ECON 4261: Introduction to Econometrics
Fall 2009

Problem Set 1 – Answer Key

Exercise 1 Let A be an $(m \times n)$ matrix, with $m > n$. Show that the matrix $A'A$ is symmetric, and that if A has full rank, $A'A$ is positive definite.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$.

Then

$$\begin{aligned} A'A &= \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{mn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^m a_{j1}^2 & \sum_{j=1}^m a_{j1}a_{j2} & \dots & \sum_{j=1}^m a_{j1}a_{jn} \\ \sum_{j=1}^m a_{j2}a_{j1} & \sum_{j=1}^m a_{j2}^2 & \dots & \sum_{j=1}^m a_{j2}a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^m a_{jn}a_{j1} & \sum_{j=1}^m a_{jn}a_{j2} & \dots & \sum_{j=1}^m a_{jn}^2 \end{bmatrix} \end{aligned}$$

and note that the elements off the diagonal of this matrix are the same. Hence $A'A$ is symmetric.

To prove $A'A$ is positive definite, we must prove that $q = x'(A'A)x > 0$ for all $x \neq 0$. Suppose this does not happen, i.e. we have $x'(A'A)x \leq 0$ for some $x \neq 0$.

Note that $q = x'(A'A)x = (x'A')(Ax) = (Ax)'(Ax)$ and Ax is an $(m \times 1)$ vector. So $(Ax)'(Ax)$ is the inner product of the vector Ax with itself. Hence, it's a sum of squares. So it cannot be the case that $q < 0$. Hence the only possibility is to have $q = 0$. Since q is a sum of squares, the only possibility is that all the components of the sum are zero. This would imply that $Ax = 0$. But since we are assuming that A has full rank (and $m > n$), we must have $x = 0$. But this contradicts our initial hypothesis that $x \neq 0$. Hence we must have $q = x'(A'A)x > 0$ for all $x \neq 0$. So $A'A$ is positive definite.

Exercise 2 Characterize each of the following statements as true or false. If a statement is true, show the proof. If it is false, give a counterexample. Assume that A and B are square matrices.

(i) If A and B are non-singular, then AB is non-singular.

This statement is TRUE. Since A and B are non-singular, we have

$\det(A) \neq 0$ and $\det(B) \neq 0$.

We know that

$\det(AB) = \underbrace{\det(A)}_{\neq 0}\underbrace{\det(B)}_{\neq 0} \neq 0.$

Hence AB is non-singular.

(ii) $(AB)' = A'B'$.

This statement is FALSE. Consider the following matrices A and B:

$A = \begin{bmatrix} 2 & 9 \\ 5 & 14 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 8 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 2 & 9 \\ 5 & 14 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 15 & 80 \\ 19 & 132 \end{bmatrix} \Rightarrow (AB)' = \begin{bmatrix} 15 & 19 \\ 80 & 132 \end{bmatrix}$,
but

$A'B' = \begin{bmatrix} 2 & 5 \\ 9 & 14 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 26 & 42 \\ 83 & 121 \end{bmatrix}$

Hence $(AB)' \neq A'B'$.

(iii) If $AB + BA = 0$, then $A^2B^3 = B^3A^2$.

This statement is TRUE. Note that $AB + BA = 0 \Rightarrow AB = -BA \Rightarrow BA = -AB$.

Now,

$$\begin{aligned}
A^2B^3 &= A(AB)B^2 = A(-BA)B^2 \\
&= (-AB)AB^2 = BAAB^2 \\
&= BA(AB)B = BA(-BA)B \\
&= B(-AB)AB = B(BA)AB \\
&= B^2A(AB) = B^2A(-BA) \\
&= B^2(-AB)A) = B^2BAA \\
&= B^3A^2.
\end{aligned}$$

(iv) If $A^2 = 0$, then $A = 0$.

This statement is FALSE. Consider

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Then $A^2 = AA = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So $A^2 = 0$, but $A \neq 0$.

(v) If A and B are symmetric, then AB is symmetric.

This statement is FALSE. Consider the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that these matrices are symmetric (i.e. $A = A'$ and $B = B'$). Moreover, we have

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and hence } (AB)' = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

So AB is not symmetric.

Exercise 3 Let X and Y be two random variables. Consider the following joint density function

$$f(x,y) = \begin{cases} A(x^2+y^2) & \text{for } x \in [0,1] \text{ and } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(i) Find A.

Note that we must have $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dxdy = 1$.

Then

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= \int_0^1\int_1^1 A(x^2+y^2)dxdy \\
&= A\int_0^1\int_1^1 (x^2+y^2)dxdy \\
&= A\int_0^1 \left[\frac{x^3}{3}+xy^2\right]_0^1 dy \\
&= A\int_0^1 (\frac{1}{3}+y^2)dy \\
&= A\left[\frac{1}{3}+\frac{y^3}{3}\right]_0^1 \\
&= A(\frac{2}{3}) = 1 \\
&\Rightarrow A = \frac{3}{2}.
\end{aligned}
$$

(ii) Find the marginal density $f(x)$.

By definition,

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_0^1 \frac{3}{2}(x^2+y^2)dy \\
&= \frac{3}{2}\int_0^1 (x^2+y^2)dy \\
&= \frac{3}{2}\left[x^2y+\frac{y^3}{3}\right]_0^1 \\
&= \frac{3}{2}(x^2+\frac{1}{3}) \\
&= \frac{3}{2}x^2+\frac{1}{2} \text{ for } x\in[0,1] \text{ and 0 otherwise.}
\end{aligned}
$$

(iii) Find the conditional density $f(y \mid x)$.

By definition, $f(y \mid x) = \frac{f(x,y)}{f(x)}$.

Then $f(y \mid x) = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}x^2+\frac{1}{2}} = \frac{3(x^2+y^2)}{(3x^2+1)}$ for $x \in [0,1]$ and 0 otherwise.

(iv) Find $Cov(X,Y)$.

Note first that $f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 \frac{3}{2}(x^2+y^2)dx = \frac{3}{2}y^2+\frac{1}{2}$ for $y \in [0,1]$ and 0 otherwise.

Then

$$\begin{aligned}
Cov(X,Y) &= E(XY) - E(X)E(Y) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy - \int_{-\infty}^{\infty} xf(x)dx \int_{-\infty}^{\infty} yf(y)dy \\
&= \frac{3}{2}\int_0^1\int_0^1 xy(x^2+y^2)dxdy - \int_0^1 x(\frac{3}{2}x^2+\frac{1}{2})dx \int_0^1 y(\frac{3}{2}y^2+\frac{1}{2})dx \\
&= (\frac{3}{2})(\frac{1}{4}) - (\frac{5}{8})(\frac{5}{8}) \\
&= \frac{3}{8} - \frac{25}{64} \\
&= -\frac{1}{64}.
\end{aligned}$$

(v) Are X and Y independent?

No, because if X and Y were independent, then $Cov(X,Y)$ should be equal to zero, but this is not the case.

(vi) Find $P(\frac{1}{2} \leq x \leq 1, \frac{1}{2} \leq y \leq 1)$.

$$\begin{aligned}
P(\frac{1}{2} &\leq x \leq 1, \frac{1}{2} \leq y \leq 1) = \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 f(x,y)dxdy \\
&= \frac{3}{2}\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (x^2+y^2)dxdy \\
&= \frac{3}{2}\int_{\frac{1}{2}}^1 \left[\frac{y^2}{2} + \frac{7}{24}\right] dy \\
&= \frac{3}{2}\left[\frac{y^3}{6} + \frac{7}{24}y\right]_{\frac{1}{2}}^1 \\
&= \frac{7}{16}.
\end{aligned}$$

Exercise 4 Let

$$A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 2 \\ 0 & 7 & 3 \end{bmatrix}$$

(i) Find $A'A$. Show your work.

$$A'A = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 7 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 2 \\ 0 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 10 & 17 \\ 10 & 69 & 39 \\ 17 & 39 & 38 \end{bmatrix}$$

(ii) Find A^{-1}. Show your work.

First, note $\det(A) = 23$.

Then $A^{-1} = \frac{1}{\det(A)} adj(A) = \begin{bmatrix} -0.0869 & 1.2608 & -0.6956 \\ -0.1304 & 0.3913 & -0.0434 \\ 0.3043 & -0.9130 & 0.4347 \end{bmatrix}$

(iii) Calculate the *trace* of A.

Trace$(A) = 3 + 4 + 3 = 10$.

Exercise 5 Let X be a random variable with the following density function

x	$P(X = x)$
2	0.08
5	0.12
7	0.22
9	0.03
14	0.28
21	0.17
25	0.10

(i) Calculate $E(X)$. Show your work.

$$\begin{aligned} E(X) &= 2(0.08) + 5(0.12) + 7(0.22) + 9(0.03) + \\ &\quad +14(0.28) + 21(0.17) + 25(0.1) \\ &= 12.56. \end{aligned}$$

(ii) Calculate $Var(X)$.Show your work.

First, note that

$$\begin{aligned} E(X^2) &= 4(0.08) + 25(0.12) + 49(0.22) + 81(0.03) + \\ &\quad +196(0.28) + 441(0.17) + 625(0.1) \\ &= 208.88. \end{aligned}$$

We know

$$\begin{aligned} Var(X) &= E(X^2) - (E(X))^2 \\ &= 208.88 - (12.56)^2 \\ &= 51.1264. \end{aligned}$$

(iii) Calculate the Standard Deviation of X. Show your work.

$Std.\ Dev(X) = (Var(X))^{\frac{1}{2}} = 7.1502.$

Exercise 6 Let X, and Y be two random variables such that $X \sim U[0,1]$, and $Y = 3X + 2$. Calculate the values for: $(E[X])^2$, $E[X^2]$, $E[Y]$, $Var[X]$, $Var[Y]$, $Cov[X,Y]$, and $Var[(2X+Y)^2]$. Show your work.

$$\begin{aligned} E(X) &= \int_0^1 xf(x)dx \\ &= \int_0^1 xdx \\ &= \frac{1}{2}, \text{ since } f(x) = 1 \text{ because } X \sim U[0,1]. \end{aligned}$$

This in turn implies that $(E[X])^2 = (\frac{1}{2})^2 = \frac{1}{4}$.

We also calculate

$$\begin{aligned} E(X^2) &= \int_0^1 x^2 f(x)dx \\ &= \int_0^1 x^2 dx \\ &= \frac{1}{3}. \end{aligned}$$

$$
\begin{aligned}
E[Y] &= E[3X+2] \\
&= 3E[X]+2 \\
&= \frac{3}{2}+2 \\
&= \frac{7}{2} \\
&= 3.5.
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= E[X^2]-(E[X])^2 \\
&= \frac{1}{3}-\frac{1}{4} \\
&= \frac{1}{12}.
\end{aligned}
$$

$$
\begin{aligned}
Var[Y] &= Var[3X+2] \\
&= Var[3X] \\
&= 9Var[X] \\
&= \frac{3}{4} \\
&= 0.75.
\end{aligned}
$$

$$
\begin{aligned}
Cov[X,Y] &= E[XY]-E[X]E[Y] \\
&= E[X(3X+2)]-(\frac{1}{2})(\frac{7}{2}) \\
&= E[3X^2+2X]-\frac{7}{4} \\
&= 3E[X^2]+2E[X]-\frac{7}{4} \\
&= 3(\frac{1}{3})+2(\frac{1}{2})-\frac{7}{4} \\
&= \frac{1}{4} \\
&= 0.25.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
Var[(2X+Y)^2] &= Var(2X+3X+2)^2 = Var(5X+2)^2 \\
&= Var(25X^2+20X+4) \\
&= Var(25X^2+20X) \\
&= Var(25X^2)+Var(20X)+2Cov(25X^2,20X) \\
&= 625Var(X^2)+400Var(X)+1000Cov(X^2,X) \\
&= 625[E(X^4)-(E(X^2))^2]+400(\frac{1}{12})+1000[E(X^3)-E(X^2)E(X)] \\
&= 625(\frac{1}{5}-\frac{1}{9})+400(\frac{1}{12})+1000(\frac{1}{4}-(\frac{1}{3})(\frac{1}{2})) \\
&= \frac{2500}{45}+\frac{400}{12}+\frac{1000}{12} \\
&= 172.2222.
\end{aligned}
$$

Exercise 7 Give an example of two random variables X and Y such that $Cov(X,Y)=0$, but X and Y are not independent.

Consider the following joint distribution for the random variables X and Y :

$P(X=-1,Y=1)=\frac{1}{3}$
$P(X=0,Y=0)=\frac{1}{3}$
$P(X=1,Y=1)=\frac{1}{3}$

Note $E(XY)=(-1)(1)(\frac{1}{3})+0(\frac{1}{3})+(1)(1)(\frac{1}{3})=0$.

Also $P(X=1)=\frac{1}{3}$, $P(X=0)=\frac{1}{3}$, $P(X=-1)=\frac{1}{3}\Rightarrow E(X)=0$, and

$P(Y=1)=\frac{2}{3}$, $P(X=0)=\frac{1}{3}\Rightarrow E(Y)=\frac{2}{3}$.

Hence $Cov(X,Y)=E(XY)-E(X)E(Y)=0-(0)(\frac{2}{3})=0$.

However, note that for example $P(X=1)=\frac{1}{3}$ and $P(Y=1)=\frac{2}{3}$, but $P(X=1,Y=1)=\frac{1}{3}$.

So $\underbrace{P(X=1,Y=1)}_{\text{joint density of X and Y}} \neq \underbrace{P(X=1)}_{\text{marginal of X}} \times \underbrace{P(Y=1)}_{\text{marginal of Y}}$.

Hence X and Y are not independent.

Exercise 8 Show that if X and Y are two independent random variables, then $Var(X+Y)=Var(X)+Var(Y)$.

Note that $Var(X+Y)=Var(X)+Var(Y)+2Cov(X,Y)$. But if X and Y are independent, then $Cov(X,Y)=0$. So we have that

$$
\begin{aligned}
Var(X+Y) &= Var(X)+Var(Y)+2\underbrace{Cov(X,Y)}_{\text{=0 if X and Y are independent}} \\
&= Var(X)+Var(Y).
\end{aligned}
$$

Exercise 9 β^* is an unbiased estimator. To see this calculate $E(\beta^*)$.

$E(\beta^*) = E(\alpha\hat{\beta} + (1-\alpha)\tilde{\beta}) = \alpha E(\hat{\beta}) + (1-\alpha)E(\tilde{\beta}) = \beta$

since both $\hat{\beta}$ and $\tilde{\beta}$ are unbiased estimators and $\alpha \in [0,1]$.

$var(\beta^*) = var(\alpha\hat{\beta} + (1-\alpha)\tilde{\beta}) = \alpha^2 var(\hat{\beta}) + (1-\alpha)^2 var(\tilde{\beta})$

Note that we have used the fact that $\hat{\beta}$ and $\tilde{\beta}$ are independent estimators of β , i.e. $cov(\tilde{\beta}, \hat{\beta}) = 0$.

To find the value of α, we will choose the one which minimizes the variance of β^*.

$\frac{\partial var(\beta^*)}{\partial \alpha} = 2\alpha var(\hat{\beta}) - 2(1-\alpha)var(\tilde{\beta})$

Equating the derivative to zero we obtain:

$$\alpha^* = \frac{var(\tilde{\beta})}{var(\hat{\beta}) + var(\tilde{\beta})}$$