SPECIAL PROBLEMS ARISING IN THE STUDY OF ECONOMIES WITH INFINITELY MANY COMMODITIES

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1. Introduction

In recent years, researchers in economics and related fields have become increasingly interested in social structures in which the natural choice sets for individual agents are infinite dimensional. Examples include:

(i) The use of L_∞ and ℓ_∞ to model economies with continuous time with either finite and infinite horizon, infinite horizon discrete time, and uncertainty with an infinite state space (see Bewley [8] and Brown and Lewis [11]).

(ii) The use of ca(T) where T is a compact metric space to model economies in which goods with a continuum of characteristics or qualities are available (see Mas-Colell [35] and Jones [29] and [30]).

(iii) The use of L_2 and Martingale theory to model certain economies with financial instruments under uncertainty (see Harrison and Kreps [27] and Duffie and Huang [21]).

The infinite dimensional character of these economies creates several technical problems not met in the standard finite dimensional treatments with which economists are familiar (see Debreu [18]). The purpose of this paper is to discuss the effects of these problems on the classical results of economic theory. Although attention will be restricted to problems arising in a price-taking context, it is to be expected that similar problems will arise in more general settings for similar reasons (e.g., the interplay between compactness and continuity in the choice of topology).

We will adopt Debreu [18] as our standard for the results of economic theory. Accordingly, the results we will be interested in are:

(a) The existence and continuity of supply.

(b) The existence and continuity of demand.

(c) The existence of competitive equililbrium.

(d) The existence of Pareto optimal allocations.

(e) The Pareto optimality of competitive equilbria.

(f) The fact that Pareto optimal allocations can be supported as competitive equilibria after suitable redistribution of the ownership of initial resources and firms.

The strategy that we will adopt in this paper is to present and analyze a series of examples. The nature of each of the examples is that it provides a counterexample to one or more of the most direct translations of the above results from the finite dimensional to the infinite dimensional setting. The point of this exercise is not to say that these problems are unsolvable. Indeed, an active area of current research is in showing just how these problems can be solved. Rather, the point is to show that the extent to which these problems can be solved is limited. Hopefully, the examples will help us understand exactly what these limitations are. (Note that many of the problems can be solved for the examples presented here through a clever choice of the consumption and/or price space. This does not seem like a useful observation, however, since these are usually regarded as primitives in the mathematical statements that the desired results always boil down to.)

There are, of course, many results from mathematical economics other than those listed above which one would like to extend to an infinite dimensional setting but which will not be discussed here. These include the representation of preference orderings by utility functions, the continuity of the equilibrium correspondence, the existence of equilibrium with infinitely many consumers, the relationship between Nash equilibria of games with many players and competitive equilibria, the equivalence of the core of an economy and its competitive equilibria with infinitely many consumers, and the genericity of the local uniqueness of competitive equilibria. Some of the literature concerning these issues is briefly mentioned in section 3.

The remainder of this paper is organized as follows. In section 2, our notation and few key mathematical preliminaries are introduced along with the collection of examples. In section 3 a brief summary of the existing literature and results for infinite dimensional economies is presented.

2. Notation and Examples

2.1 Notation and Mathematical Preliminaries

We will follow the notation of Debreu [18] as much as is possible.

Throughout, L is a locally convex topological vector space with topology τ. It will always be assumed that τ is Hausdorff. (See Dunford and Schwartz [23] or Schaefer [41] for details on topological vector spaces.)

Consumers will be indexed by i when necessary and producers will be indexed by j.

Consumption sets will be denoted by X_i and production sets will be denoted by Y_j. Typical elements will be denoted by x and y, respectively. Subscripts will be dropped when they are not needed.

Following Debreu [16], prices will be assumed by lie in the topological dual of L, $L' = (L,\tau)'$ and will be denoted by p. Note that there are typically many topologies consistent with L' being the dual of L. Further, given L, many different choices of L' are possible (τ must be adjusted accordingly). Of course, these alternative choices give rise to different collections of compact subsets, continuous preference orderings and closed production sets. Thus, the choice of both L' and the topology for L will be an important consideration in what follows.

In all of the examples we will consider, L will have a natural order structure. To preserve the similarity between the results in Debreu [18] and the examples to be considered here, attention will be restricted to the case where $X_i = L_+$, the nonnegative elements of L. Similarly, L'_+ will denote the nonnegative elements of L'. Finally, $L'_{++} = \{p \in L'_+ | x \in L_+, x \neq 0 \Rightarrow px > 0\}$.

As is usual, $\succsim_i \subset L_+ \times L_+$ will denote the agent's preferences. Throughout, we will assume that the $\succsim_i$ are complete, transitive, reflexive, convex, and weakly monotone. Agents' endowments will be denoted by ω_i. We will always assume that $\omega_i \in X_i$ for all i.

If L is a topological vector space and L' is a vector space of linear functionals on L, $\sigma(L,L')$ is the weakest topology on L such that each element of L' is continuous. Note that $\sigma(L,L')$ is Hausdorff as long as L' separates the points of L.

Following Hildenbrand [28], recall that a correspondence ϕ from the topological space S to the topological space T is upper hemicontinuous (u.h.c.) if for every open set $G \subset T$, $\{s \in S | \phi(s) \subset G\}$ is an open subset of S. (Equivalently, for every closed set $F \subset T$, $\{s \in S | \phi(s) \cap F \neq \emptyset\}$ is closed in S.)

In many of the examples considered below the space and the topology will be such that (L,τ) is not metrizeable. Normally, this situation requires the use of generalized sequences or nets (see Kelly [33] for definitions and a discussion) for a discussion of topological properties rather than just sequences. For example, in a general topological space, it does not follow that a set is closed if it contains the limit point of all its convergent sequences. That is, this property is a necessary but not sufficient one for the set under consideration to be closed. It does follow, however, that to conclude that a set is not closed it is sufficient to produce a sequence in that set which converges to a point outside of it. Since we are solely concerned with the construction of counterexamples, this will be an extremely useful fact in what follows.

To economize on notation, unless explicitly stated otherwise, L_p will always mean $L_p([0,1],\underline{F},\lambda)$ where $\underline{F}$ is the Borel subsets of $[0,1]$ and λ is Lebesgue measure on $[0,1]$.

We now turn to the examples. To simplify the presentation, we will deviate somewhat from the order listed in section 1.

2.2 Properties of Demand

There are two properties of demand that are of interest here. These are nonemptiness and continuity.

For $(p,\omega) \in L'_+ \times L_+$ define $\gamma_i(p,\omega)$ by $\gamma_i(p,\omega) = \{x \in L_+ | p \cdot x = p \cdot \omega$ and $x' \in L_+$ and $x' \succ_i x \Rightarrow p \cdot x' > p \cdot x\}$. When $L = R^k$ for some k, one can show (Debreu [18], 4.10, (10)):

(A) If $\succsim_i$ is continuous, γ_i is nonempty valued and upper hemi-continuous at $(p,\omega) \in L'_{++} \times L_+$.

First, we consider the nonempty valuedness of γ. It is straightforward to show that if $\succsim$ is τ-upper semicontinuous $(\{x' \in L_+ | x' \succsim x\}$ is τ-closed for all $x)$ and the budget set is τ-compact, then γ is nonempty. The compactness of the budget set is quite difficult to obtain even in what seems to be very reasonable circumstances. This can be seen in Example 1.

Example 1. Let $L = L_\infty$, $L' = L_1$, $\tau = \sigma(L_\infty, L_1)$. Suppose $\omega = x(t)$ where $x(t) \equiv 1$. Two examples of preferences both given by utility functions will be considered:

$$U_1(x) = \int_0^1 tx(t)dt$$

and

$$U_2(x) = \int_0^1 u(\int_t^1 x(s)ds)dt$$

where u is continuous, strictly increasing, strictly concave, and $u(0) = 0$.

It can be shown that both U_1 and U_2 are τ continuous. Each of the two utility functions has relative advantages. The first is linear in x which will make the point more transparent, but it is only weakly convex. The second is slightly more complex but has the advantage that it gives rise to strictly convex preferences.

Let $p(t) \equiv 1$. Then, $p \in L'_{++}$, but it is easy to see that both $\gamma_1(p,\omega)$ and $\gamma_2(p,\omega)$ are empty. This is because both sets of preferences prefer having consumption "piled up" at 1 when prices are constant across goods. In essence, demand for both of these agents is δ_1, where δ_t is the Dirac measure at t, when $p(t) \equiv 1$. Unfortunately, this possibility has been ruled out by the choice of L_∞ as the consumption set.

However, note that this consideration need not cause problems with the existence of equilibrium. In fact, economies with preferences such as these are covered by the existence result in Bewley [8]. Thus, there are equilibria with price systems in L_1 (in fact, in $C[0,1]$) for the one consumer exchange economies with the preferences and endowments given above. Essentially, prices adjust away from $p(t) \equiv 1$ so that demand does lie in L_+. Thus, the existence of equilibrium does not require that demand be nonempty for all prices. Rather it need only be true that prices exist such that both demand exists and markets are cleared.

Next is a discussion of the continuity properties of demand. This is a more difficult problem than the existence of demand since the topologies for both L and L' will have to be selected. One of the most fundamental problems that must be faced is that the map $\bullet: L' \times L \to \mathbb{R}$: $(p,x) \to p \cdot x$ is not jointly continuous for some choices of the topologies τ and τ' on L and L'.

In the examples we will consider, ω will be held fixed.

Let $ca(T)$ and $C(T)$ be the countably additive measures on T and the continuous real valued functions on the topological space T.

<u>Example 2</u>. Let $L = ca[0,1]$ and let τ be the variation norm topology on L (this is essentially setwise convergence of measures). Let L' be the norm dual of L. Note that although no useful characterization of L' exists, it follows that $C[0,1] \subset L'$.

For $x \in L_+$, define $U(x) = \int 1dx = x[0,1]$. Let

$$p^n(t) = \begin{cases} 2 - \dfrac{2n}{n-2}\, t & \text{for } 0 \le t < 1/2 - 1/n, \\ 2 - \dfrac{2n}{n+2} + \dfrac{2n}{n+2}\, t & 1/2 - 1/n \le t \le 1. \end{cases}$$

See Figure 1.

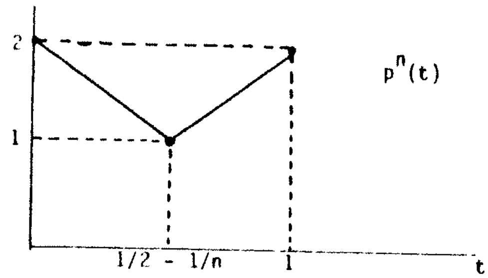

Figure 1

Then, $\|p^n - p\| \to 0$ where

$$p(t) = \begin{cases} 2 - 2t & \text{for } 0 \le t \le 1/2, \\ 2t & \text{for } 1/2 \le t \le 1. \end{cases}$$

Furthermore, when $\omega = \delta_0$, $p^n \cdot \omega = p \cdot \omega = 2$ for all n. Yet $\gamma(p^n,\omega) = 2\delta_{1/2 - 1/n}$ and $\gamma(p,\omega) = 2\delta_{1/2}$. Thus, $\|\gamma(p^n,\omega) - \gamma(p,\omega)\| \not\to 0$ so that γ is not norm to norm continuous. Note that γ: $L'_{++} \to L_+$ is $\sigma(L,L') \times \|\cdot\|$ to $\sigma(L,L')$ continuous, however.

At first sight one might think that the problem is that one is asking too much by putting the norm topology on L in this example. In fact, γ is not even nonempty for all $p \in L'_{++}$ (e.g., $p(t) = t + 1$ for $t > 0$, $p(0) = 2$). Indeed, as noted above, if one sets $\tau = \sigma(L,C[0,1])$ and $L' = C[0,1]$, τ' = the norm topology on $C[0,1]$, γ is nonempty and upper hemicontinuous on L'_{++}. It is important that the topology on L' be the norm topology, however, as the next example shows.

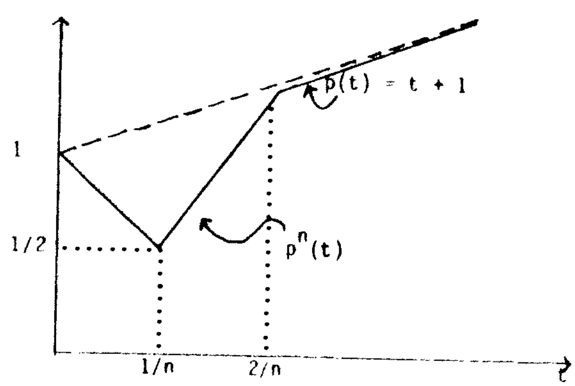

Figure 2

Example 3. Let $L = ca[0,1]$, $L' = C[0,1]$, $\tau = \sigma(L,L')$ and $\tau' = \sigma(L',L)$. As in Example 2, the utility function is given by $U(x) = x([0,1])$ and the endowment is $\omega = \delta_0$. Consider the sequence p^n, p as given in Figure 2, above. Then, using IV.6.4 of Dunford and Schwarz [23], it follows that $p^n \to p$ $\sigma(L',L)$. Further, $p^n \cdot \omega = p \cdot \omega = 1$ for all n. Yet $\gamma(p^n,\omega) = 2\delta_{1/n}$ and $\gamma(p,\omega) = \delta_0$ whence $\gamma(p^n,\omega) \not\to \gamma(p,\omega)$ in the $\sigma(L,L')$ topology.

Thus, the problem of discontinuities of demand do not revolve solely around the choice of topology for L. Given this, a natural question to ask is whether there always exists a choice of topologies for L and L' such that γ is continuous. To see that this is not true in general, it is sufficient to given an example in which τ' is the strongest topology on L' consistent with (L',L) and τ is the weakest topology on L consistent with (L,L').

Define χ_A to be the characteristic function of the set A, $\chi_A(t) = 1$ if $t \in A$, 0 otherwise.

Example 4. Let $L = L' = L_2$, τ' = the L_2 norm topology, and $\tau = \sigma(L_2,L_2)$. As above, let the utility function be given by $U(x) = \int_0^1 x(t)dt$ for $x \in L_+$ and let $\omega = 2\chi_{[1/2,1]}$.

Define p^n and p by

$$p^n(t) = 1/2\chi_{[0,1/n)} + \chi_{[1/n,1]}$$
$$p(t) = \chi_{[0,1]} .$$

Then, $p^n \cdot \omega = p \cdot \omega = 1$ for all $n > 2$ and $\|p^n - p\|_2 \to 0$. Yet, $x^n \cdot \chi_{[0,1]} = 2$ for all $x^n \in \gamma(p^n,\omega)$ and $x \cdot \chi_{[0,1]} = 1$ for all $x \in \gamma(p,\omega)$. Thus, γ is not τ' to τ upper hemicontinuous since $\{x \in L_+ | 0 < x \cdot \chi_{[0,1]} < 1\frac{1}{2}\}$ is a τ-open subset of L_+ containing $\gamma(p,\omega)$ but having null intersection with $\gamma(p^n,\omega)$ for all n.

Note that this example uses the unboundedness of the budget set in a very important way. In fact, this is crucial since it can be shown that the truncated demand correspondence (obtained by truncating the budget set) is norm to $\sigma(L,L')$ continuous in general. This fact is not of much use as far as proving existence is concerned, however, since the natural set of prices is not norm compact.

2.3 Properties of Supply

Next is a discussion of the supply correspondence.

For $p \in L'_+$ define $\eta(p) = \{y \in Y | p \cdot y \geq p \cdot y' \text{ for all } y' \in Y\}$.

When $L = \mathbb{R}^k$, one can show (Debreu [18], 3.5, (3)):

(B) If Y is compact, $\eta(p)$ is nonempty and upper hemicontinuous on L'_+.

It is straightforward to show that if Y is compact in any topology compatible with (L,L'), $\eta(p)$ is nonempty for all $p \in L'_+$. However, notice that the compactness of Y is quite strong here. That is, in the finite dimensional case, for most economies of interest, $(Y + \{\omega\}) \cap L_+$ is bounded and hence even if Y itself is not compact, it can, under relatively weak conditions, be replaced by another production set $\hat{Y}$ such that the equilibria are not altered and Y is compact. This fact does not hold in general when L is infinite dimensional as the following example demonstrates.

Example 5. Let $L = L_\infty \times R$, $L' = L_1 \times R$ and $\tau = \sigma(L_\infty, L_1) \times d)$ where d is the usual Euclidean topology on R. Let

$$Y = \{(y_1, y_2) \in L_\infty \times \mathbb{R} | y_2 \leq 0 \quad \text{and} \quad \int_0^1 y_1(t)dt + y_2 \leq 0\}.$$

Suppose that there is one consumer who has an endowment given by $\omega = (0,r)$ where $r > 0$. Then, it follows that $x^n = (nr\chi_{[0,1/n]}, 0) \in (Y + \{\omega\}) \cap L_+$ for all n and yet $\|x^n\|_\infty \to \infty$.

One can replace this production set by one which is norm bounded as a partial solution to this problem, but note that there are situations in which this will change the economic substance of the problem (see the example in section 2.5, below).

Next the continuity properties of $\eta(\cdot)$ will be considered. We will be brief since the examples closely parallel the discussion of continuity of demand given in section 2.2.

Example 6. As in Example 3, let $L = ca[0,1]$, $L' = C[0,1]$, $\tau = \sigma(L,L')$ and $\tau' = \sigma(L',L)$. Consider the production set

$$Y = \{y \in L | y(1) \geq 0 \ , \ B \subset [0,1/2] \Rightarrow y(B) \leq 0 \ ,$$
$$y(1) \leq -3y[0,1/2] \text{ and } \|y\| \leq 4\} \ .$$

Then 1 is the only output of the firm and all commodities between 0 and 1/2 (inclusive) are inputs. Consider the sequence of prices as defined in Example 3 (Figure 2). It is easy to see that Y is τ-compact and that for n finite, the unique profit maximizing decision is given by

$$y^n = \delta_1 - 3\delta_{1/n} \ .$$

However, the unique profit maximizing decision when prices are p is zero. Thus, η is not τ' to τ continuous.

As in the case of demand, it can be shown that if Y is $\sigma(L,L')$ compact and L is the Banach space dual of L', η is norm to $\sigma(L,L')$ continuous. Again, it should be emphasized that this is not a useful fact for the proof of existence of equilibrium since the natural price space is not in general norm compact.

2.4 Pareto Optimality of Equilibrium

There is little to be said on this topic. As has been shown by Debreu [16] in his classic paper on economies with infinitely many commodities, competitive equilibria are Pareto optimal under very general circumstances. In fact, Debreu shows that this result holds in our framework as long as preferences are non-satiated.

It follows from this fact that if the economy has no equilibria it has no optima and vice versa. This observation is an important one as far as the construction of examples is concerned. It will be used repeatedly below.

Notice, however, that Debreu's result depends critically upon the assumption of finitely many agents. A natural extension of the Arrow-Debreu model which becomes possible when infinitely many commodities are allowed is to include infinitely many atomic (in the measure theoretic sense) consumers as well. Equilibria need not be optimal in this setting as Samuelson [40] has shown in the context of the overlapping generations model.

2.5 Existence of Pareto Optima

In this section we will consider the failure of the existence of Pareto optima. In this regard, it can be shown (given the assumptions that $X_i = L_+$ and $\omega_i \in X_i$) (Debreu [18], 6.2, (1)):

(C) If $L = \mathbb{R}^k$ for some k, Y is closed and convex, $Y \cap L_+ = \{0\}$, and the $\succsim_i$ are continuous, a Pareto optimum exists.

Since (C) is institution independent (i.e., there are no markets) it should not be surprising that, roughly speaking, the only types of failures possible are due to a lack of compactness of the set of attainable states for the economy. Two examples will be considered. The first has one consumer with production and the second has two consumers with no production.

Example 7. In this example, we will combine the preferences given in Example 1 with the production set given in Example 5. Let $L = L_\infty \times R$ and let $\tau = \sigma(L_\infty, L_1) \times d$. Define Y by

$$Y = \{(y(t),y) \in L \mid y(t) \geq 0,\ y \leq 0 \text{ and } \textstyle\int_0^1 y(t)dt \leq -y\}.$$

Consider a one-consumer world with endowment given by $\omega = (x(t),1)$ where $x(t) \equiv 0$ and preferences are given by

$$V_1(x(t),x) = \int_0^1 tx(t)dt + x/2$$

or

$$V_2(x(t),x) = \int_0^1 u(\textstyle\int_t x(s)ds)dt + v(x)$$

where u and v are strictly increasing, strictly concave and continuously differentiable with $u'(0) > v'(1)$.

It is easy to see that some production should take place in this economy, yet that given any allowable production plan the consumer's welfare can be improved by leaving the input level the same but increasing the index of the output.

The problem here is that $(Y + \{\omega\}) \cap L_+$ is not bounded and hence it is not compact. This problem can be solved by replacing Y by

$$\hat{Y} = \{y \in Y \mid \|y(t)\|_\infty + |y| \le K\}.$$

However, note that this significantly changes the feasible allocations and that the constraint, $\|y(t)\|_\infty + |y| \le K$, will be binding at the optimum.

<u>Example 8</u>. Let $L = C[0,1]$, $L' = ca[0,1]$ and $\tau = \sigma(L,L')$.

Consider two consumers with preferences given by

$$U_1(x) = \int_0^1 tx(t)dt$$

and

$$U_2(x) = \int_0^1 (1 - t)x(t)dt \ .$$

Suppose that initial endowments are given by $\omega_1 = \omega_2 = 1/2\chi_{[0,1]} \in L$. It is easy to see that no nontrivial optima exist (i.e., optima in which neither consumer gets 0) for this economy. This follows because consumer 1 should get all commodities to the right and consumer 2 should get all commodities to the left but there is no decomposition of the aggregate endowment as the sum of two <u>continuous</u> functions with these properties. This example will be used again below.

2.6 <u>Supportability of Optima by Price System</u>

This section deals with the problem of decentralizing a given Pareto optimal allocation through the use of a price system. For this section, the disucssion will be limited to the exchange case. For this case, the following result holds (Debreu [18], 6.4., (1)).

(D) If $L = \mathbb{R}^k$, k finite, the $\succsim_i$ are continuous and convex, and x_i^* is an optimum such that $\succsim_i$ is nonsatiated at x_i^* for some i, there is a $p^* \ne 0$ supporting x_i^*.

Two examples will be examined -- one with one consumer and one with two consumers. The first example shows how the fact that L_+ has an empty interior in many examples can cause problems. The second arises essentially due to a noncompactness problem in the price space.

Example 9. Let $L = L' = \ell_2$, the square summable sequences, and $\tau = \sigma(\ell_2, \ell_2)$. Define U by

$$U(x) = \sum_{i=1}^{\infty} u(x(i);i)$$

where

$$u(s;i) = i^{-2}(1 - e^{-i^2 x}), \text{ for } i = 1,2,\ldots$$

See Figure 3.

Finally, define ω by $\omega(i) = 1/i^2$. It can be shown that U is τ-continuous. However, there is clearly no price system in ℓ_2 for this economy. That is, the only prices which clear the market are those where $p(i) = u'(\omega(i);i) = e^{-1}$ for all i. However, this sequence clearly is not square summable.

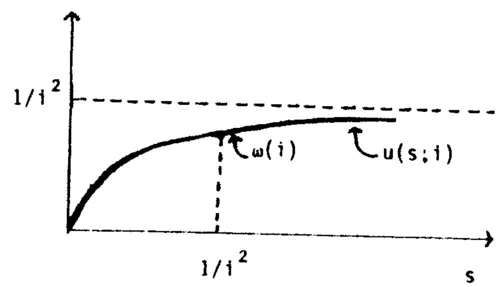

Figure 3

Note that this example is basically the same as both Example 1 in Jones [30] and the example of section IV in Mas-Colell [35]. The basic intuition behind all three of these examples is that prices are measures of relative marginal values and that continuity assumptions on preferences do not, in and of themselves, place sufficiently strong restrictions on the sequence of marginal utilities to guarantee that it lies in the dual of the commodity space.

Next, a two-person example is considered.

Example 10. Let $L = L_\infty$, $L' = C^1(0,1)$, the continuous functions on $(0,1)$ which have continuous and bounded derivatives, $\tau = \sigma(L,L')$. Following Example 8, consider the two-person economy with preferences given by

$$U_1(x(t)) = \int_0^1 tx(t)dt$$

$$U_2(x(t)) = \int_0^1 (1 - t)x(t)dt \ .$$

These preferences are clearly continuous (since they are linear).

Let $\omega = \chi_{(0,1)}$.

It is easy to see that $x_1 = \chi_{(0,1/2)}$ and $x_2 = \omega - x_1$ is an optimal distribution of ω. Yet, there is no price system in L' which supports this allocation. However, there is a price system with continuous prices. This is given by $p(t) = \max(t, 1 - t)$.

Notice that the failure in this example is of a fundamentally different nature than that of the previous one. In Example 9, the problem is that prices cannot be found in L' to support an individual's allocation. This does not cause a problem here. For example, for any $x \in L_+$, the price $p(t) = t$ lies in L'_+ and supports x for consumer 1. Simiarly $p(t) = 1 - t$ supports any allocation in L_+ for consumer 2. The problem in this example lies in the process through which individual prices are aggregated into social prices.

2.7 Existence of Competitive Equilibrium

This section will be brief since most of the examples have already been presented in the previous sections. We will simply note that these examples give rise to problems with the existence of equilibrium.

For completeness, one statement of existence is given for the finite dimensional case (Debreu [18], 5.7, (1)). Let $Y = \Sigma Y_j$.

(E) If $L = \mathbb{R}^k$ for some finite k, the $\succsim_i$ are nonsatiated, continuous and convex, $0 \in Y_j$ for all j, Y is closed and convex, $Y \cap -Y = \{0\}$, $-\mathbb{R}^k_+ \subset Y$ and $\omega_i >> 0$ for all i, a competitive equilibrium exists.

In the present context, there is some ambiguity as to how to translate the statement "$\omega_i >> 0$ for all i". There seem to be two possibilities. The first is that ω_i is in the τ interior of L_+. The problem with this version is that it rules out almost all of the interesting economies from the start since they have positive cones with empty interiors (e.g., the L_p spaces $1 \leq p < \infty$). For this reason, the following translation is used which is equivalent to the statement above that if $L = \mathbb{R}^k$: for all i, $\omega_i \cdot p > 0$ for all $p \in L'_+$ such that $p \neq 0$.

As far as counterexamples to (E) are concerned, note that Example 7 gives two counterexamples in the case with one consumer and production. As was pointed out, the problem arises due to the lack of compactness of $(Y + \{\omega\}) \cap L_+$. There are two potential remedies in this case. The first is to bound Y. If this is done, it is straightforward to check that the resulting economies satisfy the assumptions in Bewley [8]. Again, it should be emphasized that this constraint will be binding in equilibrium. The second remedy is to reinterpret L as $L = ca[0,1] \times R$. (Note that both utility functions are well defined in this case.) Then, letting $L' = C[0,1] \times R$ and $\tau = \sigma(L,L')$, it is easy to check that the resulting economies satisfy the assumptions in Jones [30] with $T = [0,1] \cup \{2\}$ (i.e., $(Y + \{\omega\}) \cap L_+$ is a bounded collection of measures), and hence, equilibria exist with prices in $C[0,1] \times R$.

Example 8 also gives rise to an example in which equilibria do not exist. In this case, letting $Y = -L_+$, we see that although $(Y + \{\omega\}) \cap L_+$ is closed and bounded, it is not compact. This suggests that it might be enought to guarantee somehow that $(Y + \{\omega\}) \cap L_+$ is compact. This cannot work in general, however, as this example shows. Letting $Y = \{0\}$, we see that $(Y + \{\omega\}) \cap L_+ = \{\omega\}$ which is indeed compact, yet still no equilibrium exists. Again, the problem can be easily resolved for the example by reinterpreting L as either L_∞ or $ca[0,1]$ and appealing to the results in Bewley [8] or Jones [30].

Example 9 shows yet another way in which (E) can fail and illustrates one of the main distinctions between finite and infinite dimensional economies. This lies in the difference between the assumptions of separation theorems for convex sets in the two cases. In finite dimensions, the relevant result is Minkowski's theorem while in infinite dimensions, the relevant result is some version of the Hahn-Banach theorem. The primary difference between these two results is that the Hahn-Banach theorem requires that the separated set have a nonempty interior.

This will not be satisfied in general since the upper contour sets of preference orderings are subsets of L_+ which has an empty interior in most examples of interest in economics. Note that this example cannot be adjusted in a simple way so that it fits in the framework of either Bewley [8] (since ω is not bounded away from 0) or Jones [30]. As of yet, no technique has been developed to resolve the type of problem suggested by this example other than to rule it out by considering only economies in which preferences are such that situations such as this do not arise (this is the solution adopted by both Jones [30] and Mas-Colell [36]).

As a final point concerning this example, one might think that a solution to this problem can always be found by expanding the price space. That is, if we made L' large enough, we could get a topology on L which is strong enough so that L_+ has a nonempty interior. The example shows that this cannot work in general since in this case the supporting prices do not even lie in the algebraic dual of L (i.e., $p \cdot x$ is not finite for all $x \in L_+$).

This is, I think, a very important point and as such should be emphasized further. Although we have restricted our attention to the topological dual in our search for equilibrium prices, there is nothing intrinsically topological about the notion of equilibrium. This suggests that the algebraic dual might be the correct place to look for prices. As Example 9 shows, this cannot be a successful endeavor in complete generality.

Example 10 shows yet another way that (E) can fail when L is infinite dimensional. In this case it seems that the chosen price space is not rich enough to allow for the aggregation of individual marginal rates of substitution that is required in equilibrium theory.

Finally, it should be pointed out that in all of the examples that have been considered in this section, the preferences that have been used are $\sigma(L,L')$ continuous. Araujo [2] has shown, by an example, that this assumption is necessary for existence to hold in Bewley's framework. That is, the example given in Araujo shows that if preferences are norm continuous but not $\sigma(L,L')$ upper semicontinuous, Pareto optima need not exist. It follows from this that equilibria need not exist as well.

3. Summary of Existing Results

The purpose of this section is to provide the reader with an outline of the existing results in the field of infinite dimensional economies. I have tried to be as complete and current as possible, but I do not doubt that I have neglected some. Further, for reasons of space, I have given only the barest clues as to the contents of the cited papers. Despite these shortcomings, I hope that the outline will offer some guidance to the reader who is interested in pursuing a particular topic further.

(1) Supportability of Optima by Prices. The first result in this regard is that of Debreu in [16]. In that paper, he shows that if either the production set or the consumption set has nonempty interior, optima can be supported as equilibria. More recently, there are two papers dealing with this subject. First is the paper by Back [4], who treats the problem when $L = L_\infty$. Second is the paper by Mas-Colell [37] who shows that optima can be supported when preferences are derived from utility functions defined on an open set including L_+.

(2) Existence of Equilibrium with Finitely Many Consumers. There is now a wide literature on this subject. The first example is that of Bewley [8] for L_∞. His arguments have been generalized and improved upon by Bojan [10], el Barkuki [6], Brown and Lewis [11], Magill [34], Toussaint [42] and [43], and Florenzano [25].

The case of $L = ca(T)$ where T is a compact metric space is covered in Jones [30]. Mas-Colell [36] gives an existence result for the exchange case when L is a Banach Lattice with predual. This result has been recently generalized by Yannelis and Zame [46] to the case of unordered preferences.

For the case of Banach Lattices with order continuous norm, Brown [13] contains an existence proof based on Kuhn-Tucker techniques. This is a generalization of an argument intiated in Debreu and Hildenbrand [20].

Aliprantis and Brown [1] contains a theorem for the case where L is a Riesz space using demand functions as primitives.

Chichilnisky and Heal [15] establish an existence result for Hilbert spaces assuming that agents have utility functions defined over the whole space.

Finally, two recent results have extended Bewley's proof to Banach spaces with preduals. These are Duffie [22] and Jones [31]. In both cases, it is assumed that the production set has a nonempty interior (Duffie's condition is slightly weaker than this). In addition, in [31], it is shown that Mas-Colell's Banach Lattice result for the exchange case [36], can be obtained as a special case of the result for an economy with a constant returns to scale production set with nonempty interior.

(3) Existence of Continuous Utility Functions. The classic results on this topic are Debreu [17] and Eilenberg [24]. In fact, these results do cover some infinite dimensional economies of interest. More recently, such a theorem has been proven by Mas-Colell [36] for bounded (order) subsets of a Banach Lattice.

(4) Continuity of the Equilibrium Correspondence. For the finite dimensional case, Hildenbrand [28] contains a very general result. To my knowledge, the only result treating this problem for the infinite dimensional case is that reported in Jones [30] where L is the countably additive measures on a compact metric space.

(5) Existence of Equilibrium with Infinitely Many Consumers and Infinitely Many Commodities. The list of references on this topic has also grown considerably in recent years. The first example of this is a result for ℓ_∞ by Bewley [7] for the case of bounded consumption sets. A similar result for hyperfinite exchange economies appears to Brown and Lewis [12].

For the case $L = ca(T)$, where T is a compact metric space, two results have appeared. The first is that of Mas-Colell [35] with bounded consumption sets featuring indivisibilities. The second is that in Jones [29] which treats the model of [35] without bounding consumptions sets or introducing indivisibilities.

A recent result by Ostroy [38] contains a result which covers, for example, the L_p spaces and is based in Vind's [44] approach to modeling consumers.

Finally, there is the literature on the existence of equilibrium in overlapping generations models including the papers by Balasko, Cass, and Shell [5] and Wilson [45].

(6) The Equivalence of the Core and Competitive Equilibria. The classic theorem in this regard for the finite dimensional case is, of course, the result of Aumann [3].

There are several infinite dimensional versions of this result in the literature. The first is due to Gabszewicz [26] and covers the case where $L = C(S)$ the continuous real valued functions on a compact metric space. In addition, Bewley [9] has provided a similar result for the L_∞ case and Mas-Colell [35] contains an equivalence result when $L = ca(T)$ and consumption sets have indivisibilities.

Finally, there is the recent very general result by Ostroy [39].

(7) Local Determinateness of Equilibria. The classic result for the finite dimensional model is due to Debreu [19] and has been extended in many ways. To my knowledge there have been only two results of this type for the infinite dimensional case. These are in the papers by Brown and Genakoplos [14] and Kehoe and Levine [32] and both deal with the overlapping generations model.

There are many other topics in economic theory which are relevant to modeling infinite dimensional economies but which we have neglected here. Most notable is the entire field of capital theory, which almost always deals with infinite horizon models. This is a literature which is sufficiently distinct in aim that the omission seemed appropriate.

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