Econ 8105

MACROECONOMIC THEORY

Class Notes: Part II

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1 <Properties of the Growth Model>

Above, we have seen that the standard growth model has a much richer interpretation than it first appears. In certain cases, it is equivalent to a complex environment with heterogeneity with many consumers, sectors and firms, each of which is taking prices as exogenous to its own decision problem. This does require assumptions however, and they are often quite strong.

What are the benefits of this? All of the properties of the standard model that come from its formulation:

- The characterization of the problem as a Dynamic Programming problem if preferences and production functions satisfy certain key assumptions.

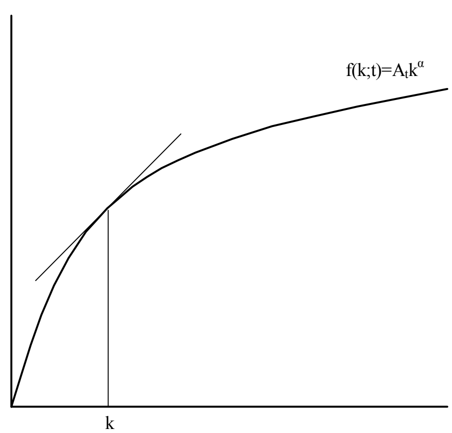
- The characterization of the solution as a first order difference equation in the optimal choice of all of the variables as functions of the current, and only the current value of the state variable k_t.
- Uniqueness of the steady state of the solution.
- Global convergence of the system to the steady state under stronger assumptions.
- The explicit analytic solution of the problem under even stronger assumptions.
- The host of numerical techniques available for the solution of DP's that have been developed over the years.

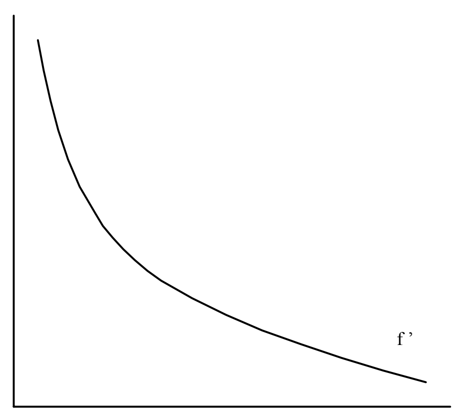
I probably should add more detail to this discussion at some point.

One question that has come up in past discussions is: How fast does the solution to these problems converges to their steady state values? There are two ways to approach this precisely. For global issues, we can look at either numerical simulations, or those special cases where analytic solutions exist. It is also possible to get some idea of the answer to this question by linearizing the system around the steady state to get some idea of what the policy function looks like in a neigborhood of its steady state value.

Typically this convergence is quite rapid. The following discussion is meant to give you some feeling for why.

$$\frac{U'(c_t)}{U'(c_{t+1})} = \beta\left(1-\delta+f_t'\left(k_{t+1}\right)\right)$$
$$\frac{U'(c_t)}{U'(c_{t+1})} = \frac{c_{t+1}}{c_t} = \beta\left(1-\delta+f_t'\left(k_{t+1}\right)\right) \quad \text{this is under log preferences}$$





If $f\ '(k)$ is very high then interest rate is also high. People save more and consume less. This accounts for the fact that the transition is really fast.

$$1+R = 1-\delta+f\ '$$

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1+R$$

Can you use this to get some idea about cross country comparisons?

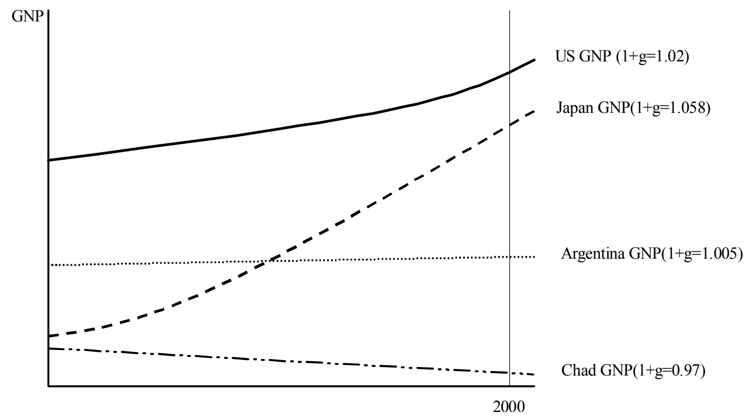


Figure 1:

What happens if you fit in same coefficient for USA into other three countries?

According to the model, we will be able to tell when Japan catches US.

But the implied interest rate differentials are quite extreme? This would imply very high growth rates in consumption where countries are at a lower level of development.

Thus, it would say Japan had higher interest rate in the beginning. Actually it is true that poor countries have higher interest rates than rich ones as a rule, but these differences are not large typically. Also, it is difficult to know to what extent this is due to the fact that k is lower, and to what extent it is related to the fact that investments in poor countries seem to be riskier.

An alternative hypothesis might be that the production functions are different in different countries. It is hard to know what this means. literally, it says that something are possible in countries with high A's, that are NOT possible in countries with low A's. Thus, it would say that poor countries are poor because it is not POSSIBLE for them to be rich. For example, suppose:

$$y_{t(US)} = A_{t(US)} k^{\alpha}_{t(US)}$$

$$y_{t(JAP)} = A_{t(JAP)} k^{\alpha}_{t(JAP)}$$

What part of the differences in y_t should be traced to differences in k_t and what part to differences in A_t?

Note the Main point however: because we are explicit, we can solve the model for different assumptions and generate the time-series of the solution to compare them with actual data. Thus, at least we can have a sensible discussion about it!

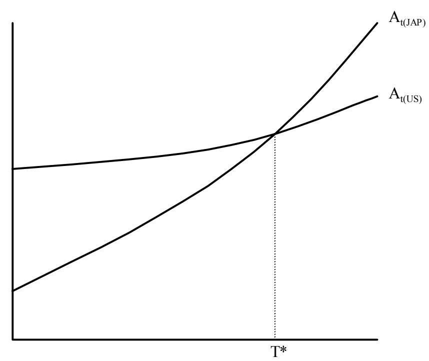


Figure 2:

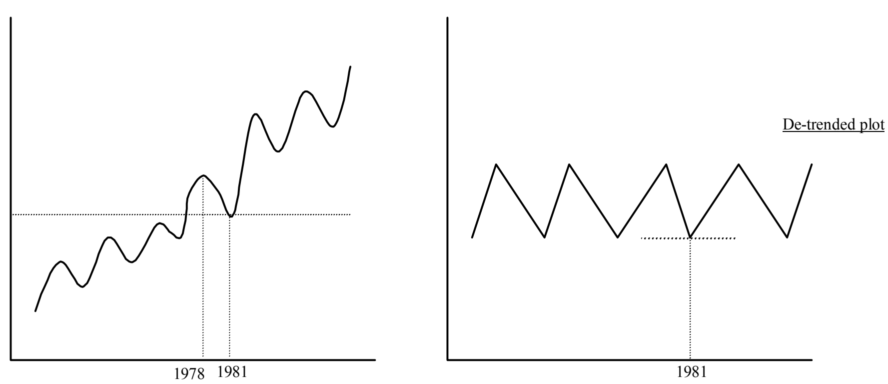


Figure 3:

2 $\boxed{\text{<Policy in the Growth Model>}}$

2.0.1 Remark:

Why do we need a model or a theory at all? Why don't we just look at data to ask the questions that we are interested in? One problem is the difficulty with doing controlled experiments. But even beyond this, (i.e., in fields where they can do controlled experiments), models/theories provide useful devices for organizing our thinking. For this, the theory needs to be sufficiently 'concrete' so that we can solve it explicitly to:

- To see if the theory is right.
- To check 'What if' policy questions. That is to answer the question: What would happen if we did X? When we have no data on situations where X has been done.
- To ask what policy 'should be'– to characterize optimal policy.

2.0.2 Examples:

What if? (Policy changes)

1) We changed the current US tax system to one in which there was a flat rate tax on income from the current progressive system?

2) We changed the way we fund social security payments from the current system to one in which social security accounts are run like individual pension accounts?

3) We changed from an income tax based system to a consumption tax based system?

What effects would these changes have on y_t, x_t, c_t, etc?

What effects would these changes have on $U_i\left(\underset{\sim}{c}, \underset{\sim}{l}\right)$? Would they improve welfare? Would they lessen it? Would they increase utility for some people and lower it for others? If so, who would benefit, who would be hurt? Can we find other changes that might improve everyone's welfare?

3 A Price Taking Model of Equilibrium with Taxes and Spending

In the notes that follow, we will examine the formulation and effects of taxes and spending in our infinite horizon neoclassical growth model. If you haven't seen things like this before, it is probably very useful to you to do some simpler examples as you go along. For example, construct a static model with one consumer and only labor income and using graphs, analyze the effects of changes in labor income tax rates, how this depends on how the revenue is used (e.g., lump-sum rebated vs. spent on purchases of goods and services by the government). Doing a couple of simple examples like this for yourself will greatly help you understand the mechanisms behind the more complex treatment we will develop in what follows.

To adress these issues we'll develop a version of the model CE, price taking model we described above and introduce taxes and government spending to the mix:

We'll want to add taxes:

1) on c_t - τ_{ct}

2) on x_t - τ_{xt}

3) on labor income, $w_t n_t$ - τ_{nt}

4) on capital income, $r_t k_t$ - τ_{kt}

and

5) lump sum transfers - T_t^i

6) spending - g_t

3.0.3 <Definition of TDCE>

A Tax Distorted Competitive Equilibrium CE with taxes and spending is given by the sequences $\tau_{ct}, \tau_{xt}, \tau_{nt}, \tau_{kt}, T_t^i, g_t$ is:

(i) Plans for households $(c_t^i, x_t^i, n_t^i, l_t^i, k_t^i)_{t=0}^{\infty}$

(ii) Plans for firms (assuming there is only one) $(c_t^f, x_t^f, n_t^f, g_t^f, k_t^f)_{t=0}^{\infty}$

(iii) Prices (p_t, r_t, w_t)

such that,

a) Firms and houses are maximizing given prices, taxes, transfers and spending and

b) the usual accounting identities for quantities hold.

<u>Maximization</u>

(HH) $Max\ U_i \left(\underset{\sim}{c}, \underset{\sim}{l} \right)$

s.t.

i)$\sum_{t=0}^{\infty} \left[p_t \left(1+\tau_{ct}\right) c_t^i + p_t \left(1+\tau_{xt}\right) x_t^i\right] \leq \sum_{t=0}^{\infty} \left[\left(1-\tau_{nt}\right) w_t n_t^i + \left(1-\tau_{kt}\right) r_t k_t^i + T_t^i\right]$

ii) $k_{t+1}^i \leq (1-\delta) k_t^i + x_t^i$

iii) $n_t^i + l_t^i \leq \bar{n}_t^i = 1$

and k_0^i is fixed.

(FIRM) $(c_t^f, x_t^f, g_t^f, k_t^f, n_t^f)_{t=0}^{\infty}$ solves

$Max\ p_t \left[c_t^f + x_t^f + g_t^f\right] - r_t k_t^f - w_t n_t^f$

s.t. $c_t^f + x_t^f + g_t^f \leq F_t(k_t^f, n_t^f)$

<u>Markets Clear</u>

i) $\forall t \quad \sum_{i=1}^{I} n_t^i = n_t^f$

ii) $\forall t \quad \sum_{i=1}^{I} k_t^i = k_t^f$

iii) $\forall t \quad \sum_{i=1}^{I}(c_t^i + x_t^i + g_t^i) = F_t(k_t^f, n_t^f)$

<u>The Budget of the Government is Balanced in Present Value</u>

$$\sum_{t=0}^{\infty} \left[p_t \tau_{ct} \left(\sum_{i=1}^{I} c_t^i\right) + p_t \tau_{xt} \left(\sum_{i=1}^{I} x_t^i\right) + \tau_{nt} w_t \left(\sum_{i=1}^{I} n_t^i\right) + \tau_{kt} r_t \left(\sum_{i=1}^{I} k_t^i\right)\right] = \sum_{t=0}^{\infty} \left[\sum_{i=1}^{I} T_t^i + p_t g_t\right]$$

(Revenue side = revenue from consumption tax + revenue from investment tax + revenue from income tax

Expenditure side = lump sum transfers + government expenditure)

3.0.4 Remarks:

1. Note that we have assumed that the tax system is linear– no progressivity/regressivity.

2. We have directly jumped to the assumption that $p_{c_t} = p_{x_t} = p_{g_t} = p_t$. Given our assumption that c, x, and g are perfect substitutes in the output of the firm, this would follow automatically in any equilibrium and in any period in which all three are positive.

3. We have assumed that households are the ones that are responsible for paying the taxes.

4. Note that I have set this up with an infinite horizon BC for both the HH and government and hence free, perfect lending markets are being assumed.

5. What would it mean for τ_{xt} to be negative? Or any of the other taxes?

6. In this formulation, it is assumed that consumers take prices, tax rates, and transfers as given. That is, unaffected by how they make their consumption, savings, labor supply and investment decisions.

7. If $T_t^i < 0$, then it is interpreted as a lump sum tax, if $T_t^i > 0$, then it is a lump sum transfer.

3.0.5 Problems:

1. Show that: If a price system and allocation satisfy everything except government budget balance, it must also be satisified.

2. Set up the problem with sequential BC's for both HH's and the government and show that these two ways are equivalent.

3. Set up the problem with firms paying taxes and show equivalence.

4. How should capital formation be included in this version of the model? Does it matter if the firm or HH does the investment for the properties of equilibrium?

3.1 Ricardian Equivalence

Theorem) Ricardian Equivalence

The timing of the T_t^i is irrelevant. (i.e. same equilibrium prices and allocations)

Proof: Obvious since only the present value of transfers appears in the BC.

3.1.1 Remarks:

1. That is, you can move T_t^i back and forth in time without changing the equilibrium allocations and prices– the only thing that matters is $(\sum_{t=0}^{\infty} T_t^1, \sum_{t=0}^{\infty} T_t^2, \sum_{t=0}^{\infty} T_t^3, ...)$.

2. (Stanley wrote Dirk Krueger next to this remark. I think that what that probably means is that he took the following formalization of the above from Dirks class notes, but I'm not sure.) Take as given a sequence of government spending $(g_t)_{t=0}^{\infty}$ and initial debt B_0. Suppose that allocations c_t^{*i}, prices p_t^* and taxes T_t^i, etc. form an Arrow-Debreu equilibrium. Let $\hat{T}_t^i$ be an arbitrary alternative tax system satisfying $\sum_{t=0}^{\infty} T_t^i = \sum_{t=0}^{\infty} \hat{T}_t^i \ \forall\, i$.Then c_t^{*i}, p_t^* and $\hat{T}_t^i$, etc., form an Arrow-Debreu equilibrium as well.

3. A more subtle version of this same result is due to Barro. This is that it does not matter whether you tax father or son. The idea is that if any redistributive taxation you do across generations will be undone through bequests among the different individuals in the family.

4. What if there were more than one firm in a sector but all firms within a sector had identical CRS production functions in every period. Would

our earlier aggregation results still hold? What if there were more than one sector, but with identical production functions? Would our earlier aggregation results still hold in this formulation with taxes?

5. What if all agents have the same homothetic utility function. Would our aggregation results still hold? What about non-linear tax systems (i.e., progressive or regressive systems)?

6. In some ways, this approach to policy is a bit odd. It is what is known as the 'throw it in the ocean' model of government spending. That is, g does not enter U (e.g., parks or schools), and it does not enter F (e.g., roads or bridges). Of course, it would be better to explicitly include those kinds of considerations in the model. It would also be more difficult! So, this formulation is used as a simple 'starter' version. Unfortunately.. often, no one goes beyond this version! It has kind of funny implications for policy: What is optimal policy under the assumptions made so far?

   i) $g_t = 0 \ \ \forall\ t$

   ii) $\tau_{ct} = \tau_{xt} = \tau_{nt} = \tau_{kt} = 0$ for all t

   iii) any desired redistribution can be done through T_t^i, this follows from the 2nd welfare theorem, i.e. PO$\rightarrow$CE under appropriate transfers. Although given the structure, it's not clear why redistribution would be desireable.

7. For your own sanity in thinking about this, it's probably best to either just think of g_t as being given outside the model– for some reason the government HAS to have g_t in each period. Or, you could think about ways to put g_t directly into either U or F where we implicitly assume that consumer views himself as having no influence on g and takes it given. If for example, g enters the utility function of the consumer in and additively separable way, you can check that you will get exactly the same equilibrium relationships as in the model we have outlined above.

3.2 Examples of Fiscal Policies

The definition of a TDCE allows the model to be solved for 'any' specification of fiscal policy. However, it implicitly assumes that there is an equilibrium. This can't be true in general! For example, suppose spending is positive in every period, but taxes are zero in every period! In that case, there can be no prices.... such that all are maximizing and quantities add up. Thus, the assumption that an equilibrium exists implicitly puts some restriction on the combinations of taxes, transfers and spending that the government is doing. There is no simple way of summarizing what this set of restrictions entails. A more general approach allows spending and transfers by the government to be contingent on (i.e., be functions of) the revenue raised. This in turn depends on what prices are in addition to quantities chosen and tax rates. If this function satisfies Budget Balance by the government at all revenue possibilities, then typically an equilibrium will exist. (This requires some additional assumptions.) An easy way to guarantee this is to have transfers be dependent on tax revenue and spending, so that they always make up the difference between direct tax revenue and spending. Under some further assumptions on g_t this is sufficient to guarantee that an equilibrium will exist.

(*FP1*) What would the behavior of the economy be if $\tau_{c3} = 0.2$ (i.e., a 20% tax on consumption at period 3), $\tau_{ct} = 0 \;\; \forall \, t \neq 3$, $\tau_{xt} = \tau_{kt} = \tau_{nt} = 0 \;\; \forall \; t$, $g_t = 0$, $T_3^i = \tau_{c3} \times c_3^i$? That is, what would happen if we taxed consumption in period 3, and used the revenue to finance lump sum transfers back to the consumer in the same period? (Note, as above, it doesn't matter if it's T_6^i

due to Ricardian Equivalence.)

(*FP2*) Is the TDCE for this economy the same as $\tau_{ct} = \tau_{xt} = \tau_{kt} = \tau_{nt} = 0 \;\; \forall\, t$, $g_t = 0$, $T_t^i = 0$?

That is, is FP1 the same as a fiscal policy where you do nothing?

Answer) No.

(FOC) $\frac{U_{c3}^i}{(1+\tau_{c3})p_t} = \frac{U_{l3}^i}{(1-\tau_{n3})w_t}$

In *FP1*, $\frac{U_{c3}^i}{U_{l3}^i} = \frac{(1+\tau_{c3})p_t}{w_t} = \frac{(1.2)p_t}{w_t} = 1.2 \times \frac{1}{F_{n3}}$

In *FP2*, $\frac{\hat{U}_{c3}^i}{\hat{U}_{l3}^i} = \frac{\hat{p}_t}{\hat{w}_t} = \frac{1}{\hat{F}_{n3}}$

Thus, these cannot be the same since in this case, $F_{n3} = \hat{F}_{n3}$ would hold, and hence $MRS_1 = MRS_2$ would have to hold too. Contradiction $\Longrightarrow\!\Longleftarrow$

That is:

* If you take any stuff from i in one way & give it back in another way, he i doesn't take into account that he gets back the tax.revenue, since he is taking tax rates and transfers as fixed in his maximization problem.

* Thus, if the consumer thought perfectly that $T_t^i = \tau_{ct} \times c_t^i$, then τ_{ct} is irrelevant.

For example, you might want to consider what would happen if instead of giving back the revenue as a lump-sum transfer, what happens if you subsidize leisure in a way that is balanced budget in equilibrium? Or:

(*FP3*) $\tau_{c3} = 0.2$, $\tau_{xt} = \tau_{nt} = \tau_{kt} = 0 \;\; \forall \; t$, $\tau_{ct} = 0 \;\; \forall \; t \neq 3, 4$, τ_{c4} chosen to balance the budget–.$\tau_{c4} c_4^i + \tau_{c3} c_3^i = 0$– i.e., tax in period 3 and subsidize in period 4)

Is the equilibrium allocation same as *FP2* in this case? NO.

$$\frac{U_{c3}^i}{(1+\tau_{c3})p_3} = \frac{U_{c4}^i}{(1+\tau_{c4})p_4}$$

$$\frac{U_{c3}^i}{U_{c4}^i} = \frac{(1+\tau_{c3})p_3}{(1+\tau_{c4})p_4}$$

but $(1 + \tau_{c3}) > 1$ and $(1 + \tau_{c4}) < 1$.

Therefore, $LHS > \frac{p_3}{p_4}$ and hence the allocations must be different.

Question: Are TDCE Pareto Optimal? In general, NO. (Look at the FOC, MRS=MRT needed for PO)

Of course, we need a definition of PO here, but it is the obvious one:

3.2.1 Definition

An allocation $(c_t^i, x_t^i, n_t^i, l_t^i, k_t^i)_{t=0}^{\infty}$, $(c_t^f, x_t^f, n_t^f, g_t^f, k_t^f)_{t=0}^{\infty}$ is PO given $(g_t)_{t=0}^{\infty}$ is there does not exist another feasible allocation $(\hat{c}_t^i, \hat{x}_t^i, \hat{n}_t^i, \hat{l}_t^i, \hat{k}_t^i)_{t=0}^{\infty}$, $(\hat{c}_t^f, \hat{x}_t^f, \hat{n}_t^f, \hat{g}_t^f, \hat{k}_t^f)_{t=0}^{\infty}$ given $(\hat{g}_t)_{t=0}^{\infty}$ such that $U_i\left(\hat{c}_t^i, \hat{l}_t^i\right) \geq U_i\left(c_t^i, l_t^i\right) \;\; \forall \; i$ with strict inequality for at least one i and $\hat{g}_t \geq g_t \;\; \forall \; t$.

As is probably not surprising, there is a tight relationship between PO given g and the use of lump-sum taxes:

3.3 Welfare Theorems Again

<First Welfare Theorem> **Theorem** If $\tau_{ct} = \tau_{xt} = \tau_{nt} = \tau_{kt} = 0 \ \ \forall$ t,(there are no distortionary taxes), then the TDCE allocation is PO. given g_t.

Proof: Obvious

Corollary: Optimal financing of government expenditures is to use lump sum transfers(T) only.

<Second Welfare Theorem> Let $(c_t^i, x_t^i, k_t^i, n_t^i, l_t^i)_{t=0}^{\infty}$, $(c_t^f, x_t^f, g_t^f, k_t^f, n_t^f)_{t=0}^{\infty}$ be PO given $(g_t)_{t=0}^{\infty}$

Then $\exists\ T_t^i$ s.t.

$(c_t^i, x_t^i, n_t^i, l_t^i, k_t^i)_{t=0}^{\infty}$ is TDCE

given the fiscal policy $(g_t)_{t=0}^{\infty}$, $(T_t^i)_{t=0}^{\infty}$, $\tau_{ct} = \tau_{xt} = \tau_{nt} = \tau_{kt} = 0 \ \ \forall\ t$.

Proof: Obvious.

3.3.1 Remarks:

1. This is the analogue of the 1st welfare theorem.

2. (Alice Schoonbroodt) There are also other ways of getting allocations that are like lump sum tax allocations given our set up. For example: suppose $\tau_{ct} = \tau_{xt}$ & $1 + \tau_{ct} = 1 - \tau_{nt} = 1 - \tau_{kt} = \alpha$. Then, this is the same as having lump sum transfers equal to $\frac{T}{\alpha}$.

4 <Solving the Model- Representative Consumer Case>

1. System of equations approach. $\rightarrow$ representative consumer (either all identical or homothetic)

2. Planner's problem equivalence? This won't work in general since TDCE is not PO in general! You can if $\tau = 0$, i.e., you only use transfers to raise revenues; with distortionary taxes, this will typically not work.

   However, sometimes this works!.

Proposition:

(A) Assume that there is a representative consumer and that $\tau_{ct} = \tau_{xt} = \tau_{nt} = \tau_{kt} = 0 \;\; \forall\ t$. (Only using lump sum tax to finance spending). Then, the TDCE allocation solves:

$$Max \;\; U(\underset{\sim}{c}, \underset{\sim}{l})$$

s.t. $c_t + x_t + g_t \leq \hat{F}_t\left(k_t, n_t\right) \;\; \forall\ t$

$$k_{t+1} \leq (1-\delta)\, k_t + x_t$$

$$l_t + n_t \leq 1 \quad\; \forall\ t$$

k_0 fixed,

where

$$\hat{F}_t\left(k_t, n_t\right) = F_t\left(k_t, n_t\right) - g_t, \;\; \forall\ t$$

(B) Assume that there is a representative consumer and $\tau_{ct} = \tau_{xt} = T_t^i = 0 \;\; \forall\ t$, but, $\tau_{nt} = \tau_{kt} = \tau_t > 0 \;\; \forall\ t$, and $p_t g_t = w_t \tau_{nt} n_t + r_t \tau_{kt} k_t \;\; \forall\ t$, i.e., period by period budget balance,

Then the TDCE allocation solves:

$$Max \;\; U(\underset{\sim}{c}, \underset{\sim}{l})$$

s.t. $c_t + x_t \leq (1-\tau_{nt})\, F_t\left(k_t, n_t\right) \;\; \forall\ t$

$$k_{t+1} \leq (1-\delta)\, k_t + x_t$$

$l_t + n_t \leq 1 \quad \forall\, t$

k_0 fixed

Proof: Obvious

Corollary: Under these conditions, the equilibrium allocation is also unique.

Proof: Concave maximization problems have unique solution.

Question) If $g_t = 0$ instead and $T_t = \tau_t \left(w_t n_t + r_t k_t\right)$ (lump sum rebate), is the THM still true?

No! Why not? g takes out real goods and services but transfers don't. It follows that feasibility is messed up.

That is, feasibility in the TDCE is

$c_t + x_t + g_t \equiv c_t + x_t = F_t\left(k_t, n_t\right) \neq \left(1 - \tau_t\right) F_t\left(k_t, n_t\right)$

In the planner's problem, feasibility is:

$c_t + x_t = \left(1 - \tau_t\right) F\left(k_t, n_t; t\right)$

and thus these are not the same.

If the revenue is used to purchase goods and services however, the feasibility constraint in the planner's problem is

$c_t + x_t \leq (1 - \tau_t) F(k_t, n_t; t) \;\; \forall \; t.$

Since $g_t \equiv \tau_t F(k_t, n_t; t)$, this is the same as that in the equilibrium:

$c_t + x_t + g_t \leq F(k_t, n_t; t) \;\; \forall \; t. \Leftrightarrow c_t + x_t + \tau_t F(k_t, n_t; t) \leq F(k_t, n_t; t) \;\; \forall \; t \Leftrightarrow$

$c_t + x_t \leq (1 - \tau_t) F(k_t, n_t; t) \;\; \forall \; t.$

Are there other results like this? A few....

Proposition: Suppose there is a representative consumer, and that:

i) $\tau_{xt} = \tau_{nt} = \tau_{kt} = T_t^i = 0 \quad \forall \;\; t, \, i$ –no transfers,

ii) $p_t g_t = \tau_{ct} p_t c_t \quad \forall \;\; t$ –consumption tax revenue is spent on g_t in every period,

then TDCE solves

$Max \;\; U(\underset{\sim}{c}, \underset{\sim}{l})$

s.t. $x_t \leq F_t(k_t^x, n_t^x) \;\; \forall \; t \;\; (*)$

$(1 + \tau_{ct}) \, c_t \leq F_t(k_t^c, n_t^c) \;\; \forall \; t \;\; (**)$

$k_{t+1} \leq (1 - \delta) \, k_t + x_t$

$n_t^x + n_t^c + l_t \leq 1$

$k_t^x + k_t^c \leq k_t$

k_0 given.

Proof: Sorta obvious.

Corollary: Equilibrium is unique.

$(*)\&(**) \rightarrow (1+\tau_{ct})\, c_t + x_t \leq F_t\,(k_t, n_t) \quad (CRS)$

Many other, related results follow from these characterizations. For example in the inelastic labor supply case, it follows that the system in the TDCE is globally asymptotically stable, with the capital stock converging to the unique level it has under the distorted planner's problem:

$$1 = \beta\left[1 - \delta + (1-\tau)F_k(k^*, 1)\right].$$

That is, every result that we found about the solution to the Planner's Problem version of the single sector growth model holds here as well.

5 First Order Conditions in General Tax Problems

A standard and useful way to analyze the effects of taxes is to examine the FOC's from the firm and consumer problems:

(CP) $Max \quad \sum \beta^t U\,(c_t, 1-n_t)$

s.t. $\sum p_t\left[(1+\tau_{ct})\, c_t + (1+\tau_{xt})\, x_t\right] \leq \sum\left[(1-\tau_{nt})\, w_t n_t + (1-\tau_{kt})\, r_t k_t + T_t\right]$

$\dashrightarrow$ multiplier λ

$$k_{t+1} \leq (1-\delta)\, k_t + x_t \quad \dashrightarrow \quad \text{multiplier } \beta^t \mu_t$$

(FOC)

- w.r.t. c_t $\quad \beta^t U_c(t) - \lambda p_t (1+\tau_{ct}) = 0$

$$\beta^t U_c(t) = \lambda p_t (1+\tau_{ct})$$

$$\lambda p_t = \frac{\beta^t u_c(t)}{(1+\tau_{ct})}$$

$$\rightarrow \frac{\beta^t U_c(t)}{\beta^0 U_c(0)} = \frac{p_t(1+\tau_{ct})}{p_0(1+\tau_{co})} \quad \text{(wlog let } p_0 = 1)$$

Note: if $\tau_{c_t} > \tau_{c_0}$ then people artificially consumes less in period t. $\because$tax distorts people's choice. Without tax, $\frac{MU}{MU} = p_t$.

- w.r.t. n_t $\quad -\beta^t U_l(t) + \lambda (1-\tau_{nt})\, w_t = 0$

$$\beta^t U_l(t) = \lambda (1-\tau_{nt})\, w_t$$

$$\rightarrow \frac{\beta^t U_l(t)}{\beta^t U_c(t)} = \frac{w_t(1-\tau_{nt})}{p_t(1+\tau_{ct})}$$

- w.r.t. x_t $\quad -\lambda p_t (1+\tau_{xt}) + \beta^t \mu_t = 0$

$$\beta^t \mu_t = \lambda p_t (1+\tau_{xt})$$

- w.r.t. k_{t+1} $\quad -\beta^t \mu_t + (1-\delta)\, \beta^{t+1} \mu_{t+1} + \lambda (1-\tau_{kt+1})\, r_{t+1} = 0$

$$\text{or } \lambda p_t \left(1+\tau_{xt}\right) = \left(1-\delta\right)\lambda\left(1+\tau_{x_{t+1}}\right)p_{t+1} + \lambda\left(1-\tau_{kt+1}\right)r_{t+1}$$

$(\star)$

This last condition is called an *Arbitrage condition* it constrains how prices and taxes must move together in any TDCE.

If there were no tax, this would be, $p_t = (1-\delta)\, p_{t+1} + r_{t+1}$

or $\quad p_t = p_{t+1}\left(1-\delta+\frac{r_{t+1}}{p_{t+1}}\right) = p_{t+1}\left(1-\delta+F_k\left(t+1\right)\right)$

and then $\frac{\beta^t U_c(t)}{U_c(0)} = \frac{\beta^{t+1}U_c(t+1)}{U_c(0)}\left(1-\delta+F_k\left(t+1\right)\right)$

thus $(\star)$ is tax-distorted version of Euler equation

Recall that investment is CRS, and hence, $(\star)$ is the condition that implies that the after-tax profits from investment is zero (no quantities involved).

5.0.2 Euler equation for Tax Model

$$\frac{U_c(t)(1+\tau_{xt})}{(1+\tau_{ct})} = \frac{\beta U_c(t+1)}{(1+\tau_{ct+1})}\left[\left(1-\delta\right)\left(1+\tau_{xt+1}\right)+\left(1-\tau_{kt+1}\right)\frac{r_{t+1}}{p_{t+1}}\right]$$

5.0.3 Remark:

This is a necessary condition for an interior solution. Are they sufficient? Need transversality condition. BC holding with equality.

(Firm's problem)

$Max \qquad p_t F_t\left(k_t, n_t\right) - w_t n_t - r_t k_t$

$F_k\left(t\right) = \frac{r_t}{p_t}$

$F_n\left(t\right) = \frac{w_t}{p_t}$

So a TDCE is $\left(p_t, w_t, r_t\right), \left(c_t, x_t, n_t, k_t\right)$ such that

i) $\forall\, t \quad p_t\left(1 + \tau_{ct}\right) = \frac{\beta^t U_c(t)}{U_c(0)};\ \ p_0 = 1$

ii) $F_k\left(t\right) = \frac{r_t}{p_t} \qquad \forall\, t$

iii) $F_n\left(t\right) = \frac{w_t}{p_t} \qquad \forall\, t$

iv) $\frac{U_c(t)}{U_l(t)} = \frac{(1+\tau_{ct})}{(1-\tau_{nt})}\frac{1}{F_n(t)} \qquad \forall\, t$

v) $\frac{U_c(t)(1+\tau_{xt})}{(1+\tau_{ct})} = \frac{\beta U_c(t+1)}{(1+\tau_{ct+1})}\left[\left(1-\delta\right)\left(1+\tau_{xt+1}\right) + \left(1-\tau_{kt+1}\right)\frac{r_{t+1}}{p_{t+1}}\right] \quad \forall\, t$

vi) $c_t + x_t + g_t = F_t\left(k_t, n_t\right) \quad \forall\, t$

vii) $\sum p_t\left[\left(1+\tau_{ct}\right)c_t + \left(1+\tau_{xt}\right)x_t\right] = \sum\left[\left(1-\tau_{nt}\right)w_t n_t + \left(1-\tau_{kt}\right)r_t k_t + T_t\right]$

$\forall\, t$

viii) $k_{t+1} \leq \left(1-\delta\right)k_t + x_t \quad \forall\, t$

This is the system of equations that has to be solved.

What happened to Budget balance by government? As noted above, it automatically follows from the other conditions.

$$\sum_{t=0}^{\infty}\left[p_t\tau_{ct}\left(\sum_{i=1}^{I}c_t^i\right)+p_t\tau_{xt}\left(\sum_{i=1}^{I}x_t^i\right)+\tau_{nt}w_t\left(\sum_{i=1}^{I}n_t^i\right)+\tau_{kt}r_t\left(\sum_{i=1}^{I}k_t^i\right)\right]=$$

$$\sum_{t=0}^{\infty}\left[\sum_{i=1}^{I}T_t^i+p_tg_t\right]$$

5.1 < Steady State with and without taxes >

Suppose $\tau_{ct}\rightarrow\tau_c^\star$

$\tau_{nt}\rightarrow\tau_n^\star$

$\tau_{xt}\rightarrow\tau_x^\star$

$\tau_{kt}\rightarrow\tau_k^\star$

$g_t\rightarrow g^\star$

Can we characterize the steady state?

What should $c^\star, n^\star, k^\star, x^\star$ have to satisfy?

$i')\frac{U_l(c^\star,1-n^\star)}{U_c(c^\star,1-n^\star)}=\frac{(1-\tau_n^\star)}{(1+\tau_c^\star)}F_n\left(k^\star,n^\star\right)$

$ii')\frac{(1+\tau_x^\star)}{(1+\tau_c^\star)}=\frac{\beta}{(1+\tau_c^\star)}\left[\left(1-\delta\right)\left(1+\tau_x^\star\right)+\left(1-\tau_k^\star\right)F_k\left(k^\star,n^\star\right)\right]\iff\left(1+\tau_x^\star\right)=$

$\beta\left[\left(1-\delta\right)\left(1+\tau_x^\star\right)+\left(1-\tau_k^\star\right)F_k\left(k^\star,n^\star\right)\right]$

$iii')$ $c^\star+x^\star+g^\star=F\left(k^\star,n^\star\right)$

$iv')$ $k^\star\leq\left(1-\delta\right)k^\star+x^\star\quad\iff\quad x^\star=\delta k^\star$

Four unknowns $(c^\star, n^\star, k^\star, x^\star)$ and four equations $i'), ii'), iii'), iv')$

Note: $\tau_c^\star$ doesn't appear in $ii\prime$).

5.1.1 Remarks:

1. At Steady State(SS) $\frac{k^\star}{n^\star}$ doesn't depend on $\tau_c^\star$ or $\tau_n^\star$ (only on $\tau_x^\star, \tau_k^\star, \delta, \beta$).

2. That is, different countries with different $\tau_c^\star$ or $\tau_n^\star$ will still have same $\frac{k^\star}{n^\star}$.

3. If $n^\star = 1$ (inelastic labor supply with $U_l = 0$), then

   i) $i\prime$) disappears,

   ii) Only equations $ii\prime) \sim iv\prime$) will determine the SS

   iii) Since $F\left(k^\star, n^\star\right) = F\left(k^\star, 1\right)$, $k^\star$ depends on $\tau_x^\star, \tau_k^\star, \delta, \beta \ \rightarrow \ k^\star\left(\tau_x^\star, \tau_k^\star, \delta, \beta\right)$

   $\rightarrow$ enables comparative statics

What is $\frac{\partial k^\star}{\partial \tau_k^\star}$?

$$1 = \beta\left[(1-\delta) + \frac{\left(1-\tau_k^\star\right)}{\left(1+\tau_x^\star\right)} F_k\left(k^\star, 1\right)\right]$$

$$\left[\frac{1}{\beta} - (1-\delta)\right] \frac{\left(1+\tau_x^\star\right)}{\left(1-\tau_k^\star\right)} = F_k\left(k^\star, 1\right) \left(= f'\left(k^\star\right)\right)$$

Thus, $\frac{\partial k^\star}{\partial \tau_k^\star} < 0$. Similarly, $\frac{\partial k^\star}{\partial \tau_x^\star} < 0$.

Since $x^\star = \delta k^\star$, it follows that we have the same signs for $\frac{\partial x^\star}{\partial \tau_k^\star}$ and $\frac{\partial x^\star}{\partial \tau_k^\star}$.

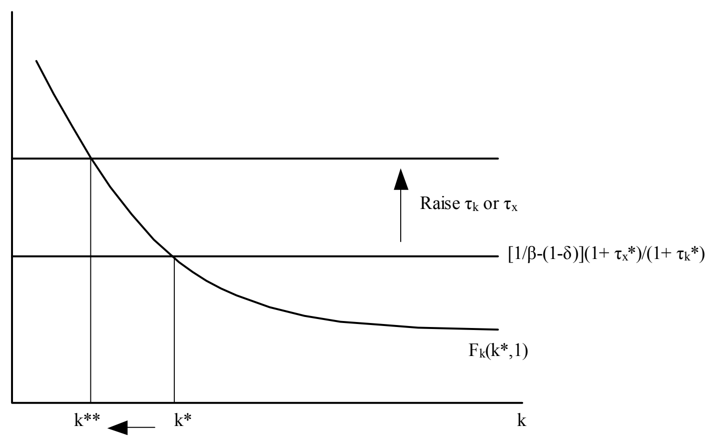


Figure 4:

What about $c^{\star}$? $\frac{\partial c^{\star}}{\partial \tau_k^{\star}} = f'(k^{\star}) \frac{\partial k^{\star}}{\partial \tau_k^{\star}} - \delta \frac{\partial k^{\star}}{\partial \tau_k^{\star}} = \left(f'(k^{\star}) - \delta\right) \frac{\partial k^{\star}}{\partial \tau_k^{\star}} < 0$

5.1.2 Remarks:

1. In a sense, this formulation of the problem has too many taxes. That is: Show that given any TDCE with $T_t = 0$ but $\tau_{ct}, \tau_{nt}, \tau_{xt}, \tau_{kt} > 0$ (or any subset....) there is another tax system $\hat{\tau}_{ct}, \hat{\tau}_{nt}, \hat{\tau}_{xt}, \hat{\tau}_{kt}$ with $\hat{\tau}_{ct} = \hat{\tau}_{xt} = 0$ $\forall\, t$ but the same TDCE allocation. That is, you can support the same allocation through a tax system in which consumption and investment taxes are zero. In this sense, these taxes are redundant.

2. Note: This does not say that $\tau_{kt}, \tau_{nt} > 0, \tau_{xt} = 0, \tau_{ct} = 0$ supports the

same allocation. That is, you may have to adjust τ_{kt}, τ_{nt} (to $\hat{\tau}_{nt}, \hat{\tau}_{kt}$) to support the same allocation.

3. Can you think of other versions of this?

4. How would you include a provision for depreciation allowances in the tax code? $Taxes_t = \tau_{ct} p_t c_t + \tau_{xt} p_t x_t + \tau_{nt} w_t n_t + \tau_{kt} (r_t k_t - \delta_\tau k_t)$

   with the BC then being:

   $$\sum(p_t c_t + p_t x_t) \leq \sum(r_t k_t + w_t n_t + T_t - Taxes_t)$$

5. How would you include progressive (or regressive) tax systems?

   $Taxes_t = \tau_t (w_t, r_t, n_t, k_t)$

   with the budget constraint being:

   $\sum(p_t c_t + p_t x_t) \leq \sum(r_t k_t + w_t n_t + T_t - Taxes_t)$.

6 What 'should' taxes be?

For a given streams of expenditures g_t would consumers be better off under

System A) Choose τ_{ct} so that $p_t g_t = \tau_{ct} p_t c_t \;\; \forall \;\; t$

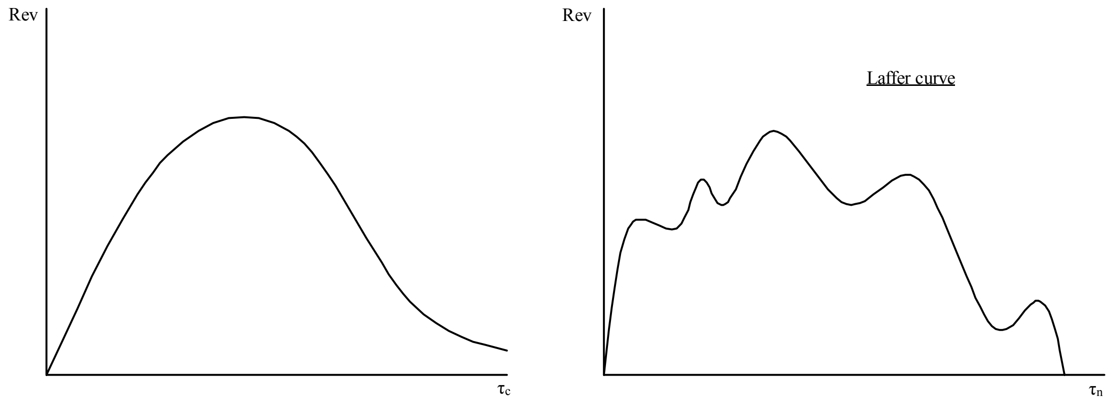


Figure 5:

OR

System B) Choose τ_{nt} so that $p_t g_t = \tau_{nt} w_t n_t \;\; \forall \;\; t$.

Note that in both of these systems would require that the relevant prices (resp. wages) would depend also depend on the tax code.

Remark: At this point, it is not even obvious that we can find such a system? That is, the first question is: When can I find a system of τ_{ct}, τ_{nt}, etc., such that the there is an equilibrium supporting the given sequence of government expenditures, g_t (and hence, in particular, such that the government budget balances)?

More generally, what should τ be?

Ramsey Problems

This is the name given to a class of optimal policy problems:

Maximize utility of consumers given revenue requirements and instrument "availability."

That is, the Ramsey Problem is to choose tax rates to maximize the welfare of the representative agent subject to the constraints that the government budget be balanced in PV in the resulting CE.

They generally consist of three distinct elements:

1) What is the Objective Function of the Tax Designer? Revenue Maximization? Representative Consumer Utility Maximization? Etc.

2) What are the 'instruments' available to the Tax Designer? Linear Income taxes? Non-Linear Income Taxes? Lump-Sum Taxes? Direct seizure or control of decision making of the individuals in the economy?

3) What is the mapping between the the setting of the instruments in 2 and the planner utility in 1)? E.g., a competitive market system in between them so that $U(\tau)$ from the Tax Designer perspective is $U(c(\tau), \ell(\tau))$ where $(c(\tau), \ell(\tau))$ is the TDCE allocation that results from the tax system τ.

Historically, this approach to policy choice comes from a classic paper by Ramsey (1928). Ramsey took g as given and asked what combination of excise taxes(taxes on consumption goods) should be used to finance a given level of expenditures, g. He phrased this as maximizing Consumer's Surplus and found this by integrating under the demand curves:

$Max \qquad CS(\tau)$

s.t. $\quad \sum_{i=1}^{n} \tau_i q_i = R \quad$ i.e., the tax revenue from consumption good $i = 1, ..., n$ covers the required Revenue, R.

(Mechanism) Pick $\tau's \;\rightarrow\;$ CE given $\tau \;\rightarrow\;$ Revenue raised $(R(\tau))$ CS obtained $(CS(\tau))$

The more modern version of this problem is stated as:

Given any set of taxes, τ_i, Consumers solve

$Max \qquad U(q_1, ..., q_n)$

s.t. $\sum (1 + \tau_i)\, p_i q_n \leq W$

6.0.3 Remarks:

1. In the TA session Mike Golosov showed:

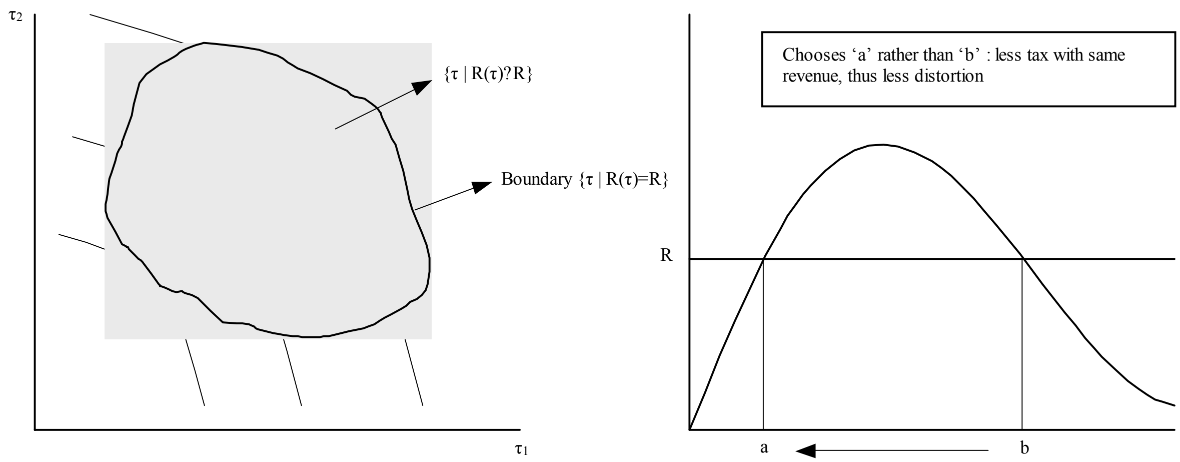


Figure 6:

In the case $\sum (1+\tau_i) p_i q_i \leq W - T$, i.e., lump sum taxation is allowed, then solution is $\tau_i = 0$ & $T = R$.

If $T = 0$ is assumed, then the solution is $\tau_i = \tau \;\; \forall \;\; i$ where $W - \frac{w}{1-\tau} = R$.

In this case, the BC becomes $\sum p_i q_i \leq \frac{W}{(1+\tau)}$ and this is equivalent to lump sum taxes.

2. Ramsey solved:

$$Max \qquad U\left(q_1(\tau), q_2(\tau), ..., q_n(\tau)\right)$$

subject to:

i) $\sum q_i(\tau)\tau_i + T = R$,

ii) $T = 0$,

iii) $\tau_1 = 0$,

where $\tau = (\tau_1, \tau_2, ..., \tau_n, T)$

Ramsey Result) Tax goods according to their elasticity of demand. That is, low $\varepsilon \rightarrow$ high τ.

3. Ramsey did not really have a tight justification for not allowing lump sum taxation, or assuming that $\tau_1 = 0$. The only reason is that if you don't make these restrictions, the solution will be rather simple, use lump sum taxes, or the equivalent. He viewed this as unrealistic, but gave no formal justification. The modern solution to this problem would be to assume that there is private information about earning abilities. This approach was pioneered by Mirlees. Including this at this point would complicate matters considerably and hence we won't do it here. It's also not true that this more complex approach is equivalent to disallowing lump sum taxes, and assuming that $\tau_1 = 0$.

6.1 $<$ Macro Version $>$

Ramsey problem, Ramsey planner:

Choose $\tau = (\tau_{kt}, \tau_{nt})_{t=0}^{\infty}$

to maximize $U\left(c\left(\underset{\sim}{\tau}\right), l\left(\underset{\sim}{\tau}\right)\right)$

s.t. $c(\tau), l(\tau)$ is a TDCE allocation for the economy with tax system τ and $\underset{\sim}{g} = (g_0, ...)$ given.

6.1.1 Remark:

1. As in Ramsey, we will not allow lump sum taxes. Also, we will assume that $\tau_{kt} \leq 1 \ \ \forall \ \ t$. The reason for this assumption is two fold. First, if we allow $\tau_{k0} > 1$, this is equivalent to lump sum taxation. Second, if we do not assume this, it is questionable that individual household supply of capital is equal to the stock that they have on hand.

6.1.2 <Solving the Ramsey Problem>

We will solve this Ramsey Problem in three steps:

Step1) Characterize the set of τ's that raise enough revenue in equilibrium, and the allocations that go along with them. Thus, we want to find the set:

$A^0 = \{(\tau; p, r, w; c, x, k, n, l) | (p, r, w; c, x, k, n, l) \ is\ a\ TDCE\ given$
$(\tau, g)\}$

Step2) Rewrite A^0 in terms of those variables that a benevolent Planner would care about... I.e., those variables that enter the utility of the representative agent.

That is, a benevolent Planner would solve the maximization problem:

Maximize $U\left(\underset{\sim}{c}, \underset{\sim}{l}\right)$

subject to:

$$\exists(\tau, p, r, w, x, k, n) \ such\ that\ (\tau, p, r, w, c, x, k, n, l) \in A^0.$$

That is, the planner sets τ and then the private economy responds with (p, r, w, c, x, k, n, l), a TDCE price system and allocation given τ. By construction, the planner is only allowed to choose fiscal policies which will raise sufficient revenue to finance the given sequence of expenditures, $\{g_t\}$.

In this step, what we will do is to start with the maximization problem as given above, and then systematically remove all variables in the problem that do not enter the utility function. That is, if you notice the problem above, only $\left(\underset{\sim}{c}, \underset{\sim}{l}\right)$ enters the objective function, but $(\tau, p, r, w, c, x, k, n, l)$ are the decision variables. The characterization found in Step 1, will allow us to 'eliminate' all the extraneous variables in the maximzation problem and rewrite it as:

Maximize $U\left(\underset{\sim}{c}, \underset{\sim}{l}\right)$

subject to:

$\left(\underset{\sim}{c}, \underset{\sim}{l}\right) \in A^1,$

where $A^1 = \{\left(\underset{\sim}{c}, \underset{\sim}{l}\right) | \exists (\tau, p, r, w, x, k, n) \text{ } such \text{ } that \text{ } (\tau, p, r, w, c, x, k, n, l) \in A^0\}$.

I.e., we're taking the 'projection' of A^0 onto $\left(\underset{\sim}{c}, \underset{\sim}{l}\right)$, the set of variables we care about.

Step3) Use the fact that this new version of the maximization problem looks a lot like the maximization problem of a standard one sector growth model to characterize the behaviour of the solution first in terms of the quantities that enter the utility function of the representative consumer, and then in terms of the taxes that that implies.

Step 1: ¿From the Firm's problem, we have:

i) $F_k(k_t, n_t) = \frac{r_t}{p_t} \quad \forall \ t$

ii) $F_n(k_t, n_t) = \frac{w_t}{p_t} \quad \forall \ t$

iii) $c_t + x_t + g_t = F_t(k_t, n_t)$

¿From the Consumer's problem, we have:

i) $p_t = \frac{\beta^t U_c(t)}{U_c(0)}$

ii) $\frac{U_l(t)}{U_c(t)} = (1-\tau_{nt}) \frac{w_t}{p_t}$

iii) $U_c(t) = \beta U_c(t+1) [(1-\delta) + (1-\tau_{kt+1}) F_{kt}(k_{t+1}, n_{t+1})] \quad \forall\ t$

iv) $k_{t+1} \leq (1-\delta) k_t + x_t$

v) $\sum [p_t c_t - (1-\tau_{nt}) w_t n_t] = \sum [(1-\tau_{kt}) r_t k_t - p_t (k_{t+1} - (1-\delta) k_t)]$

(RHS) $= (1-\tau_{k_0}) r_0 k_0 + (1-\delta) k_0 + \sum_{t=1}^{\infty} k_t [p_t (1-\delta + (1-\tau_{k_t}) F_k(t)) - p_{t-1}]$

$= k_0 (1-\tau_{k_0}) F_k(0) + (1-\delta) k_0$

(LHS) $= \sum \frac{\beta^t}{U_c(0)} [U_c(t) c_t - U_l(t) n_t]$

Thus, v) becomes

Implementability Condition

$$\boxed{[k_0 (1-\tau_{k_0}) F_k(0) + (1-\delta) k_0] U_c(0) = \sum \beta^t [U_c(t) c_t - U_l(t) n_t]}$$

Conversely, if $(c_t, x_t, k_t, n_t, l_t)$ satisfy

FP 3 $\quad c_t + x_r + g_t \leq F_t(k_t, n_t) \ \ \forall\ t$

CP 4 $\quad k_{t+1} \leq (1-\delta)k_t + x_t \ \ \forall\ t$

and CP 5 – The Implementability condtion above written purely in terms of quantities

Then, $\exists\ (\tau_{nt}, \tau_{kt}), (p_t, r_t, w_t)$ s.t.

(p_t, r_t, w_t) & $(c_t, x_t, k_t, n_t, l_t)$ is a TDCE for the policy

$[(\tau_{nt}, \tau_{kt}), g_t]_{t=0}^{\infty}$.

That is, pick any quantities that satisfy the feasibility conditions and the implementability condition, CP5 and you can construct a system of taxes, such that the given allocation is a TDCE allocation given those taxes.

Step 2: Thus, the RP is equivalent to:

$$Max \qquad U\left(c\,(\underset{\sim}{\tau}), l\,(\underset{\sim}{\tau})\right)$$

s.t. $c_t + x_t + g_t \leq F_t(k_t, n_t)$

$k_{t+1} \leq (1-\delta) k_t + x_t$

CP5 $\rightarrow [k_0 (1-\tau_{k_0}) F_k(0) + (1-\delta) k_0] U_c(0) = \sum \beta^t [U_c(t) c_t - U_l(t) n_t]$

That is,

Proposition:

$(p_t^\star, r_t^\star, w_t^\star)$ and $(c_t^\star, n_t^\star, k_t^\star, x_t^\star)$ is a TDCE with taxes (τ_{nt}, τ_{kt}) supporting g_t.

$\Leftrightarrow$

i) $F_{kt}(k_t^\star, n_t^\star) = \frac{r_t^\star}{p_t^\star}$

ii) $F_{nt}(k_t^\star, n_t^\star) = \frac{w_t^\star}{p_t^\star}$

iii) $p_t^\star = \frac{U_c(t)}{U_c(0)} \beta^t$

iv) $\frac{U_l(t)}{U_c(t)} = (1 - \tau_{nt}) \frac{w_t^\star}{p_t^\star}$

v) $U_c(t) = \beta U_c(t+1) \left[(1-\delta) + (1 - \tau_{kt+1}) F_k(t+1)\right]$

vi) $k_{t+1}^\star \leq (1-\delta) k_t^\star + x_t^\star$

vii) $c_t^\star + x_t^\star + g_t^\star \leq F_t(k_t^\star, n_t^\star)$

viii) $U_c(0) k_0 \left[(1 - \tau_{k_0}) F_{k0}(k_0^\star, n_0^\star) + (1-\delta)\right] = \sum \beta^t \left[U_c(t) c_t^\star - U_l(t) n_t^\star\right]$

Remark: You can think of this as saying that i)˜v) "determine" $(p_t^\star, r_t^\star, w_t^\star)$ & (τ_{nt}, τ_{kt}) from an allocation determined by vi)˜viii). Note that vi)-viii) depend on quantities only.

¿From i)˜ii), it follows that Firms are maximizing.

¿From iii)˜v), and vii) Consumers are maximizing, assuming the solution is interior.

vi) and vii) are accounting identities. They merely make sure that physical feasibility is satisfied.

viii) is the Implementability constraint. This is what differentiates tax distorted equilibria from other feasible allocations. This is also what makes this problem different from standard growth model without distortions.

Ramsey planner's problem:

$\boxed{(RP\ I)}$

$$\underset{\tau}{Max} \qquad U\left(c\left(\underset{\sim}{\tau}\right), l\left(\underset{\sim}{\tau}\right)\right)$$

s.t. $c\left(\underset{\sim}{\tau}\right), l\left(\underset{\sim}{\tau}\right)$ is the TDCE allocation given τ.

$RP\ I$ is equivalent to $RP\ II$.

$\boxed{(RP\ II)}$

$$\underset{c,n,x,k}{Max} \qquad U\left(\underset{\sim}{c}, \underset{\sim}{l}\right)$$

s.t. $(RPA)\ c_t + x_t + g_t = F_t(k_t, n_t)$

$(RPB)\ k_{t+1} \leq (1-\delta) k_t + x_t$

$(RPC)\ \sum \beta^t \left[U_c(t) c_t - U_l(t) n_t\right] = U_c(0) k_0 \left[(1-\tau_{k_0}) F_k(0) + (1-\delta)\right]$

Remark:. RPC is real version of BC after substitution. Implementability constraint. τ_{k0} is the tax rate on initial capital. It is equivalent to a lump sum tax. Because of this, it is typically assumed that $\tau_{k0} = 1$ $(or\ 0)$.

Step 3: Let λ denote the multiplier on RPC in this maximization problem.

Then, the Lagrangian is (letting $\tau_{k0} = 1$)

$$\sum \beta^t U(c_t, 1-n_t) + \lambda\left[U_c(0) k_0 (1-\delta) - \sum \beta^t \left[U_c(t) c_t - U_l(t) n_t\right]\right]$$

$+other\ terms$

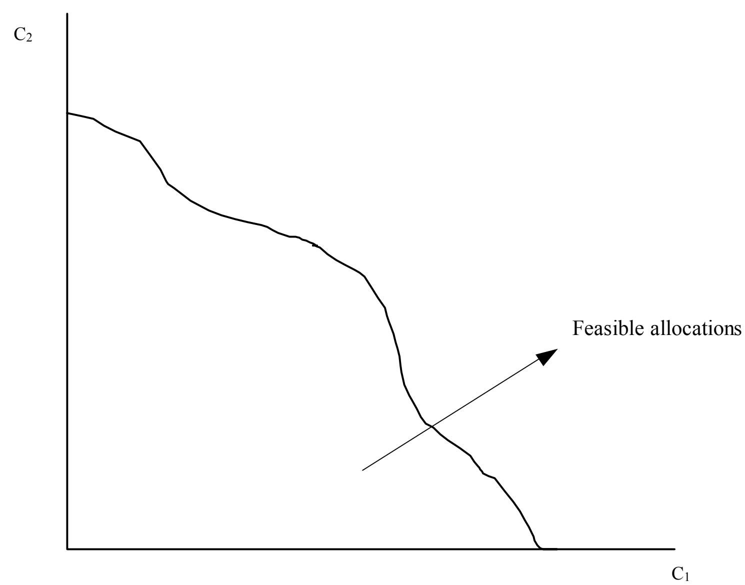


Figure 7:

$=U(c_0, 1-n_0) + \lambda U_c(0)\, k_0 (1-\delta) - \lambda [U_c(0)\, c_0 - U_l(0)\, n_0]$

$+\sum_{t=1}^{\infty} \beta^t [U(c_t, 1-n_t) - \lambda U_c(t)\, c_t + \lambda U_l(t)\, n_t] + \textit{other terms.}$

Let $V(c, n; \lambda) = U(c, 1-n) - \lambda U_c(c, 1-n)c + \lambda U_l(c, 1-n)n$. Then, define

$W_0(c_0, n_0, k_0, \lambda) = U(c_0, 1-n_0) + \lambda U_c(c_0, 1-n_0)\, k_0 (1-\delta) - \lambda U_c(c_0, 1-n_0)\, c_0 + \lambda U_l(c_0, 1-n_0)\, n_0$

Thus $\hat{V}\left(\underset{\sim}{c}, \underset{\sim}{l}\right) = W_0(c_0, n_0, k_0, \lambda) + \sum_{t=1}^{\infty} \beta^t V(c_t, n_t; \lambda)$

Thus, we can rewrite the Ramsey Problem as :

$\boxed{RP\ III}$

$$\underset{\underset{\sim}{c}, \underset{\sim}{x}, \underset{\sim}{k}, \underset{\sim}{n}, \underset{\sim}{l}, \lambda}{Max} \qquad \hat{V}\left(\underset{\sim}{c}, \underset{\sim}{l}\right)$$

s.t. $c_t + x_t + g_t = F_t(k_t, n_t; t)$

$k_{t+1} \leq (1-\delta)\, k_t + x_t$

$RP\ III$ is a standard one-sector growth model where period utility is $V(c, n; \lambda)$ not $U(c, 1-n)$. And of course, λ is endogenous.

It follows from the standard reasoning that the solution to $RP\ III$ satisfies:

i) $\frac{V_l(t)}{V_c(t)} = F_n(t) \quad t = 1, 2, ...$

ii) $V_c(t) = \beta V_c(t+1)(1 - \delta + F_k(t+1))$

iii) $k_{t+1} \leq (1-\delta) k_t + x_t$

iv) $c_t + x_t + g_t = F_t(k_t, n_t)$

In what follows, we will assume that the production function does not depend on time, $F_t = F$.

Let $\left(c_t^{RP}, x_t^{RP}, k_t^{RP}, n_t^{RP}\right)$ solve $RP\ III$.

Assume $c_t^{RP} \to c^{RP}$

$x_t^{RP} \to x^{RP}$

$k_t^{RP} \to k^{RP}$

$n_t^{RP} \to n^{RP}$

so that the solution to this problem converges to a steady state.

Note: we also know that g also converges to a constant in this case.

(Steady State)

i)' $\frac{V_l^{RP}}{V_c^{RP}} = F_l\left(k^{RP}, n^{RP}\right)$

ii)' $1 = \beta\left(1 - \delta + F_k\left(k^{RP}, n^{RP}\right)\right)$ (rmk. Describes after tax savings)

iii)' $x^{RP} = \delta k^{RP}$

iv)' $c^{RP} + x^{RP} + g^* = F\left(k^{RP}, n^{RP}\right)$

Recall that if $(c_t^*, x_t^*, k_t^*, n_t^*)$ is a TDCE allocation supporting g, it satisfies

(Euler equation) $U_c(c_t^*, 1 - n_t^*) = \beta U_c\left(c_{t+1}^*, 1 - n_{t+1}^*\right)\left[1 - \delta + (1 + \tau_{kt+1}) F_k\left(k_{t+1}^*, n_{t+1}^*\right)\right]$

Hence, $U_c\left(c_t^{RP}, 1-n_t^{RP}\right) = \beta U_c\left(c_{t+1}^{RP}, 1-n_{t+1}^{RP}\right)\left[1-\delta+\left(1+\tau_{kt+1}^{RP}\right)F_k\left(k_{t+1}^{RP}, n_{t+1}^{RP}\right)\right]$

If $t\to\infty$, then $1=\beta\left[1-\delta+\left(1+\tau_{k\infty}^{RP}\right)F_k\left(k^{RP}, n^{RP}\right)\right]$.

This, along with ii)' implies $\boxed{\tau_{k\infty}^{RP}=0}$.

That is, in the limit, the tax rate on capital income is zero.

What about limiting tax on labor? From the FOC's for the consumer in the TDCE problem, we have that:

$$\frac{U_l\left(c^{RP},1-n^{RP}\right)}{U_c\left(c^{RP},1-n^{RP}\right)} = \left(1-\tau_{n\infty}^{RP}\right)F_n\left(k^{RP}, n^{RP}\right).$$

¿From the Ramsey Problem, we get:

$$\frac{V_l\left(c_t^{RP},1-n_t^{RP};\lambda\right)}{V_c\left(c_t^{RP},1-n_t^{RP};\lambda\right)} = F_n\left(k_t^{RP}, n_t^{RP}\right)$$

Combining the two,

$$\frac{V_l\left(c_t^{RP},1-n_t^{RP};\lambda\right)}{V_c\left(c_t^{RP},1-n_t^{RP};\lambda\right)} = \frac{1}{1-\tau_{nt}^{RP}}\cdot\frac{U_l\left(c_t^{RP},1-n_t^{RP};\lambda\right)}{U_c\left(c_t^{RP},1-n_t^{RP};\lambda\right)}$$

¿From ($\star$), we see that

$$\frac{U_l\left(c_t^{RP},1-n_t^{RP};\lambda\right)-\lambda[U_{cl}(c,1-n)c-U_{ll}(c,1-n)n+U_l(c,1-n)]}{U_c\left(c_t^{RP},1-n_t^{RP};\lambda\right)-\lambda[U_c(c,1-n)+U_{cc}(c,1-n)c-U_{lc}(c,1-n)n]} = \frac{1}{1-\tau_{nt}^{RP}}\cdot\frac{U_l\left(c_t^{RP},1-n_t^{RP};\lambda\right)}{U_c\left(c_t^{RP},1-n_t^{RP};\lambda\right)}$$

It follows that if $\lambda=0$, then $\tau_{nt}^{RP}=0$.

Summarizing: $\underline{\tau_{k\infty}^{RP}=0 \quad \& \quad \tau_{n\infty}^{RP}>0}$

6.1.3 Remarks:

1. What is the interpretation of λ? It is the multiplier on the BC in Ramsey problem.

   $\lambda = 0 \;\Leftrightarrow\;$ BC in RP is "slack", i.e., using distortionary taxes to finance g has no welfare effects. This means that can drop this constraint and the solution would be unchanged. But, if you drop that constraint, the resulting problem has only feasibility. This means that in this case, the solution is the same as if you had lump sum taxes at your disposal.

2. For early $t's$ the tax rate on capital income is 100%, but it decreases over the transition period, and in the limit goes to zero.

   $Rev_t = \tau_{kt}^{RP} r_t^{RP} k_t^{RP} + \tau_{nt}^{RP} w_t^{RP} n_t^{RP}$,

   $Rev_{nt} = \tau_{nt}^{RP} w_t^{RP} n_t^{RP}$.

3. If $U_l = 0$ (inelastic labor supply), $n_t = 1$ for all t (as long as $\tau_{nt} \leq 1$), there is no distortion, but still generates revenue.

4. There are few examples of lump sum taxation in practice. One such is what was known as the 'head tax,' in which everyone had to pay the same fee. Another partial example might be the approach to expendi-

ture finance seen in Alaska and in some Arab oil countries. There, no taxes are collected (and sometimes there are even transfers BACK to citizens). Rather, revenues from oil sales are used to finance g.

Special Case with Inelastic Labor Supply:

6.1.4 <Some Related Problems>

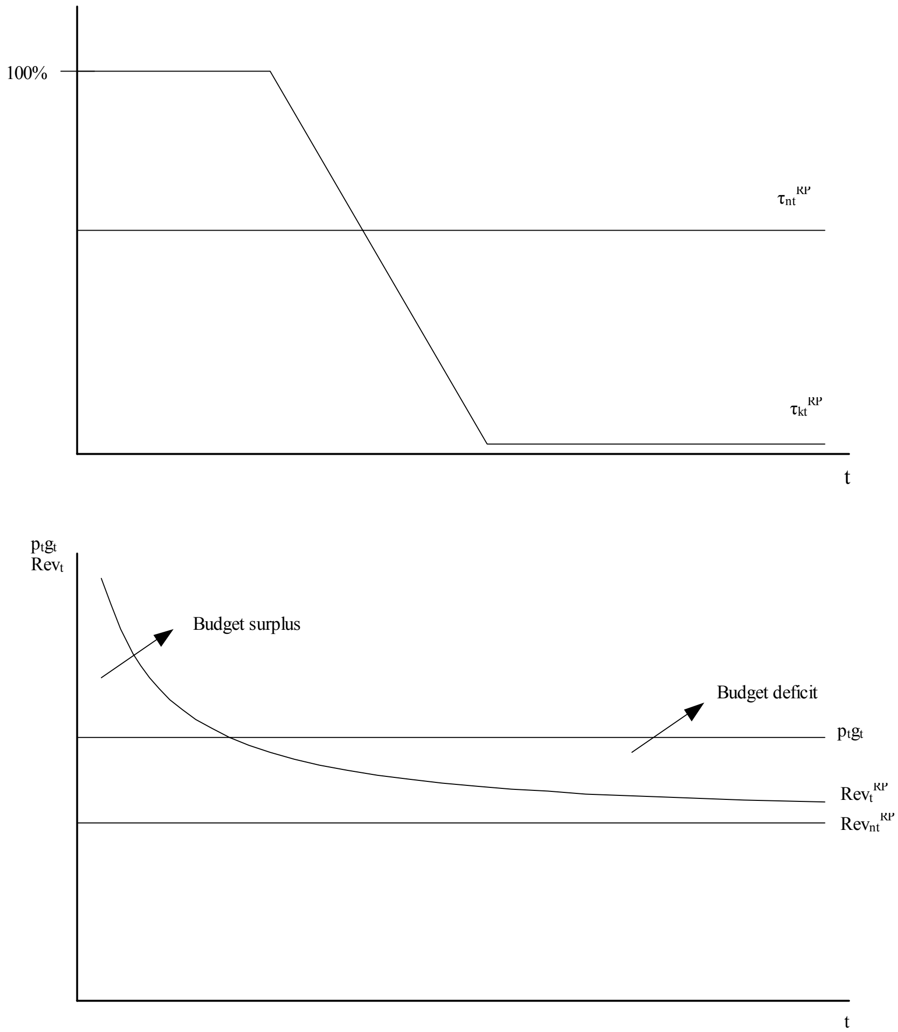
1. Show that the RP with τ_{kt} and τ_{nt} is equivalent to the one with τ_{ct} and τ_{nt}. Note that τ_{nt} will not necessarily be the same in the two formulations.

2. Show that the TDCE with τ_{nt} and τ_{kt} is implemented with a sequence with $\tau_{kt} \to 0$ $\Leftrightarrow$the TDCE is implemented with τ_{nt} and τ_{ct} such that $\tau_{ct} \to \tau_{c\infty} < \infty$.

3. Show that if we had formulated the Ramsey problem in terms of τ_{nt} and τ_{ct}, we would have concluded that:

   $\tau_{nt} \to \tau_{n\infty} < \infty$ and $\tau_{ct} \to \tau_{c\infty} < \infty$.

4. Show that the TDCE with τ_{nt} and τ_{kt} is implemented with a sequence with $\tau_{kt} \to \tau_{k\infty} > 0$



100%
τ_{nt}^{RP}
τ_{kt}^{RP}
t
$p_t g_t$
Rev_t
Budget surplus
Budget deficit
$p_t g_t$
Rev_t^{RP}
Rev_{nt}^{RP}
t

Figure 8:

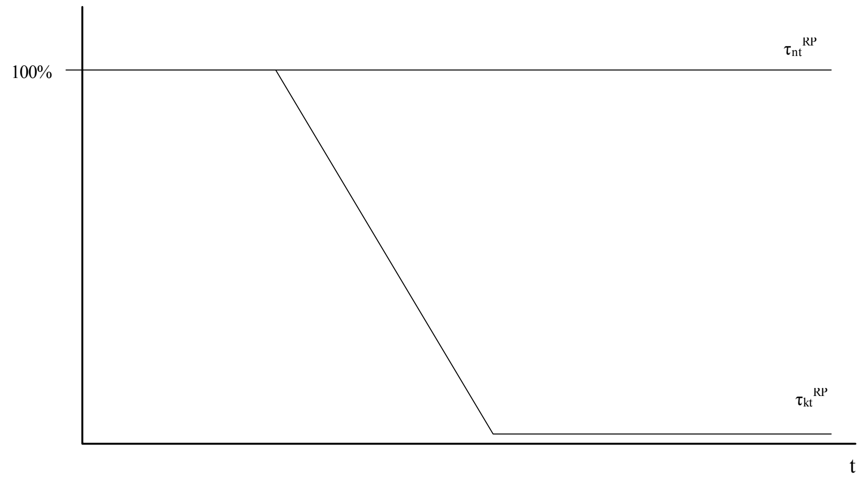


Figure 9:

$\Leftrightarrow$the TDCE is implemented with τ_{nt} and τ_{ct} such that $\tau_{ct} \to \tau_{c\infty} = \infty$.