounds D^i are *self enforcing* for agent i at p given default utilities $\bar{V}_d^i$ if

$$U_{s_t}^{*i}(p, D^i, -D^i(s_t)) \geq \bar{V}_d^i(s_t) \tag{21}$$

for every s_t. Debt bounds D^i are *not too tight* for agent i at p given default option $\bar{V}_d^i$ if (21) holds with equality. Equilibrium with not-too-tight debt constraints is an equilibrium with any debt bounds D^i such that D^i are not too tight at equilibrium price p for every i.

The property of being not too tight does not determine debt constraints in a unique way. This is explained as follows. We say that a real-valued process $\{\kappa_t\}$ *lies in the asset span* if

$$\kappa(s_t^+) \in span\{p_j(s_t^+) + x_j(s_t^+) : j \in J\} \tag{22}$$

for every s_t. Process $\{\kappa_t\}$ is a *discounted martingale*, if there exists a strictly positive event price process q such that the discounted martingale property (11) holds for $\{\kappa_t\}$ with respect to q.

The relation $\simeq_c$ between any two sets of consumption-portfolio plans indicates that consumption plans in those sets are the same. We have

Lemma 1: *If $\{\kappa_t\}$ is a discounted martingale and lies in the asset span, then*

$$B^i_{s_t}(p, D^i, \Phi^i(s_t)) \simeq_c B^i_{s_t}(p, D^i + \kappa, \Phi^i(s_t) - \kappa(s_t)) \qquad (23)$$

for every s_t.

PROOF: We prove (23) for s_0. Let $(c, h) \in B^i_0(p, D^i, \Phi^i(s_0))$. Since $\{\kappa_t\}$ lies in the asset span, there is a portfolio strategy $\hat{h}$ such that $[p(s_{t+1}) + x(s_{t+1})]\hat{h}(s_t) = \kappa(s_{t+1})$ for every $s_{t+1} \subset s_t$ and every s_t. Since κ is a discounted martingale, it follows that $p(s_t)\hat{h}(s_t) = \kappa(s_t)$. It is easy to see now that $(c, h + \hat{h}) \in B^i_0(p, D^i + \kappa, \Phi^i(s_0) - \kappa(s_0))$. The same argument shows that if $(c, \tilde{h}) \in B^i_0(p, D^i + \kappa, \Phi^i(s_0) - \kappa(s_0))$, then $(c, \tilde{h} - \hat{h}) \in B^i_0(p, D^i, \Phi^i(s_0))$. This concludes the proof. $\square$

Lemma 1 implies that if debt bounds D^i are not too tight, then $D^i + \kappa$ are not too tight as well, for any discounted martingale κ that lies in the asset span. Indeed, it follows that $U^{*i}_{s_t}(p, D^i, -D^i(s_t)) = U^{*i}_{s_t}(p, D^i + \kappa, -(D^i(s_t) + \kappa(s_t)))$.

For default utilities (20), zero bounds $D^i \equiv 0$ are not too tight. This implies that $D^i = \kappa$ is not too tight for every discounted martingale that lies in the asset span. Hellwig and Lorenzoni (2009, Theorem 1) and Bidian and Bejan (2012) show that the converse holds, too, if markets are complete: If debt bounds D are not too tight for default utilities (20), then D is a discounted martingale. In Example 1, event prices are equal to one and debt bounds are constant, which implies that they are discounted martingale. Consequently, this equilibrium has not-too-tight debt constraints.

Next, we present a method of injecting price bubbles on infinitely-lived assets. Let $\{\epsilon_t\}$ by a $\Re^J$-valued process. We say that $\{\epsilon_t\}$ is *asset-span preserving* if

$$span\{p_j(s_t^+) + x_j(s_t^+) : j \in J\} = span\{p_j(s_t^+) + \epsilon_j(s_t^+) + x_j(s_t^+) : j \in J\} \qquad (24)$$

for every s_t. The property of asset-span preservation (24) and the spanning condition (22) are closely related. If ϵ_t is asset-span preserving, then ϵ^j_t lies in the asset span at p for every j. The converse holds for almost every ϵ_t. We elaborate on this in Appendix C, see also Bejan and Bidian (2013). It is important to note that the set of asset-span preserving processes is always non-empty. If markets are complete at p, then almost every $\{\epsilon_t\}$ is asset-span preserving.

The $\Re^J$-valued process $\{\epsilon_t\}$ is a discounted martingale, if $\{\epsilon^j_t\}$ is a discounted martingale for every j. We have

Lemma 2: *If $\{\epsilon_t\}$ is a positive asset-span preserving discounted martingale, then*

$$B_0^i(p, D^i, \Phi^i(s_0)) \simeq_c B_0^i(p + \epsilon, D^i, \Phi^i(s_0)) \tag{25}$$

for every $\Phi^i(s_0)$.

PROOF: Let $(c, h) \in B_0^i(p, D^i, \Phi^i(s_0))$. Since $\{\epsilon_t\}$ is asset-span preserving, there exists portfolio strategy $\hat{h}$ such that

$$[p(s_{t+1}) + x(s_{t+1})]h(s_t) = [p(s_{t+1}) + \epsilon(s_{t+1}) + x(s_{t+1})]\hat{h}(s_t) \tag{26}$$

for every $s_{t+1} \subset s_t$ and every s_t. Further, since $\{\epsilon_t\}$ is a discounted martingale, it follows

$$p(s_t)h(s_t) = [p(s_t) + \epsilon(s_t)]\hat{h}(s_t) \tag{27}$$

It is easy to see now that $(c, \hat{h}) \in B_0^i(p + \epsilon, D^i, \Phi^i(s_0))$. The same argument shows that if $(c, \hat{h}) \in B_0^i(p + \epsilon, D^i, \Phi^i(s_0))$, then $(c, h) \in B_0^i(p, D^i, \Phi^i(s_0))$. This concludes the proof. $\square$

It follows from Lemmas 1 and 2 that if a $\Re^J$-valued process $\{\epsilon_t\}$ is an asset-span preserving discounted martingale, then

$$B_0^i(p, D^i, p_0\alpha_0^i) \simeq_c B_0^i(p + \epsilon, D^i - \alpha_0^i\epsilon, [p_0 + \epsilon_0]\alpha_0^i) \tag{28}$$

Observation (28) leads to the following

Theorem 2: *Let p and $\{c^i\}$ be an equilibrium with not-too-tight debt constraints D^i for default utilities (19) or (20). For every positive asset-span preserving discounted martingale $\{\epsilon_t\}$ such that $\alpha_0^i\epsilon_t \leq D_t^i$, price process $p+\epsilon$ and consumption allocation $\{c^i\}$ are an equilibrium with not-too-tight debt constraints, too.*

PROOF: It follows from (28) that $p + \epsilon$ and $\{c^i\}$ are an equilibrium under debt constraints with positive bounds $D^i - \alpha_0^i\epsilon$. Debt bounds $D^i - \alpha_0^i\epsilon$ are not too tight by Lemma 1. Further, default utilities (19) and (20) remain the same at prices $p + \epsilon$. For (20) this is so because $U_{s_t}^{*i}(p, 0, 0) = U_{s_t}^{*i}(p + \epsilon, 0, 0)$ by Lemma 2. $\square$

The discounted martingale $\{\epsilon_t\}$ is a price bubble in the equilibrium of Theorem 2. If assets are in strictly positive supply and utility functions are uniformly impatient (12), then the present value of the aggregate endowment must be infinite for otherwise we would have a contradiction to Theorem 1. Condition $\alpha_0^i\epsilon_t \leq D_t^i$

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guarantees that the adjusted debt bounds $D^i - \alpha_0^i \epsilon$ are positive. Bejan and Bidian (2013) and - for complete markets - Kocherlakota (2008) proved results similar to Theorem 2 without the condition $\alpha_0^i \epsilon \leq D_t^i$ but allowing for negative debt bounds. A sufficient condition guaranteeing that there exists ϵ_t such that $\alpha_0^i \epsilon_t \leq D_t^i$ for every i is that the discounted value of debt bounds, i.e., $\rho_t D_t^i$ is bounded away from zero for every agent i whose initial portfolio α_0^i is non-zero, and the risk-free payoff lies in the one-period asset span for every s_t. Bidian and Bejan (2012) provide further sufficient conditions.

Hellwig and Lorenzoni (2009, Example 1) present an example of an equilibrium with not-too-tight debt constraints for default utilities (20) such that debt limits are bounded away from zero and present value of the aggregate endowment is infinite. Theorem 2 implies that there are equilibria with price bubbles in that example. We demonstrate such equilibria in Example 2.

Example 2: Uncertainty is described by a binomial event-tree with the two successor events of every s_t indicated by *up* and *down*. The (Markov) transition probabilities are

$$\text{Prob}(up|s_t) = 1 - \alpha, \quad \text{Prob}(down|s_t) = \alpha, \tag{29}$$

whenever $s_t = (s_{t-1}, up)$, and

$$\text{Prob}(up|s_t) = \alpha, \quad \text{Prob}(down|s_t) = 1 - \alpha, \tag{30}$$

for $s_t = (s_{t-1}, down)$, where $0 < \alpha < 1$. Initial event is $s_0 = up$.

There are two consumers with the same expected utility functions with logarithmic period-utility and discount factor $0 < \beta < 1$. Endowments depend only on the current state and are given by $w^1(up) = A$, $w^2(up) = B$, and $w^1(down) = B$, $w^2(down) = A$. It is assumed that $A > B > 0$. Note that there is no aggregate risk. Traded assets are one-period Arrow securities at every date-event and an infinitely-lived asset with zero dividends (fiat money). Arrow securities are in zero supply. Fiat money is in strictly positive supply with each agent holding one share at date 0.

We first find an equilibrium with zero price of the fiat money. There exists a stationary Markov equilibrium such that, at every event, the debt constraint is binding for the agent who receives high endowment. The equilibrium is as follows:

Prices of Arrow securities are

$$p_c(s_t) = 1 - \beta(1-\alpha), \quad p_{nc}(s_t) = \beta(1-\alpha), \tag{31}$$

where subscript c stands for "change" of state (for example, from up to down) and nc for "no change." Consumption allocation is

$$c^1(s_t) = \bar{c}, \quad c^2(s_t) = \underline{c} \tag{32}$$

whenever $s_t = (s_{t-1}, up)$, and

$$c^1(s_t) = \underline{c}, \quad c^2(s_t) = \bar{c} \tag{33}$$

whenever $s_t = (s_{t-1}, down)$. Consumption plans $\bar{c}$ and $\underline{c}$ are such that $\bar{c} + \underline{c} = A + B$ and

$$1 - \beta(1-\alpha) = \beta\alpha\frac{\bar{c}}{\underline{c}}. \tag{34}$$

The solution must satisfy $\bar{c} < A$, which is a restriction on parameters α and β. Debt bounds are

$$D(s_t) = \bar{D} = \frac{A - \bar{c}}{2[1 - \beta(1-\alpha)]}, \tag{35}$$

constant over time and across events.

Equilibrium holdings of Arrow securities are such that agent 1 always holds the maximum possible short position $-\bar{D}$ in the Arrow security that pays in the up-event (where she gets high endowment A) and long position $\bar{D}$ in the Arrow security that pays in the down-event. The opposite holds for agent 2. Transversality condition (see (41) in Appendix A) holds. Date-0 endowments must be $w^1(s_0) = A - \bar{D}$ and $w^2(s_0) = B + \bar{D}$.

Event prices are products of Arrow securities prices (31). They have the property that, at any date, the sum of event prices for all date-t events equals one. In other words, there is no discounting (i.e., $\rho(s_t) = 1$). Consequently, the present value of the aggregate endowment is infinite. Further, constant debt bounds $\bar{D}$ are a martingale, and therefore they are not too tight.

Since debt bounds (35) are strictly positive, we can take any positive martingale $\{\epsilon_t\}$ such that $\epsilon_t \leq \bar{D}$ and "inject" it as a bubble on fiat money as described in Theorem 2. For simplicity, we take a deterministic process $\epsilon_t = \bar{\epsilon}$ for $\bar{\epsilon} \leq \bar{D}$. Price

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process for fiat money given by $p_t = \bar{\epsilon}$ together with consumption allocation $\{c^i\}$ given by (34) and debt bounds $\bar{D} - \bar{\epsilon}$ for each agent constitute an equilibrium with non-too-tight debt constraints.

6. Bubbles on Assets in Zero Supply

Theorem 2 implies that if assets are in zero supply, then there exist equilibria with not-too-tight debt constraints with price bubbles on infinitely-lived assets. This result extends to equilibria under debt constraints with arbitrary bounds.

Theorem 3: *Suppose that all infinitely-lived assets are in zero supply. If p and $\{c^i\}$ are an equilibrium under debt constraints, then $p + \epsilon$ and $\{c^i\}$ are an equilibrium, too, for every positive asset-span preserving discounted martingale ϵ.*
PROOF: Lemma 2 implies that

$$B_0^i(p, D^i, 0) \simeq_c B_0^i(p + \epsilon, D^i, 0)$$

Since $p_0 \alpha_0^i = [p_0 + \epsilon_0]\alpha_0^i = 0$, it follows that $p + \epsilon$ and $\{c^i\}$ are an equilibrium. $\square$

Related results have been previously established in Magill and Quinzii (1996) for the transversality constraint on portfolio holdings and in Huang and Werner (2004) for the natural borrowing constraint and complete markets.

Equilibria with price bubbles in Theorem 3 have the same asset span (and the same consumption allocation) as their no-bubble counterparts. There may exist equilibria with price bubbles on assets in zero supply such that the injection of price bubble changes the asset span and leads to different consumption allocation. We present an example which is a variation of Example 1.

Example 3: Consider the economy of Example 1. The Arrow-Debreu equilibrium in this economy has time-independent consumption plans $c_t^i = \bar{c}^i$ for $i = 1, 2$ and event prices equal to the discount factor, that is,

$$q_t = \beta^t, \tag{36}$$

for every t. Consumption plans $\bar{c}^i$ are given by

$$\bar{c}^i = \sum_{t=0}^{\infty} \beta^t w_t^i, \tag{37}$$

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that is $\bar{c}^1 = \frac{1}{1-\beta^2}[A\beta + B] + \eta$ and $\bar{c}^2 = \frac{1}{1-\beta^2}[A + B\beta] - \eta$.

The Arrow-Debreu equilibrium allocation can be implemented by trading fiat money in zero supply under the natural debt constraints. The natural debt constraints have bounds equal to the present value of current and future agent's endowments, that is, $D_t^i = \frac{1}{q_t}\sum_{\tau=t}^{\infty} q_\tau w_\tau^i$. The price of fiat money equals bubble σ_t that must satisfy the discounted-martingale property (11), that is

$$q_t\sigma_t = q_{t+1}\sigma_{t+1}. \tag{38}$$

We take $\sigma_t = \beta^{-t}$ so that the price of fiat money is

$$p_t = \beta^{-t}. \tag{39}$$

The present value of agent 1 future endowments at event prices (36) is $\frac{1}{1-\beta^2}[A\beta + B]$ for even dates, and $\frac{1}{1-\beta^2}[A + B\beta]$ for odd dates. The reverse holds for agent 2. Debt bounds are equal to those present values.

7. Concluding Remarks

The standard theorem establishing sufficient conditions for non-existence of price bubbles in equilibrium under borrowing constraints is extended to debt constraints including endogenous debt constraints. The no-bubble theorem leaves two possibilities (beside violations of uniform impatience) for price bubbles to arise in equilibrium: infinite present value of aggregate resources, and zero supply of assets. We argued that equilibria with endogenous debt constraints are prone to generate infinite present value of aggregate endowments. Two examples were given to illustrate that assertion. Further, we showed that there always exist equilibria with price bubbles on assets in zero supply.

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Appendix

A. First-Order and Transversality Conditions.

Assuming that the utility function u^i is differentiable, the necessary first-order conditions for an interior solution to the consumption-portfolio choice problem under debt constraints are

$$p(s_t) = \sum_{s_{t+1} \subset s_t} [p(s_{t+1}) + x(s_{t+1})] \Big[\frac{\partial_{s_{t+1}} u^i}{\partial_{s_t} u^i} + \frac{\gamma^i (s_{t+1})}{\partial_{s_t} u^i} \Big]. \tag{40}$$

for all s_t, where $\gamma^i(s_t) \geq 0$ is the Lagrange multiplier associated with debt constraint (3).

First-order conditions (40) together with transversality condition are sufficient to determine an optimal consumption-portfolio choice for concave utility function. For the discounted time-separable expected utility $\sum_{t=0}^{\infty} \beta^t E[v(c_t)]$ with concave period-utility v, the transversality condition for (c, h) is

$$\lim_{t \to \infty} \sum_{s_t \in F_t} \beta^t \pi(s_t) v'(c(s_t))[(p(s_t) + x(s_t))h(s_t^-) + D(s_t)] = 0. \tag{41}$$

B. Proof of Theorem 1. We show first that the value $p(s_t)h^i(s_t)$ of each agent's equilibrium portfolio cannot exceed fraction $1/(1-\gamma^i)$ of the aggregate endowment for every s_t. If it did, the agent could scale down her portfolio holdings and consumption at all successor events of s_t by factor γ^i and use the retained wealth to purchase the aggregate endowment in s_t. Because of uniform impatience, doing so would result is a strictly preferred consumption plan contradicting optimality of the equilibrium plan. Since the present value of the aggregate date-t endowment converges to zero as t goes to infinity, the discounted price of the aggregate portfolio converges to zero. This will be shown to imply that $\sigma_0 = 0$. The discounted martingale property (11) implies then that $\sigma(s_t) = 0$ for every s_t.

Let γ^i be be such that the uniform impatience condition (12) holds for agent i. We claim that

$$(1 - \gamma^i)p(s_t)h^i(s_t) \leq \hat{w}(s_t), \tag{42}$$

for every s_t, every i. To prove (42), suppose that there exists s_t such that

$$(1 - \gamma^i)p(s_t)h^i(s_t) > \hat{w}(s_t), \tag{43}$$

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for some i. Consider consumption plan

$$\tilde{c}^i = (c^i(-S^t), c^i(s_t) + (1 - \gamma^i)p(s_t)h^i(s_t), \gamma^i c^i(S^{t+})). \tag{44}$$

Consumption plan $\tilde{c}^i$ together with portfolio strategy $\tilde{h}^i = (h^i(-S^t), \gamma^i h^i(S^t))$ satisfy budget constraints (2). Further, because bounds D^i are positive, $\tilde{h}^i$ satisfies debt constraints. By assumption (12), $u^i(\tilde{c}^i) > u^i(c^i)$ which is a contradiction to optimality of c^i. This proves (42).

It follows from (42) that

$$(1 - \bar{\gamma})p(s_t)\bar{\alpha}_0 \leq I\hat{w}(s_t), \tag{45}$$

for every s_t, where $\bar{\gamma} = \max \gamma^i$. Multiplying both sides of (45) by $q(s_t)$ and summing over all date-t events, we obtain

$$\sum_{s_t \in F_t} q(s_t)p(s_t)\bar{\alpha}_0 \leq \frac{I}{(1 - \bar{\gamma})} \sum_{s_t \in F_t} q(s_t)\hat{w}(s_t). \tag{46}$$

Assumption (13) implies that $\sum_{t=1}^{\infty} \sum_{s_t \in F_t} q(s_t)\hat{w}(s_t) < \infty$ since the present value of each asset, and hence of the portfolio $\bar{\alpha}_0$, is finite. It follows that

$$\lim_{t \to \infty} \sum_{s_t \in F_t} q(s_t)\hat{w}(s_t) = 0. \tag{47}$$

Summing eq. (7) recursively from 0 to t and taking the limit as t goes to infinity we obtain

$$\sigma_0 = \lim_{t \to \infty} \sum_{s_t \in F_t} q(s_t)p(s_t). \tag{48}$$

Using (46), (47) and (48), we obtain

$$\sigma_0 \bar{\alpha}_0 \leq 0, \tag{49}$$

Since $\sigma_0 \geq 0$ and $\bar{\alpha}_0 >> 0$, it follows $\sigma_0 = 0$. $\square$

Remark 1: The assumption of uniform impatience in Theorem 1 can be dispensed with if markets are complete and debt limits are the "natural" bounds.
PROOF: The natural debt bounds are

$$\tilde{D}^i(s_t) = \frac{1}{q(s_t)} \sum_{\tau=t}^{\infty} \sum_{s_\tau \in s_t} q(s_\tau)w^i(s_\tau) \tag{50}$$

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for every s_t, $t \geq 1$. Following the arguments in Huang and Werner (2004) for the natural borrowing constraints, one can show that if asset markets are complete, then consumption plans in the budget set $B_0(p, \tilde{D}^i, p_0\alpha_0^i)$ with natural debt bounds $\tilde{D}^i$ are the same as the consumption plans satisfying a single budget constraint

$$\sum_{t=0}^{\infty} \sum_{s_t \in F_t} q(s_t)c(s_t) \leq \sum_{t=0}^{\infty} \sum_{s_t \in F_t} q(s_t)w^i(s_t) + p_0\alpha_0^i. \tag{51}$$

In equilibrium, constraint (51) holds with equality for every agent i. Summing those constraints over all agents and using market clearing, $\sum_i c^i = \sum_i w^i + \bar{\alpha}_0 x$, we obtain $\sigma_0 \bar{\alpha}_0 = 0$. As in the proof of Theorem 1, this implies $\sigma_0 = 0$ and hence $\sigma(s_t) = 0$ for every s_t. □

C. Asset-span preserving process $\{\epsilon_t\}$.

Here, we discuss the properties of asset-span preservation and lying in the asset span introduced in Section 5. Let $\{\epsilon_t\}$ be a $\Re^J$-valued process. First, it is easy to see that if $\{\epsilon_t\}$ is asset span preserving at p, then $\{\epsilon_t^j\}$ lies in the asset span at p for every j. The converse holds under a minor rank condition which we explain next.

Suppose that $\{\epsilon_t^j\}$ lies in the asset span at p. For any s_t, let $h^j(s_t)$ be a portfolio such that $\epsilon(s_t^+) = [p(s_t^+) + x(s_t^+)]h^j(s_t)$. It follows that $p(s_t^+) + \epsilon(s_t^+) + x(s_t^+) = [p(s_t^+) + x(s_t^+)](h^j(s_t) + I^j)$, where I^j denotes a portfolio consisting of one share of asset j. Let $H(s_t)$ be the $J \times J$ matrix with rows $h^j(s_t)$ for all j and I be the $J \times J$ identity matrix. If matrix $H(s_t) + I$ is invertible, then $\{\epsilon_t\}$ is asset span preserving at p.

Summing up, if $\{\epsilon_t^j\}$ lies in the asset span at p for every j. and $H(s_t) + I$ is invertible for every s_t, then $\{\epsilon_t\}$ is asset span preserving at p.

We conclude with construction of an asset-span preserving process: For each asset j, consider a strategy that starts with buying 1 share of asset j at date 0 and consists of rolling over the payoff at every event after date 0. This may by called a reverse Ponzi scheme on asset j. Formally it is a real-valued process γ_j defined by

$$p_j(s_t)\gamma_j(s_t) = [p_j(s_t) + x_j(s_t)]\gamma_j(s_t^-)$$

for every s_t, and $\gamma_j(s_0) = 1$. Note that γ_j is positive. Let ϵ_t be an $\Re^J$-valued

process defined by

$$\epsilon_j(s_t) = [p_j(s_t) + x_j(s_t)]\gamma_j(s_t^-).$$

Since

$$p(s_t) + \epsilon(s_t) + x(s_t) = [p(s_t) + x(s_t)](I + \gamma(s_t^-)),$$

it follows that ϵ_t is asset-span preserving. Further, it is a discounted martingale.

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