Fluctuations in Convex Models of Endogenous Growth I: Growth Effects*

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Abstract

Is there a trade-off between fluctuations and growth? The empirical evidence is mixed, with some studies finding a positive relationship, while others find a negative one. Our objectives are to understand how fundamental uncertainty affects the long run growth rate and to identify important factors determining this relationship in a convex endogenous growth model. Qualitatively, we show that the relationship between volatility in fundamentals (or policies) and mean growth can be either positive or negative. The curvature of the utility function is a key parameter that determines the sign of the relationship. Quantitatively, an increase in uncertainty always increases the growth rate in our calibrated models. Though the changes we find are nontrivial, they are not large enough by themselves to account for the large differences in growth rates observed in the data. We also find that differences in the curvature of preferences have very substantial effects on the estimated variability of stationary objects like the consumption-output ratio and hours worked. For this reason, we expect that the models considered in this paper will provide the basis of sharp estimates of the curvature parameter.

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1 Introduction

In his celebrated 1987 book, "Models of Business Cycles," Robert Lucas presented some simple calculations to argue that the trade-off between fluctuations and growth is such that a representative agent's willingness to pay for a more stable environment, in terms of growth rates, is almost zero. Lucas' conclusion has been challenged by studying models that relax some of the details in his basic environment.[1] However, none of these analyses question a fundamental implicit assumption: that the factors that affect fluctuations do not affect long run growth.[2]

Is there any evidence that the volatility of shocks – both policy and productivity shocks – has an impact on long run growth? Since it is difficult at best to directly measure volatility in fundamentals, most analyses study the relationship between some measure of variability of the growth rate of output and mean, or average, growth. In an early study, Kormendi and Meguire (1985) find that variability is positively related to mean growth in a cross section of countries. More recently, Ramey and Ramey (1995) find that higher volatility decreases growth, also in a cross section of countries. Empirical work that relates policy variability (mostly inflation variability) and growth also seems to point to a negative relationship (see Judson and Orphanides, 1996). Simple regressions of mean growth rates on measures of volatility of growth rates in cross section from the Penn World Table suggest a U-shape relationship, with an "upward sloping" segment only at very high levels of volatility.[3]

Our objective in this paper is to evaluate the proposition that differential levels of volatility in fundamentals can account for the observed cross-sectional differences in growth rates. To this end we study a class of models in the neoclassical tradition, in which fundamental uncertainty can affect the long run growth rate.[4] Our analysis includes both theoretical and numerical results. Qualitatively, if shocks are i.i.d. and depreciation is full, we show that the relationship between mean growth and volatility in fundamentals and policies can be either positive or negative. The key factor is the curvature of the utility function. If utility is more concave than the log case, an

[1] These range from the specification of preferences to the details of the market structure. For the former see Manuelli and Sargent (1988), and for the latter, Imrohoroglu (1989) and Atkeson and Phelan (1994).

[2] The current standard in the real business cycle literature, is to view long run growth as exogenous and, hence, independent of the fundamental shocks. For an explicit discussion see Cooley and Prescott (1995). The recent paper by Barlevy (2004) studies the relationship between growth and cyclical fluctuations in an endogenous growth model and obtains an estimate of the welfare costs of business cycles that is larger than that of Lucas.

[3] More recent work seems to suggest that even the results in Ramey and Ramey are not robust. They seem to depend on both the sample period as well as the collection of countries included. See Chatterjee and Shukayev (2004).

[4] Although we emphasize a "technology shock" interpretation of the type used in the real business cycle literature in our model (see Cooley, 1995, for a good survey of this literature), the shocks that we model can also be interpreted as random fiscal policies; for an equivalence result, see Jones and Manuelli (1999).

increase in shock volatility increases the savings rate and the average growth rate. If it is less concave than the log, the opposite occurs. This is in keeping with findings in earlier papers (see Phelps, 1962; Levhari and Srinivasan, 1969; Rothschild and Stiglitz, 1971; Leland, 1974; and de Hek, 1999). However, our result is substantively more general in that it allows for a non-trivial role for labor supply, so that interest rates are, even on average, endogenous.

Quantitatively, in contrast to the theoretical results above, the relationship between shock volatility and growth is positive in all of our calibrated examples. This is true even for preference specifications with less curvature than log utility, due to our inclusion of partial depreciation and autocorrelated shocks. For variations in the degree of volatility in fundamentals calibrated to match the range of growth rate volatility seen in the data, we obtain changes in mean growth rates on the order of 0.3% per year. Although this is a nontrivial change, it could not by itself account for a significant fraction of the differences in growth rates between countries. In particular, since increased uncertainty increases mean growth, we find that these types of models are not capable of reproducing the growth performances seen at the low end of the distribution as a sole result of high volatility in fundamentals.

To better understand the interplay between model specification, volatility, and growth, we conduct two types of sensitivity analysis. We vary preference parameters (the degree of risk aversion) and consider alternative decompositions of shock volatility between innovation variance and autocorrelation. We show that the relationship between the degree of risk aversion and mean growth is inverse U-shaped. Moreover, we find that differences in the curvature of preferences have very substantial effects on the estimated variability of stationary objects like the consumption-output ratio and hours worked. For this reason, we expect that the class of models considered in this paper will provide the basis of sharp estimates of the curvature parameter. This is in contrast with the results in exogenous growth models in which curvature has only a small effect. We also show that the class of models that we study can generate positively autocorrelated growth rates of output but, for this to be the case, it is necessary that the driving shocks be positively autocorrelated themselves.

Even though our work follows the recent analyses of stochastic endogenous growth models in which the "source" of shocks is either technology (see, for example, King and Rebelo, 1988; King, Plosser and Rebelo, 1988; Obstfeld, 1994; and de Hek, 1999), policies (Eaton, 1981; Bean, 1990; Aizenman and Marion, 1993; Gomme, 1993; Hopenhayn and Muniagurria, 1996; and Dotsey and Sarte, 1997), or a combination of the two (Kocherlakota and Yi, 1997), it has a different emphasis. We are interested in understanding how volatility in fundamentals affects growth and whether, for reasonable specifications, fundamental uncertainty might explain cross-country differences in mean growth.

Section 2 presents the basic theoretical results and discusses a key property of our endogenous growth models that makes them computationally tractable. Section 3 contains numerical results for our baseline calibration and quantitative comparative statics results with respect to changing the degree of shock volatility. Section 4 contains sensitivity analysis while section 5 offers some concluding comments.

2 Stochastic Models of Endogenous Growth

In this section we lay out the basic planning problems that we study and discuss how they are solved. The equilibria of the class of models that we study can be computed as the solution to the following planner's problem:

$$\max E_t\{\sum_t \beta^t c_t^{1-\sigma} v(\ell_t)/(1-\sigma)\}, \tag{1}$$

subject to,

$$\begin{aligned} c_t + x_{zt} + x_{ht} + x_{kt} &\leq F(k_t, z_t, s_t), \\ z_t &\leq M(n_{zt}, h_t, x_{zt}) \\ k_{t+1} &\leq (1-\delta_k)k_t + x_{kt}, \\ h_{t+1} &\leq (1-\delta_h)h_t + G(n_{ht}, h_t, x_{ht}) \\ \ell_t + n_{ht} + n_{zt} &\leq 1, \end{aligned}$$

with h_0 and k_0 given. Here $\{s_t\}$ is a stochastic process which we assume is Markov with transition probability function $P(s, A)$; c_t is consumption; z_t is "effective labor," n_{zt} is hours spent working in the market, n_{ht} is hours spent augmenting human capital, and ℓ_t is leisure; x_{zt}, x_{kt} and x_{ht} are investment in effective labor, physical and human capital, respectively; k_t and h_t are the stock of physical and human capital, respectively. The depreciation rates on physical and human capital are given by δ_k and δ_h, respectively. The usual non-negativity constraints on consumption, investment, and hours worked apply.

Thus, this is a fairly standard endogenous growth model in which effective labor is made up of a combination of hours and human capital supplied to the market. It is a natural generalization of the technology shock driven RBC model modified for the growth rate to be endogenously determined. For specific choices of functional forms, many models in this literature are special cases. For example, if $M = n_z h$ and $G = G_0 n_h h$, the model corresponds to Lucas (1988) in the absence of externalities. If $M = n_z h$ and $G = x_h$, this corresponds to the two capital goods version discussed in Jones, Manuelli and Rossi (1993). Finally, note that the standard one-sector growth model with exogenous technological change is also a special case with $G = 0$ and $M = n_z$ (and the s_t process no longer stationary). Given convexity of technologies and preferences, if markets are complete (as we assume) the equilibrium allocation can be found by solving a planner's problem of this form.[5]

[5] Note that although we are formally interpreting the shocks as technology shocks, they can in certain cases (period-by-period, state-by-state budget balance, etc.) be interpreted as shocks to income tax rates. See Jones and Manuelli (1999).

The actual solution of the model does cause some technical problems. The natural choice of the state is the vector (k_t, h_t, s_t). The difficulty is that both k_t and h_t are diverging to infinity (at least for versions that exhibit growth on average). Despite this, the value and policy functions have relatively simple characterizations under some additional assumptions about the form of the utility and production functions. The key property that we will exploit is that for general versions of the models of the type described in (1) to have a balanced growth path, both preferences and technology must be restricted in a specific way (see King, Plosser and Rebelo, 1988; and Alvarez and Stokey, 1995).

It can be shown that the essential property is that the technology set be linearly homogeneous in reproducible factors. This is summarized as follows:

Condition 1 *(Linear Homogeneity)*
a) F is concave and homogeneous of degree one in (k, z),
b) M is concave and homogeneous of degree one in (h, x_z),
c) G is concave and homogeneous of degree one in (h, x_h).

These restrictions effectively imply that the choice set in this version of (1) is linearly homogeneous in the initial stocks. Further, since preferences are homothetic, holding fixed the non-reproducible choice variables (hours worked in our application) it follows that knowledge of the current shock and the current human to physical capital ratio (the two relevant pseudo-state variables) is sufficient to determine the optimal choices of hours worked and next period's human to physical capital ratio.

More formally, let $\{e_t\}$ be the entire state/date contingent plan for the reproducible factors. The plan $\{e_t, n_t\}$ is feasible from initial state $e_0 = (h_0, k_0)$, for a given s_0, if and only if $\{\lambda e_t, n_t\}$ is feasible from the initial state $\lambda e_0 = (\lambda k_0, \lambda h_0)$ $(\lambda > 0)$. Moreover, utility (i.e., the entire expected discounted sum) realized from $\{\lambda e_t, n_t\}$ is $\lambda^{1-\sigma}$ times the utility of $\{e_t, n_t\}$. Formally, consider the maximization problem:

$$\max_{e,n} U(e, n), \tag{2}$$

subject to,

$$(e, n) \in \Gamma(h_0, k_0, s_0),$$

where as noted, (e, n) is interpreted as the entire path of the endogenous variables and vector of labor supplies, and $U(\cdot)$ is the resulting expected discounted sum of utilities. Let $V(h_0, k_0, s_0)$ denote the maximized value in this problem (assuming that it exists) and let $(e^*(h_0, k_0, s_0), n^*(h_0, k_0, s_0))$ denote the optimal plan. We obtain the following result.

Proposition 2 *Assume that the utility function in (2) is homogeneous of degree $(1 - \sigma)$ in e (holding n fixed) and that the feasible set, Γ, is linearly homogeneous in (h, k) (holding n and s fixed) and that a solution exists for all (h, k, s). Then, the value function, $V(\cdot)$, for the problem (2) satisfies $V(\lambda k, \lambda h, s) = \lambda^{1-\sigma} V(k, h, s)$, for* *all $\lambda > 0$. Moreover, the optimal plans are homogeneous of degree one in e and zero in n:* $(e^*(\lambda k, \lambda h, s), n^*(\lambda k, \lambda h, s)) = (\lambda e^*(k,h,s), n^*(k,h,s))$.

Proof. See Appendix A. ■

From the point of view of numerical approximation, this result implies that it is possible to estimate the optimal decision rules for c/k and x_j/k, $j = h, k, z$, as functions of the bounded variable h/k, and then calculate:

$$\begin{aligned} k' &= \left[1 - \delta_k + \frac{x_k}{k}\right] k, \\ h' &= \left[1 - \delta_h + G\left(\frac{x_h}{k}, \frac{h}{k}, n_h\right)\right] h, \end{aligned}$$

to determine h'/k'. Thus, in this case the Euler equations corresponding to (2) are solved by functions that depend on the stationary variables, h/k and s only.

Proposition 2 applies to any planning problem that has the required linearity and homogeneity properties. These include, for example, models with multiple sectors or preferences that depend on the state (e.g., human capital determining effective leisure). A separate, but related question is under what conditions equilibrium allocations can be represented as solutions to planner's problems of the type described in (1). This class includes convex endogenous growth models with no external effects and the same class of models with proportional income taxes (see Jones and Manuelli, 1990), among others. The proposition does not apply to planner's problems in which the technology displays increasing or decreasing returns to scale in reproducible factors (see Romer, 1986, for the former; and Brock and Mirman, 1972, and the real business cycle applications for the latter), or ones that have distortions with no planning representations (e.g., a model with different tax rates on capital and labor income).

Not surprisingly, analytic characterizations of the solutions to stochastic endogenous growth models such as the one outlined above are hard to come by. However, for certain specifications, our model reduces to models often used to study optimal savings with uncertain interest rates. In particular, if the shocks are i.i.d., depreciation is full, labor is inelastically supplied (or unproductive) and there is only one capital stock (either h or k) the model reduces to those studied by Phelps (1962), Levhari and Srinivasan (1969), Rothschild and Stiglitz (1971) and is similar to that analyzed in Leland (1974). In those papers, it is shown that increasing the volatility of the interest rate shocks can either increase or decrease savings rates, giving rise to the same effect on the associated growth rate of wealth. The key factor in those results is the curvature of the utility function. If utility is more concave than the log case, an increase in shock volatility increases the savings rate and the average rate of growth of wealth. If it is less concave than the log, the opposite occurs.[6]

[6] Leland studies a finite horizon model and allows for partial depreciation and autocorrelated

Our model is more complex than those in that literature since it is a general equilibrium model with elastic labor supply, partial depreciation, and serially correlated shocks. However, in certain cases a generalization of that result does hold.[7]

Proposition 3 *Assume there is full depreciation of both k and h and that the shocks are i.i.d.. Mean preserving spreads on the distribution of the shocks increase labor supply, savings rates and average growth rates if $\sigma > 1$, and decrease them if $\sigma < 1$. There is no change if $\sigma = 1$.*

Proof. See Appendix A for a more formal statement and proof of this result. ■

Thus, in principle, increased uncertainty could either increase or decrease average growth rates. As we will see in the calculations below, this will no longer hold if depreciation is only partial and shocks are positively autocorrelated.

3 The Quantitative Effects of Uncertainty

In this section we use numerical methods to analyze the quantitative effects of variability in fundamentals upon the distribution of growth rates.

3.1 Model Specification and Calibration

We study a special case of the model of Section 2. In particular, we consider the following specification:

$$\begin{aligned}
n_h &= x_z = 0, \quad n_z = n, \\
v(\ell) &= \ell^{\psi(1-\sigma)}, \\
F(k, z, s) &= sAk^{\alpha} z^{1-\alpha}, \\
G(h, x_h) &= x_h, \\
M(n, h) &= nh, \\
s_t &= \exp[\zeta_t - \frac{\sigma_{\varepsilon}^2}{2(1-\rho^2)}], \\
\zeta_{t+1} &= \rho\zeta_t + \varepsilon_{t+1},
\end{aligned}$$

with ε's i.i.d., normal with mean zero and variance σ_{ε}^2. The specification is relatively standard. Our assumption that only x_{ht} enters in the production of new human capital amounts to an aggregation assumption – namely that the technology used

shocks. He shows that mean preserving spreads to shocks increase (decrease) consumption's share in output when preferences show more (less) curvature than log utility. This is only directly related to mean output growth rates under the additional assumption that the shocks are i.i.d., however.

[7] Eaton (1981) was the first to apply these ideas to growth models. A two technology version is in Obstfeld (1994).

to produce human capital is identical to that in the final goods sector.[8] Finally, we assume that $\delta_k = \delta_h$. This assumption simplifies the solution since it implies a constant physical-to-human capital ratio (for details see Appendix A).[9]

To calibrate the model, we choose our base case to match observations for the postwar US economy, and adjust the parameters of the shock process to determine the relationship between volatility and mean growth. In particular, we assume that capital's share, α, is given by 0.36, and hold β fixed at 0.95, so that the length of a period is taken to be one year. We set the common depreciation rate of human and physical capital to $\delta = 0.075$. This was chosen as an intermediate value, greater than those estimated for the depreciation of human capital (usually in the 1% to 4% range), but smaller than that of physical capital. A detailed discussion is provided in Jones, Manuelli and Siu (2005), JMS hereafter. For this equal depreciation case, it is straightforward to construct a time series for the unobserved shocks using the approach and data of JMS (annual US observations for 1959-2000). Doing so, we pin down the base case parameter values governing the shock process to be $\rho = 0.95$ and $\sigma_\varepsilon = 0.011$. Labor supply in the non-stochastic steady state is set at $n_{ss} = 0.17$ (see Jones, Manuelli and Rossi, 1993).[10] The value for the non-stochastic growth rate is set to $\gamma_{ss} = 2\%$; this is very close to the mean growth rate of 1.8% for the US data of JMS. It also corresponds well with the cross-country mean value of 2.0% taken from the Penn World Table, version 6.1, PWT hereafter (see Summers and Heston, 1991). In analyzing this data, we follow the standard practice of excluding countries with populations of under 1 million (in 1980), and also omit countries with fewer than 15 observations. This leaves 104 countries in our sample.

These last two restrictions ($n_{ss} = 0.17$ and $\gamma_{ss} = 2\%$) still leave one degree of freedom in the selection of preference and technology parameters, σ, ψ, and A. To solve this indeterminacy, we choose σ (the coefficient of risk aversion) to match the standard deviation of output growth found in the US data, $\sigma(\gamma_y) = 1.92\%$. As we document below, there is a strong monotonic relationship between the coefficient of risk aversion and the volatility of output growth, so proceeding in this manner allows us to pin down $\sigma = 1.07$, $\psi = 8.505$ and $A = 0.897$.

This value of $\sigma(\gamma_y) = 1.92\%$ in the US data is low by international standards. So naturally, one of the primary focuses of the experiments we consider is to increase σ_ε so that the resulting growth rate volatility is more in line with that found in cross-country data. Thus, we ask: Does changing the parameters of the stochastic process

[8] Of course, human capital investment is produced using labor and both physical and human capital through the production function $F(\cdot)$.

[9] This assumption obviously carries quantitative effects as well. For example, it implies that the fraction of x_h in output is quite large. Moving to more realistic versions of the model with $\delta_h < \delta_k$ fixes this, while having no effects on the properties studied in this paper. For example, the magnitude of the changes in average growth coming from increased uncertainty is not affected by this simplification. See Jones, Manuelli and Siu (2005).

[10] In earlier versions of the paper, we also studied calibrations with $n_{ss} = 0.3$. This had only minor effects, and hence, the results are not included here.

in such a way as to increase $\sigma(\gamma_y)$ to levels representative of countries in the PWT have a substantial impact on mean output growth, $E(\gamma_y)$?

Though our choice of base case parameters is motivated by the desire to match observations, the principal aim of the paper is to understand how differences in the variability of fundamentals affects the distribution of growth rates more generally. Thus, we study alternative parameter values to better understand this relationship. In our specification, there are two key parameters that influence the volatility of the shock process: σ_ε and ρ. Below, we study the effects on mean output growth of changes in both of these sources of uncertainty. In addition, our theoretical results indicate that some parameters – notably, σ – are important determinants of the transmission mechanism of exogenous shocks; hence, we explore the effects of varying this preference parameter as well. Finally, since we want to explore the possibility that these effects are non-linear, we study the effects of alternative calibrated non-stochastic growth rates.

In summary, we consider the following experiments:

1. we vary the standard deviation of the innovation, σ_ε, holding ρ and all other parameters fixed to match the interquartile (i.e., 25% to 75%) range of $\sigma(\gamma_y)$ found in the PWT data.

2. we vary the persistence of the shock, ρ, from 0.8 to 0.99, holding σ_ε and all other parameters fixed.

3. we vary the coefficient of risk aversion, σ, from 0.95 to 3.0.

4. we vary the calibrated growth rate, γ_{ss}, from 0% to 4% per year.

Note that experiments 3 and 4 require simultaneous adjustments of ψ and A to maintain the same first moments as the base case calibration.

To solve the model, we compute the optimal decision rules after we discretize the state space. We then draw a realization of $\{s_t\}$ of size 20,000, and compute the moments using this realization. In those cases in which the stochastic process $\{s_t\}$ is not changed, we used the same realization to facilitate comparisons.

3.2 Shock Volatility and Growth Rates

For the linear, stochastic, Markov process $\{s_t\}$, the standard deviation of the shocks is given by $\sigma_s = \sigma_\varepsilon/(1-\rho^2)^{1/2}$. Thus, σ_s depends on both the standard deviation of the innovation, σ_ε, and the autocorrelation coefficient, ρ. In this section we study the effects of these two components of shock volatility. We find that changes in σ_ε (holding ρ constant) have a substantial impact on mean growth, while changes in ρ (holding σ_ε constant) do not.

Case	σ	ρ	σ_ε	σ_s	$E(\gamma_y)$	$\sigma(\gamma_y)$	$\rho(\gamma_y)$
1 (base)	1.07	0.95	0.011	0.035	2.024	1.195	0.143
2	1.07	0.95	0.020	0.064	2.072	3.485	0.143
3	1.07	0.95	0.026	0.083	2.117	4.537	0.142
4	1.07	0.95	0.030	0.096	2.154	5.241	0.142
5	1.07	0.95	0.038	0.122	2.242	6.656	0.141
6	1.07	0.95	0.040	0.128	2.267	7.011	0.140
7	1.07	0.95	0.043	0.138	2.308	7.546	0.140
PWT mean	-	-	-	-	2.01	5.33	0.138
PWT median	-	-	-	-	2.11	4.51	0.143
PWT quartile 1	-	-	-	-	0.93	3.35	-0.012
PWT quartile 3	-	-	-	-	3.04	6.73	0.290

Table 1: The effect of σ_ε on growth rates. Note: The column labeled $E\left(\gamma_y\right)$ gives the average growth rate, $\sigma(\gamma_y)$ the standard deviation of the growth rate, and $\rho(\gamma_y)$ the autocorrelation of the growth rate. The rows correspond to model simulations with parameter values listed in columns 2 through 5, as well as the Penn World Table (PWT) data.

3.2.1 Innovation Volatility and its Effects on the Distribution of Growth Rates

Our first set of experiments study changes in σ_ε holding ρ fixed at $\rho = 0.95$. In the context of the theory developed in Section 2, an increase in σ_ε corresponds to an increase in risk. We vary σ_ε so that the range for the standard deviation of output growth produced in the model covers the corresponding interquartile range in the cross-country data. In our PWT data sample, the interquartile range is 3.35% to 6.73% (see the last two rows of Table 1). Hence, this corresponds approximately with Cases 2 through 5 in Table 1 below. We report the values of the average growth rate, $E(\gamma_y)$, the standard deviation of the growth rate, $\sigma(\gamma_y)$, and the first order autocorrelation coefficient of the growth rate, $\rho(\gamma_y)$, in the simulated data. As a reference point, it follows that if we shut down uncertainty (i.e., if $\sigma_\varepsilon = 0$), $E(\gamma_y) = \gamma_{ss}$, the non-stochastic balanced growth rate of 2% per year. Hence, any difference between the simulated values of $E(\gamma_y)$ and 2% is due to variability in the shocks.

We also present comparable statistics from the PWT measure of GDP per capita. Thus, the PWT mean of 2.01 signifies that the cross-country average of annual real output per worker growth is 2.01%, while the middle 50% of countries had average growth rates between 0.93% and 3.04%.

As can be seen in the table, increasing the innovation volatility causes mean growth rates to rise. At the low end (Case 2), $E(\gamma_y) = 2.072$, and increases to $E(\gamma_y) = 2.308$ in Case 7, an increase in mean growth of about one fourth of one percent per year. Although this is substantial, this range in values for $E(\gamma_y)$ covers only about 9% of the range in values seen in our PWT data sample. Recall that our sample is restricted by excluding countries with population under than 1 million. If we include these, increasing our sample to 126 countries, the interquartile range for $\sigma(\gamma_y)$ is 3.65 to 7.51. To match this range for output growth volatility, σ_ε must vary between 0.021 and 0.043. This corresponds to Cases 2 through 7. The interquartile range for mean growth rates in the enlarged sample is 1.03 to 3.06. Hence, the model generates about 11% of the range found in the data.

Moreover, there is difficulty for this experiment in matching the low average growth performances found in the PWT. Note that all values of $E(\gamma_y)$ exceed 2% in the table. Thus, even though preferences are very close to log utility, changes in the distribution of the shock process tend to increase average growth rates. This is because, with less than full depreciation and positively autocorrelated shocks, increases in volatility increase average growth even when $\sigma \leq 1$. This will become clear in Section 4.

Summarizing:

- There is a monotonically increasing relationship between mean growth rates and σ_ε.

- The impact is not linear, with larger effects for high levels of uncertainty. At the high end, when the standard deviation of the innovation, $\sigma_\varepsilon \approx 0.04$, the average growth rate is about 2.3%, an increase of 0.3% over the non-stochastic benchmark of 2% per year.

- Changes in σ_s due to changes in σ_ε have almost linear effects on the standard deviation of the growth rate, and very small effects on the autocorrelation of growth rates. Further evidence of this is displayed in Table 5 in the appendix.

- There is significant autocorrelation in growth rates, $\rho(\gamma_y) \approx 0.14$, independent of the size of σ_ε. This is in contrast to the typical RBC model where $\rho(\gamma_y) \approx 0$, independent of the parameterization.[11]

3.2.2 Shock Autocorrelation and its Effects

For our next set of experiments, we hold σ_ε constant, and change σ_s by varying the correlation coefficient, ρ. The major findings are presented in Table 2.

Note that increasing ρ (holding σ_ε constant) has a monotonically increasing effect on the average growth rate. However, this effect is quantitatively very small. Hence, increases in the shock volatility, σ_s, that are due to increases in ρ have smaller

[11] See, for example, Cogley and Nason (1995). For further analysis of this difference between endogenous and exogenous growth models, see JMS. In particular, we find that by dropping the simplifying assumption that $\delta_h = \delta_k$, we obtain values of $\rho\left(\gamma_y\right)$ closer to that of the US data. See also the following subsection and Section 4.

Case	σ	ρ	σ_ε	σ_s	$E(\gamma_y)$	$\sigma(\gamma_y)$	$\rho(\gamma_y)$
1	1.07	0.70	0.011	0.015	2.019	1.956	-0.124
2	1.07	0.80	0.011	0.018	2.019	1.906	-0.058
3	1.07	0.90	0.011	0.025	2.020	1.879	0.037
4 (base)	1.07	0.95	0.011	0.035	2.024	1.915	0.143
5	1.07	0.99	0.011	0.078	2.066	2.439	0.519
PWT mean	-	-	-	-	2.01	5.33	0.138
PWT median	-	-	-	-	2.11	4.51	0.143
PWT quartile 1	-	-	-	-	0.93	3.35	-0.012
PWT quartile 3	-	-	-	-	3.04	6.73	0.290

Table 2: The effect of ρ on growth rates. Note: The column labeled $E\left(\gamma_y\right)$ gives the average growth rate, $\sigma(\gamma_y)$ the standard deviation of the growth rate, and $\rho(\gamma_y)$ the autocorrelation of the growth rate. The rows correspond to model simulations with parameter values listed in columns 2 through 5, as well as the Penn World Table (PWT) data.

impacts on $E(\gamma_y)$ than those that are due to increases in σ_ε. Increases in ρ have a U-shaped effect on the standard deviation of growth rates, but again, this effect is quantitatively small. In contrast, there is a strong positive relationship between ρ the serial correlation of the growth rate, $\rho(\gamma_y)$. Moreover, at high levels of ρ, this effect is quite substantial. Indeed, to generate positively autocorrelated output growth rates, ρ must be sufficiently large.

4 Sensitivity Analysis and Other Properties of the Model

In this section, we present sensitivity analyses of the results in the previous section. We also briefly discuss some of the cyclical properties of the class of models studied.

4.1 Sensitivity Analysis

Here we study the sensitivity of our basic results to two key assumptions: the degree of risk aversion in preferences and the level of the calibrated non-stochastic growth rate. We find that the impacts of volatility on mean growth are highly sensitive to the degree of risk aversion but not at all to the calibrated mean growth rate.

Case	σ	ρ	σ_ε	σ_s	$E(\gamma_y)$	$\sigma(\gamma_y)$	$\rho(\gamma_y)$
1	0.90	0.95	0.011	0.035	2.073	2.822	0.180
2	1.00	0.95	0.011	0.035	2.033	2.146	0.155
3 (base)	1.07	0.95	0.011	0.035	2.024	1.915	0.143
4	1.50	0.95	0.011	0.035	2.017	1.423	0.106
5	2.00	0.95	0.011	0.035	2.019	1.272	0.088
6	2.50	0.95	0.011	0.035	2.023	1.206	0.078
7	3.00	0.95	0.011	0.035	2.027	1.170	0.071
8	0.90	0.95	0.026	0.083	2.376	6.705	0.182
9	1.00	0.95	0.026	0.083	2.159	5.087	0.154
10	1.50	0.95	0.026	0.083	2.079	3.369	0.106
11	2.00	0.95	0.026	0.083	2.095	3.010	0.088
12	2.50	0.95	0.026	0.083	2.117	2.854	0.078
13	3.00	0.95	0.026	0.083	2.139	2.769	0.072
PWT mean	-	-	-	-	2.01	5.33	0.138
PWT median	-	-	-	-	2.11	4.51	0.143
PWT quartile 1	-	-	-	-	0.93	3.35	-0.012
PWT quartile 3	-	-	-	-	3.04	6.73	0.290

Table 3: The effect of σ on growth rates. Note: The column labeled $E\left(\gamma_y\right)$ gives the average growth rate, $\sigma(\gamma_y)$ the standard deviation of the growth rate, and $\rho(\gamma_y)$ the autocorrelation of the growth rate. The rows correspond to model simulations with parameter values listed in columns 2 through 5, as well as the values taken from the Penn World Table (PWT) dataset.

4.1.1 Uncertainty, Risk Aversion and Growth Rates

Table 3 presents results for several specifications in which we hold γ_{ss}, ρ and σ_ε fixed at their base case values, and adjust σ from 0.90 to 3.0.[12] Again, because the non-stochastic version of all cases is calibrated to grow at 2%, any difference between the simulated values of $E(\gamma_y)$ and 2% is due to fundamental uncertainty. In particular, since $\sigma_s = \sigma_\varepsilon/(1-\rho^2)^{1/2}$, we are increasing the standard deviation of the shocks from 0% in the non-stochastic case to 3.5% in the simulations when $\sigma_\varepsilon = 0.011$, and to 8.3% when $\sigma_\varepsilon = 0.026$. Note that when $\sigma_\varepsilon = 0.026$ and σ is close to unity, the standard deviation of output growth in the simulations is approximately the mean value of $\sigma(\gamma_y)$ in the PWT data (see also Case 3 of Table 1). As such, it provides a useful benchmark in addition to the baseline US calibration of $\sigma_\varepsilon = 0.011$.

Our major findings are as follows:

[12]Note that if $\sigma < 1$, concavity of the utility function puts restrictions on what ψ can be. For each σ, we adjust A and ψ to keep the growth rate for the non-stochastic version of the model fixed at 2% and labor supply equal to 0.17. Thus, we could equally well index the cases by either A or ψ.

- The effect of a given amount of uncertainty upon the average growth rate varies with the curvature parameter σ; moreover, it is not a monotone function of curvature. Figure 1 shows that the largest impact of uncertainty occurs for values below log utility. For $\sigma > 1.5$, increases in risk aversion increase $E(\gamma_y)$. This is true for both low and high values of σ_ε ($\sigma_\varepsilon = 0.011$ and $\sigma_\varepsilon = 0.026$). Overall, the relationship between σ and $E(\gamma_y)$ is U-shaped. The range of values of $E\left(\gamma_y\right)$ is increasing in σ_ε.

- Our base case corresponds to Case 3. For this case, the impact of increased uncertainty upon mean growth is small, and approximately equal to 0.025% per year. The largest impact of uncertainty occurs for preferences that are less concave than the log ($\sigma < 1$), as in case 1. But even in this case, the change in mean growth is only 0.07% per year. Thus, for $\sigma_\varepsilon = 0.011$ and $\rho = 0.95$, there is virtually no impact on mean growth of uncertainty, for reasonable values of σ. The effect is larger for $\sigma_\varepsilon = 0.026$. Here, even in the log case, the increase in the mean growth rate (relative to certainty) is about 0.15% per year.

- Again, the average simulated growth rate exceeds the calibrated value of 2% for each value of σ. This is in contrast to the analytical result of Proposition 2, when shocks are i.i.d. ($\rho = 0$) and depreciation is full ($\delta = 1$).

- As expected, increases in the curvature of utility, σ, result in decreases in the standard deviation of growth rates, $\sigma(\gamma_y)$. This relationship is both monotone and quantitatively large. Thus, for coefficients of relative risk aversion exceeding 1.5, we find that increases in risk aversion increase mean growth and decrease its variability.

- The smaller the curvature of the utility function the higher the autocorrelation coefficient. More curvature makes investment respond less to the current shock, and this in turn implies that the growth rate is more negatively serially correlated, although the values are not significantly different from zero. At the other extreme, if the source of differences across economies is the curvature parameter, our model predicts a positive relationship between mean growth and the autocorrelation of growth rates if σ is less than 1.5. This is consistent with Fatas' (1999) finding.

*********Figure 1 goes about here.********

4.1.2 Non-Linearities and the Effect of Volatility on Mean Growth

Is it possible that uncertainty has a different effect for "high" growth and "low" growth countries?[13] To explore this possibility we adjust γ_{ss}, the calibrated non-stochastic

[13] In the context of this paper the differences in growth rates could be due to distortionary taxes and/or differences in technology.

growth rate. Relative to our base case, we considered values between 0% and 4%. Our numerical results (see Table 6 in the appendix) show that γ_{ss} has virtually no impact upon either the mean or standard deviation of output growth. That is, $E(\gamma_y) - \gamma_{ss}$ is approximately independent of γ_{ss}, and $\sigma(\gamma_y)$ while depending on σ_ε and ρ does not depend on γ_{ss}. The value of γ_{ss} does affect the autocorrelation of the growth rate, however. For example, $\rho(\gamma_y) = 0.104$ when $\gamma_{ss} = 0.0$ and $\rho(\gamma_y) = 0.188$ when $\gamma_{ss} = 4.0\%$. See Table 6 for details.

4.2 Volatility and Cyclical Behavior

Though our primary interest in this paper is to begin the exploration of the effects of uncertainty upon growth, the model delivers implications for cyclical variables. However, unlike more standard real business cycle models, we are not free to detrend the data. Our theoretical model implies that the appropriate detrending procedure is to consider the ratio of each variable (except for hours worked) to output. In the case of hours, the model implies that it is a stationary variable.

Before we confront the model's predictions with the data, it is necessary to match the notion of investment in human capital with observable quantities. In the model, the variable x_h corresponds to investment and is conceptually different from consumption. What is the counterpart in the data? One reason why this is a difficult question to answer is that it is not clear what activities constitute human capital investment. Most economists would agree that it includes education and training, but it is also likely to encompass other activities like health care, investments in mobility and the like. Even for those items in which there is consensus (e.g., education and training) there are no good measures. To say the least, training is poorly measured and, depending on its nature, may not even be part of measured output. In the case of education, and some forms of training, gross investments appear in consumption.[14] In this paper we assume that all of x_h is part of measured output, and we experiment with two notions of consumption: the "narrow" view that consumption in the data corresponds to consumption in the model, and the "broad" view that consumption in the data is the sum of consumption and investment in human capital, $c + x_h$.

In Table 4 we report the results for cyclical variables for our base case and various levels of curvature. There are several interesting features:

- As can be seen, the amount of curvature in utility has only a small effect upon the mean of the consumption-output ratio, both in its narrow version, c/y, and its broad version, $(c + x_h)/y$. However, the choice of narrow versus broad consumption has a substantial effect on the mean; the difference between

[14] Of course, it is possible to net out educational expenditures, both private and public from the data. However, other components like health care are much more difficult to allocate since not all expenditures qualify as investments in productive human capital.

Case	σ	ρ	σ_ε	$E(c/y)$	$E\left[(c+x_h)/y\right]$	$\sigma(c/y)$	$\sigma\left[(c+x_h)/y\right]$	$\sigma(n)/E(n)$
1	0.90	0.95	0.011	0.354	0.767	0.033	0.012	0.078
2	1.00	0.95	0.011	0.362	0.770	0.021	0.007	0.047
3 (base)	1.07	0.95	0.011	0.368	0.773	0.016	0.006	0.036
4	1.50	0.95	0.011	0.404	0.785	0.007	0.002	0.014
5	2.00	0.95	0.011	0.442	0.799	0.004	0.001	0.007
6	2.50	0.95	0.011	0.475	0.811	0.002	0.001	0.004
7	3.00	0.95	0.011	0.505	0.822	0.001	0.000	0.002
8	0.90	0.95	0.026	0.362	0.770	0.082	0.030	0.184
9	1.00	0.95	0.026	0.367	0.772	0.050	0.018	0.111
10	1.50	0.95	0.026	0.405	0.786	0.016	0.006	0.032
11	2.00	0.95	0.026	0.441	0.799	0.008	0.003	0.016
12	2.50	0.95	0.026	0.473	0.810	0.005	0.002	0.009
13	3.00	0.95	0.026	0.502	0.821	0.003	0.001	0.005

Table 4: The effect of σ on cyclical moments. Note: Columns 5 and 6 give, respectively, the mean of the "narrow" and "broad" consumption to output ratios; columns 7 and 8 give the standard deviation of these same objects; column 9 gives the coefficient of variation of hours worked. The rows correspond to model simulations with parameter values listed in columns 2 through 4.

columns 5 and 6 indicates that human capital investment comprises roughly 35 percent of output.[15]

- The model has very sharp implications for the effect of curvature on volatility. Increasing the degree of relative risk aversion decreases the standard deviation of the consumption-output ratio drastically using either measure. The standard deviation falls by over a factor of 25 when moving from $\sigma = 0.90$ to $\sigma = 3.0$.[16] For reference, the standard deviation of the measured consumption-output ratio in the US data is around 0.014. If we wanted the broad measure in the model to match this value for $\sigma_\varepsilon = 0.011$, the best estimate of σ is near 0.9.

- The model implies that the amount of curvature in the utility function has sharp implications for the coefficient of variation of hours worked. This is shown in the last column of Table 4. As the coefficient of relative risk aversion moves from

[15] The size of this depends critically on the calibrated magnitude of δ_h. For lower and more realistic values, this is substantially reduced. See JMS.

[16] Given our definitions, it follows that $\sigma((c + x_h)/y) = \alpha\sigma(c/y)$. Thus, "broad" consumption is less variable than the "narrow" measure because the former includes x_h which is an investment good and, as such, its ratio to output increases in good times and decreases in bad times. Curvature in the utility function implies that the c/y ratio decreases in good times and increases in bad times. Thus, roughly, c/y and x_h/y are negatively correlated, and their sum exhibits lower variability than either of the components.

0.90 to 3.0, the predicted coefficient of variation falls by a factor of almost 40 for the US calibrated shock parameters. For comparative purposes, the analogous value of the coefficient of variation of hours worked in the U.S. is 0.0481 (again, see JMS for details on the US data). Thus, in this case the "best" value of σ is something close to 1.0, or log utility for $\sigma_\varepsilon = 0.011$.

In the cases presented to this point, hours worked, $n(s)$, is strictly increasing as a function of the shock. However, it is possible to modify the model a obtain a non-monotone $n(s)$ function. Our results (not presented here) suggest that cases in which the mean growth rate is small (say less than 1.4%) and the serial correlation of the shock is large (exceeding 0.95) are consistent with an increasing response of hours worked to productivity shocks when the shock is small, and a decreasing response when the shock is large. Whether that asymmetric response can account for puzzles like the productivity slowdown and the behavior of hours worked over the cycle is yet to be determined.

5 Conclusion

For the class of convex models that we study, changes in the variability of fundamentals result in changes in average growth rates. Theoretically, we show that it is possible for increased uncertainty to decrease average growth. However, this requires parameter values that lie outside the usual range – high intertemporal substitution, zero correlation of shocks and very short lived capital. In our calibrated models, for all levels of risk aversion, eliminating cycles completely would result in lower growth rates. The size of this effect is as large as 0.3% per year, depending on the parameterization. Of course, it is likely that this only reinforces Lucas' conclusions that the payoff from eliminating cycles is not too large. For reasonable specification of exogenous uncertainty, variability in fundamentals can explain only a small part of the difference in cross country growth rates.

We also identify changes in the variability of the innovations to fundamental shocks as having a larger impact upon average growth rates than changes in the serial correlation of shocks. Finally, uncertainty in fundamentals has a large impact on the predicted standard deviation of cyclical variables (e.g., the consumption-output ratio), and the size of the impact is very sensitive to the degree of curvature of preferences.

Our finding that increased uncertainty increases average growth seems at odds with the empirical work of Ramey and Ramey (1995). However, since it is possible to interpret the shocks in our model as shocks to tax rates, our results imply that – holding average tax rates fixed – increases in the variance of tax rates increases average growth. Of course, if growth inhibiting policies (on average) are associated with volatile policies, the model could deliver a negative correlation between volatility and average growth. However, in this case, it is not the high volatility that is causing growth to be low, but the high average tax rates.[17]

Our preliminary conclusion is that, even though there is a trade-off between fluctuations and growth, bringing stochastic elements to the class of endogenous growth models that we studied does not radically improve its ability to explain "growth facts." However, it delivers very sharp implications about the effect of curvature in preferences on the variability of cyclical variables and, hence, it can use data to pin down preference parameters. The version of the model that we studied is too simple to proceed with this program. One manifestation of this is the difficulty in matching growth and cyclical observations simultaneously.

A Appendix

A.1 Proof of Proposition 2:

Fix an arbitrary initial state, (h,k,s) and let $(e^*(h,k,s), n^*(h,k,s))$ denote the solution to problem (2) from this state. Now consider the same problem when the initial state is $(\lambda k, \lambda h, s)$. It follows immediately from the linear homogeneity of Γ that $(\lambda e^*(h,k,s), n^*(h,k,s))$ is feasible for the problem with initial state $(\lambda k, \lambda h, s)$. Contrary to the conclusion of the proposition, assume that $(\lambda e^*(h,k,s), n^*(h,k,s))$ is not optimal. Then, take some alternative plan, (e,n) that is feasible and gives higher utility:

$$U(e,n) > U(\lambda e^*(h,k,s), n^*(h,k,s)). \qquad (3)$$

Since (e,n) is feasible given initial state $(\lambda k, \lambda h, s)$, it follows from the the linear homogeneity of Γ that $(e/\lambda, n)$ is feasible when the initial state is $(\lambda k/\lambda, \lambda h/\lambda, s) = (h,k,s)$. Moreover, the utility of $(e/\lambda, n)$ is given by $U(e/\lambda, n) = U(e,n)/\lambda^{1-\sigma}$. Using this and (3) we have that:

$$U(e/\lambda, n) = U(e,n)/\lambda^{1-\sigma} > U(\lambda e^*, n^*)/\lambda^{1-\sigma} = \lambda^{1-\sigma} U(e^*, n^*)/\lambda^{1-\sigma} = U(e^*, n^*).$$

That is, $(e/\lambda, n)$ is feasible when the initial state is (h,k,s) and it gives higher utility than (e^*, n^*), a contradiction.

That the value function is homogeneous of degree $(1-\sigma)$ in e (holding n fixed) follows immediately from the fact that the policy rules have the property that they do.

[17] In their work, Ramey and Ramey (1995) find that policy variability is associated with residual uncertainty. Our findings do not depend on the shock affecting all sectors. The model in Obstfeld (1994) can be used to show that for risk aversion levels greater than the log there is an "approximate" positive relationship between variability and growth. The reason why this relationship is approximate is that, in the model, the relationship between variability of output and mean output is not a function but a correspondence.

A.2 Mean Preserving Spreads with i.i.d. shocks and the Proof of Proposition 3:

Here we consider the case where shocks are i.i.d. and there is full depreciation of both capital stocks ($\delta = 1$). We assume that the distribution of the shocks is given by the measure μ_θ, where θ is an index of riskiness. More precisely, $\theta' > \theta$ means that $\mu_{\theta'}$ is dominated by μ_θ in the sense of second order stochastic dominance. Thus, a higher θ corresponds to higher volatility of the innovation to the technology shock.

To guarantee that an equilibrium exists, it must be the case that the economy is not too productive (for a discussion, see Jones and Manuelli, 1990). For this example, the relevant condition – which we assume holds – is:

$$[\beta(A(1-\alpha)^{1-\alpha}\alpha^{\alpha})^{1-\sigma}]^{1/\sigma}\left[\int_S (1+\varepsilon)^{1-\sigma}\mu_\theta d\varepsilon\right]^{1/\sigma} < 1.$$

To ensure an interior solution (in terms of n), we need stronger conditions, namely:

$$[\beta(A(1-\alpha)^{1-\alpha}\alpha^{\alpha})^{1-\sigma}]^{1/\sigma}\left[\int_S (1+\varepsilon)^{1-\sigma}\mu_\theta d\varepsilon\right]^{1/\sigma} < 1 - [(\sigma-1)(1-\alpha)v(1)/v\prime(1)], \tag{4}$$

and,

$$if\ 0<\sigma<1,\ \ \lim_{n\to 0} 1 - [(\sigma-1)(1-\alpha)v(n)/nv\prime(n)] < 0. \tag{5}$$

These two conditions guarantee that the equilibrium labor supply is strictly between 0 and 1. We assume that both hold. From now on, we will describe the conditions for the case $\sigma \neq 1$.

The equilibrium decision rules display three properties: saving is a constant fraction, φ of income; labor supply is constant; and the ex-post rates of return to physical and human capital are equal. First, if rates of return to the two forms of capital are equal (for each realization of s) then the stocks of human and physical capital must satisfy, $h_t = [(1-\alpha)/\alpha]\, k_t$. Given this, the saving rate, φ, and the level of employment, n, must solve:

$$\varphi = 1 - [(\sigma-1)(1-\alpha)v(n)/nv'(n)], \tag{6}$$

and,

$$\varphi = D\hat{s}^{1/\sigma}n^{(1-\alpha)(1-\sigma)/\sigma}, \tag{7}$$

where $D = \left[\beta(A^*)^{(1-\sigma)}\right]^{1/\sigma}$, $A^* = A(1-\alpha)^{1-\alpha}\alpha^{\alpha}$, and $\hat{s} = \int_S (1+\varepsilon)^{1-\sigma}\mu_\theta(d\varepsilon)$. Basically, (6) guarantees that at the conjectured equilibrium, the marginal rate of substitution between consumption and leisure is equal to the real wage, while (7) is the Euler equation that ensures equality between the intertemporal marginal rate of substitution in consumption and the rate of return on capital. Let the solution to (6) and (7) be a pair (φ, n), which depends on the parameters (σ, μ_θ). An equilibrium is fully characterized by this pair. The growth rate associated with this equilibrium is given by:

$$y_{t+1}/y_t = s_{t+1}A^* n^{1-\alpha}\varphi = s_{t+1}\gamma,$$

where, since $E(s_{t+1}) = 1$, γ is the mean growth rate.

Then, we have the following formal statement of Proposition 3:

Proposition 3: *Assume that conditions (4) and (5) hold. Then an equilibrium of the conjectured form exists and is unique. Moreover, if $\theta' > \theta$, the equilibrium satisfies:*

1. The effects of increases in risk:
   (a) (φ, n, γ) *increase with* θ *if* $\sigma > 1$,
   (b) (φ, n, γ) *decrease with* θ *if* $0 < \sigma < 1$,
   (c) (φ, n, γ) *are independent of* θ *if* $\sigma = 1$.

2. Amplification: *The ratio of the standard deviation of the growth rate to the standard deviation of the technology shock, σ_γ/σ_s:*
   (a) *is greater than one ($\sigma_\gamma/\sigma_s > 1$) if the growth rate is positive ($\gamma > 1$),*
   (b) *increases with* θ *if* $\sigma > 1$,
   (c) *decreases with* θ *if* $0 < \sigma < 1$,
   (d) *is independent of* θ *if* $\sigma = 1$.

Proof. We first consider the case $\sigma \neq 1$. The first order conditions for the problem are:

$$c_t v'(n_t) = (\sigma - 1)(1-\alpha)y_t v(n_t)/n_t, \tag{8}$$

$$c_t^{-\sigma} v(n_t) = \beta \int_S \left[c_{t+1}^{-\sigma} v(n_{t+1})\right]\left[A\alpha k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} n_{t+1}^{1-\alpha}(1+\varepsilon_{t+1})\right]\mu_\theta(d\varepsilon), \tag{9}$$

$$c_t^{-\sigma} v(n_t) = \beta \int_S \left[c_{t+1}^{-\sigma} v(n_{t+1})\right]\left[A\alpha k_{t+1}^{\alpha} h_{t+1}^{-\alpha} n_{t+1}^{1-\alpha}(1+\varepsilon_{t+1})\right]\mu_\theta(d\varepsilon), \tag{10}$$

and the feasibility constraints at equality. In order to find the solution to the planner's problem, we first hypothesize that (9) and (10) are satisfied by having the terms in square brackets inside the integral operator equal in each state. Second, we conjecture that consumption is a constant fraction of income. Finally, we guess that the fraction of the time allocated to working is constant as well. These conjectures imply that the solution must satisfy:

$$\begin{aligned} h_t &= [(1-\alpha)/\alpha]\, k_t, \\ (1-\varphi)v'(n) &= (\sigma-1)(1-\alpha)v(n)/n, \\ \varphi^\sigma &= \beta(A^*)^{1-\sigma} n^{(1-\alpha)(1-\sigma)} \int_S (1+\varepsilon)^{1-\sigma}\mu(d\varepsilon), \end{aligned}$$

where φ is the fraction of income, y, which is saved (and $(1-\varphi)$ is consumed), and A^* is $A(1-\alpha)^{1-\alpha}\alpha^{\alpha}$. The solution to equations (8) and (9) can be used to construct an equilibrium by letting investment in physical capital, x_k, be given by $\alpha\varphi y$, while x_h is $(1-\alpha)\varphi y$. To simplify notation, let $D = \left[\beta(A^*)^{1-\sigma}\right]^{1/\sigma}$, and let $\hat{s} = \int_S (1+\varepsilon)^{1-\sigma} d\mu_\theta(d\varepsilon)$. Then, (8) and (9) imply that the equilibrium values of φ and n solve:

$$\varphi = H(n) \equiv 1 - [(\sigma-1)(1-\alpha)v(n)/(nv'(n))],$$

and

$$\varphi = G(n) \equiv D\hat{s}^{1/\sigma} n^{(1-\alpha)(1-\sigma)/\sigma}.$$

Note that the function $G(n)$ is upward sloping if $0 < \sigma < 1$, and downward sloping if $\sigma > 1$. Moreover, increases in $\hat{s}$ increase $G(n)$. The properties of $H(n)$ depend on the function $v(\cdot)$. However, concavity of the utility function imposes some restrictions, with the nature of these restrictions dependent on σ. It is straightforward to verify that positive marginal utility of leisure and concavity imply that $v'(n)/(1-\sigma)$ and $v''(n)/(1-\sigma)$ must both be negative. In addition, concavity requires that $(\sigma/(\sigma-1))v''(n)v(n) - (v'(n))^2 > 0$. To ensure that these conditions hold for all values of σ, we will assume that $v''(n)v(n) - (v'(n))^2 > 0$. These restrictions imply that $H(n)$ is an increasing function of n. Finally note that $H(1) > G(1)$.

We first discuss existence and uniqueness for the two possible ranges of σ. Consider the case $\sigma > 1$. It follows that:

$$\lim_{n\to 0} G(n) = \infty, \quad G(1) = D\hat{s}^{1/\sigma}, \quad \textit{and } G'(n) < 0,$$

and

$$\lim_{n\to 0} H(n) < \infty, \quad H(1) > G(1), \quad \textit{and } H'(n) > 0.$$

It follows that there is a unique intersection. An example is shown in Figure A.1. Consider next the case $0 < \sigma < 1$. In this case, we have:

$$\lim_{n\to 0} G(n) = 0, \quad G(1) = D\hat{s}^{1/\sigma}, \quad \textit{and } G'(n) > 0,$$

and

$$\lim_{n\to 0} H(n) < 0, \quad H(1) > G(1), \quad \textit{and } H'(n) > 0,$$

where the first inequality corresponds to (5). Here, both $H(n)$ and $G(n)$ are upward sloping, and hence, establishing uniqueness requires a separate argument. It is possible to show (details available from the authors) that if $\tilde{n}$ satisfies $G(\tilde{n}) = H(\tilde{n})$, then $H'(\tilde{n}) > G'(\tilde{n})$. Thus, the function H can intersect the function G only from below. This, of course, suffices for uniqueness. Possible $H(n)$ and $G(n)$ functions are displayed in Figure A.2.

***********Figures A.1 and A.2 go about here. *********

In both Figures, we use G^* to denote the function G corresponding to a higher value of $\hat{s}^{1/\sigma}$. Thus, it follows that increases in $\hat{s}^{1/\sigma}$ increase both hours worked (the utilization rate of human capital), n, and the fraction of income saved, φ. It is straightforward to calculate the growth rate of output. It is given by:

$$y_{t+1}/y_t \equiv \gamma_{t+1} = s_{t+1}A^* n^{1-a}\varphi = s_{t+1}\gamma.$$

Thus, the average growth rate, γ, is simply $A^* n^{1-\alpha}\varphi$. It follows that growth rates are increasing in $\hat{s}$.

Let $\hat{s}(\theta)$ be given by $\hat{s}(\theta) = \int_S (1+\varepsilon)^{(1-\sigma)}\mu(d\varepsilon)$. Since the function $(1+\varepsilon)^{1-\sigma}$ is concave for $0 < \sigma < 1$ and convex for $\sigma > 1$, it follows that if $0 < \sigma < 1$, $\hat{s}(\theta)$ is increasing in θ, and if $\sigma > 1$, $\hat{s}(\theta)$ is decreasing in θ. This, in turn, implies that (φ, n, γ) are decreasing in θ when $0 < \sigma < 1$, and increasing if $\sigma > 1$. From, $\gamma_{t+1} = s_{t+1}\gamma$, it follows that:

$$\sigma_\gamma = \gamma\sigma_s,$$

where σ_s is the standard deviation of the shock, s_t. Thus:

$$\sigma_\gamma/\sigma_s = \gamma,$$

and our claims follow from the properties of γ.

Finally, consider the case $\sigma = 1$. The first order conditions are satisfied with $\varphi = \beta$, and n as the unique solution to:

$$nv'(n) = (\alpha - 1)/(1-\beta).$$

It is clear that, in this case, the key elements of the equilibrium are independent of θ. ■

A.3 Derivation of the First Order Conditions for the Model of Section 3

The Euler equations for an interior solution are given by:

$$u_c(t) = E_t\{u_c(t+1)[1-\delta + F_k(t+1)]\}, \tag{11}$$

and

$$u_c(t) = E_t\{u_c(t+1)[1-\delta + n_{t+1}F_z(t+1)]\}, \tag{12}$$

where u_c is the partial derivative of $u(\cdot)$ with respect to c and F_k and F_z are the partial derivatives of $F(\cdot)$ with respect to capital and effective labor.

For the Cobb-Douglas form, (11) and (12) can be combined to yield:

$$E_t\{u_c(t+1)[\alpha F(t+1)/k_{t+1} - (1-\alpha)F(t+1)/h_{t+1}]\} = 0.$$

It follows that in any interior equilibrium, we must have that $h_t/k_t = (1-\alpha)/\alpha$ for all t. This is an important property of the specification of a Cobb-Douglas production function with equal depreciation rates: the human-physical capital ratio is independent of the level of employment and the productivity shock.

Given this, and setting $A^* = (1-\alpha)^{1-\alpha}\alpha^\alpha$, it follows that:

$$c_t = k_t[s_t A^* n_t^{1-\alpha}((1-n_t)/n_t)((1-\alpha)/\alpha\psi)] \equiv k_t g_1(s_t, n_t).$$

Using this, we obtain:

$$k_{t+1} = k_t \left[s_t A^* n_t^{1-\alpha} \left(1 - \frac{1-\alpha}{\psi} \frac{1-n_t}{n_t} \right) + 1 - \delta \right] \equiv k_t g_2(s_t, n_t).$$

Finally, after substitution, the relevant Euler equation becomes:

$$\begin{aligned}[g_1(s_t,n_t)(1-n_t)^\psi]^{-\sigma}(1-n_t)^\psi &= \beta \int_S \left\{ \left[g_2(s_t,n_t) g_1(s_{t+1},n_{t+1})(1-n_{t+1})^\psi \right]^{-\sigma} \times \right. \\ & \left. (1-n_{t+1})^\psi \left[1 - \delta + s_{t+1} A^* (n_{t+1})^{1-\alpha} \right] \right\} P(s_t, s_{t+1}).\end{aligned}$$

A solution to this equation is a function $n^* : S \to [0,1]$ with $n_t = n^*(s_t)$. Note that given $n^*(\cdot)$, the optimal solution to the planner's problem is given by:

$$\begin{aligned} n_t &= n^*(s_t), \\ k_{t+1} &= k_t g_2(s_t, n^*(s_t)), \\ h_{t+1} &= ((1-\alpha)/\alpha) k_t g_2(s_t, n^*(s_t)), \\ c_t &= k_t g_1(s_t, n^*(s_t)), \end{aligned}$$

which correspond to the equations calculated in Section 3.

Case	σ	ρ	σ_ε	σ_s	$E(\gamma_y)$	$\sigma(\gamma_y)$	$\rho(\gamma_y)$
1	0.90	0.95	0.011	0.035	2.073	2.822	0.180
2	1.00	0.95	0.011	0.035	2.033	2.146	0.155
3(US)	1.07	0.95	0.011	0.035	2.024	1.915	0.143
4	1.50	0.95	0.011	0.035	2.017	1.423	0.106
5	2.00	0.95	0.011	0.035	2.019	1.272	0.088
6	2.50	0.95	0.011	0.035	2.023	1.206	0.078
7	3.00	0.95	0.011	0.035	2.027	1.170	0.071
8	0.90	0.95	0.026	0.035	2.2376	6.705	0.182
9	1.00	0.95	0.026	0.035	2.159	5.087	0.154
10	1.50	0.95	0.026	0.035	2.079	3.369	0.106
11	2.00	0.95	0.026	0.035	2.095	3.010	0.088
12	2.50	0.95	0.026	0.035	2.117	2.854	0.078
13	3.00	0.95	0.026	0.035	2.139	2.769	0.072
PWT mean	-	-	-	-	2.01	5.33	0.138
PWT median	-	-	-	-	2.11	4.51	0.143
PWT quartile 1	-	-	-	-	0.93	3.35	-0.012
PWT quartile 3	-	-	-	-	3.04	6.73	0.290

Table 5: The effect of changing σ_ϵ and σ on growth. Note: The column labeled $E\left(\gamma_y\right)$ gives the average growth rate, $\sigma(\gamma_y)$ the standard deviation of the growth rate, and $\rho(\gamma_y)$ the autocorrelation of the growth rate. The rows correspond to model simulations with parameter values listed in columns 2 through 5, as well as the Penn World Table (PWT) dataset.

A.4 Additional Simulation Results

Case	γ_{ss}	ρ	σ_ε	σ_s	$E(\gamma_y)$	$\sigma(\gamma_y)$	$\rho(\gamma_y)$
1	1.00	0.95	0.011	0.035	0.024	1.902	0.104
2	1.00	0.95	0.019	0.061	0.066	3.290	0.104
3	1.00	0.95	0.038	0.122	0.249	6.618	0.102
4(US)	1.02	0.95	0.011	0.035	2.024	1.915	0.143
5	1.02	0.95	0.019	0.061	2.065	3.310	0.143
6	1.02	0.95	0.038	0.122	2.242	6.656	0.141
7	1.04	0.95	0.011	0.035	4.047	1.934	0.188
8	1.04	0.95	0.019	0.061	4.102	3.344	0.188
9	1.04	0.95	0.038	0.122	4.311	6.721	0.187
PWT mean	-	-	-	-	2.01	5.33	0.138
PWT median	-	-	-	-	2.11	4.51	0.143
PWT quartile 1	-	-	-	-	0.93	3.35	-0.012
PWT quartile 3	-	-	-	-	3.04	6.73	0.290

Table 6: The effect of changing γ_{ss}, the callibrated, non-stochastic steady state growth rate on the distribution of growth rates, $\sigma = 1.07$. Note: The column labeled $E\left(\gamma_y\right)$ gives the average growth rate, $\sigma(\gamma_y)$ the standard deviation of the growth rate, and $\rho(\gamma_y)$ the autocorrelation of the growth rate. The rows correspond to model simulations with parameter values listed in columns 2 through 5, as well as the Penn World Table (PWT) dataset.

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Figure 1: Curvature and Mean Growth

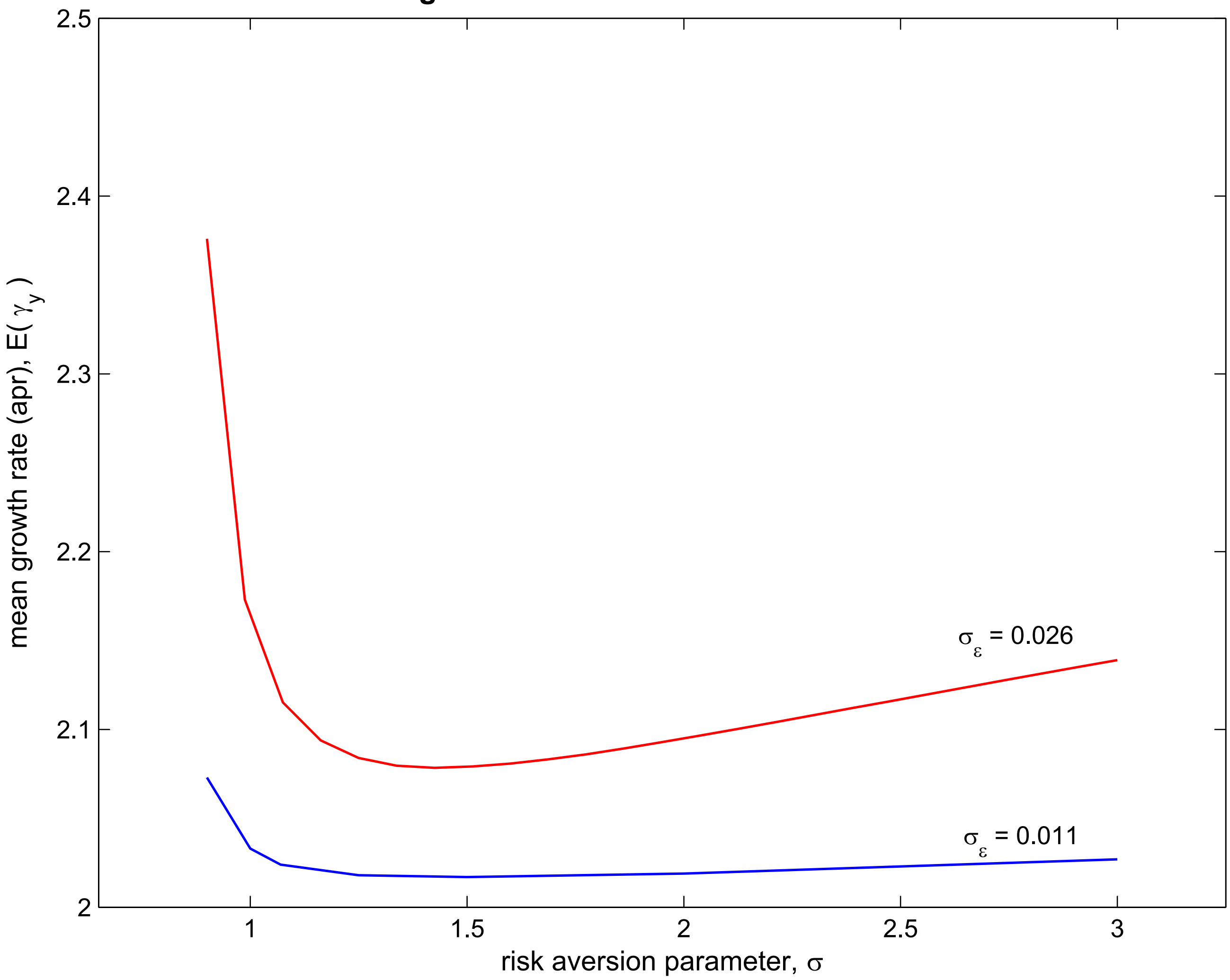

Figure A.1

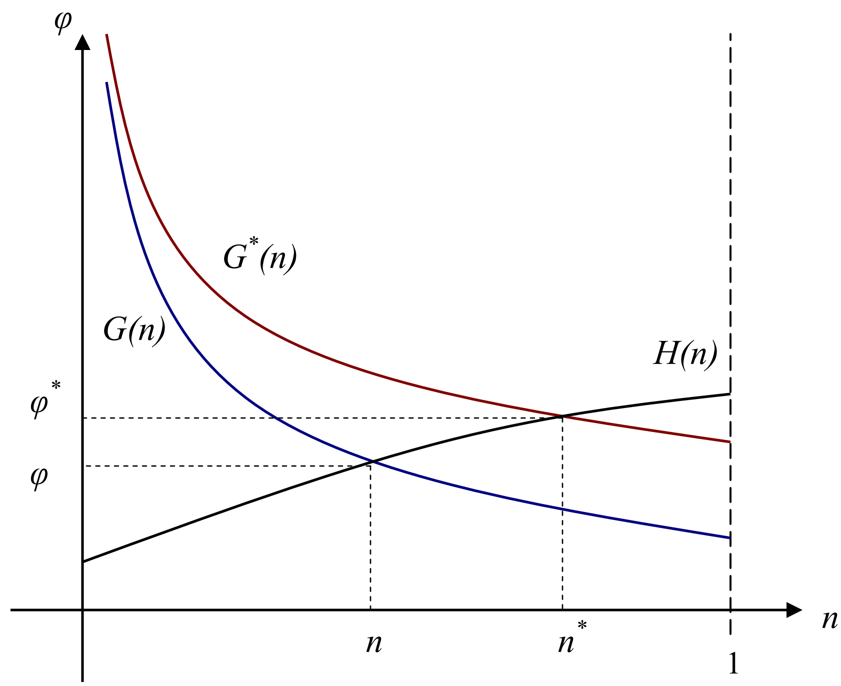

Figure A.2