Econ 8105 MACROECONOMIC THEORY
DYNAMIC PROGRAMMING FOR MACRO

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These notes are a condensed treatment of the chapters in SLP that deal with Deterministic Dynamic Programming used in conjunction with the treatment of the single sector growth model and its generalizations. More or less, this is Chapters 4-6 of the book along with some of the Mathematics that is used in those sections.

Anderson Schneider will be teaching during the first two weeks of classes and he will base his lectures on the material of this notes. A final version of this file will be posted by the beginning of the third week of classes.

Read S.L.P.

- Chapters 1 and 2 for background (skim 2.2)
- Skim Chapter 3 – Math.
- We will cover Chapter 4/parts of Chapter 5/parts of Chapter 6 in detail. Reread Chapter 3 as needed as we go along.

Go for:

1. Simple version.
2. Time stationary rep.

3. Global Dynamics (special cases)

4. Numerical procedure.

From what we've seen so far, an ADE allocation can be found as the solution to the maximization problem of the form:

$$
\begin{aligned}
P(\widehat{k}): \quad & \max_{(\widetilde{c},\widetilde{x},\widetilde{k},\widetilde{\ell},\widetilde{n})} && u(\widetilde{c},\widetilde{\ell}) \\
& s.t. && c_t + x_t \leq F_t(k_t, n_t) \\
& && k_{t+1} \leq (1-\delta)k_t + x_t \\
& && n_t + \ell_t \leq \overline{n}_t \\
& && k_0 = \widehat{k} \ \text{ fixed} \\
& && \text{non-negativity.}
\end{aligned}
$$

Assume that $\overline{n}_t$ is independent of t and F_t is independent of t.

e.g., $\overline{n}_t \equiv \overline{n} \equiv 1,\ F_t(k,n) = Ak^{\alpha}n^{1-\alpha}$

Let:

$\Gamma(\widehat{k};\overline{n},F)$ denote the set of feasible sequences for $(\tilde{c},\widetilde{x},\widetilde{k},\widetilde{n},\widetilde{\ell})$ given $\overline{n}$, F and $\widehat{k}$. That is,

$$
\begin{aligned}
(\tilde{c},\widetilde{x},\widetilde{k},\widetilde{n},\widetilde{\ell}) \in \Gamma(\widehat{k};\overline{n},F) \qquad & \Longleftrightarrow \\
c_t + x_t \leq F(k_t,n_t) \qquad & \forall t \\
k_{t+1} \leq (1-\delta)k_t + x_t \qquad & \forall t \\
n_t + \ell_t \leq \overline{n} \qquad & \forall t \\
k_0 = \widehat{k} \qquad &
\end{aligned}
$$

and write

$$(\tilde{c}, \widetilde{x}, \widetilde{k}, \widetilde{n}, \widetilde{\ell}) = (c_0, x_0, k_0, n_0, \ell_0; \widetilde{c_1}, \widetilde{x}_1, \ldots, \widetilde{\ell}_1)$$

in period 0, and period 1,2... decisions, respectively.

NOTICE:

$$\begin{aligned}
(\tilde{c}, \widetilde{x}, \widetilde{k}, \widetilde{n}, \widetilde{\ell}) \in \quad & \Gamma(\widehat{k}; n, t) \\
\Leftrightarrow \quad & c_0 + x_0 \leq F(k_0, n_0) \\
& k_1 \leq (1 - \delta)k_0 + x_0 \\
& n_0 + \ell_0 \leq \overline{n} \\
\text{and} \quad & k_0 = \widehat{k}
\end{aligned}$$

AND $\quad (\widetilde{c_1}, \widetilde{x}_1, \ldots, \widetilde{\ell}_1) \in \Gamma(k_1; \overline{n}, F).$

That is,

The constraint set for $P(\widehat{k}; \overline{n}, F)$ has a RECURSIVE structure – There is a "$t = 0$ component" and a "continuation component" and, moreover, the "continuation component" looks "just like" the original set!

Problem: Give a Max Problem where this *isn't* true!

Note: This requires infinite horizon for it to be true!

Indeed, note that $\Gamma(k; \overline{n}, F)$ is of the form

$$\left\{(\tilde{c}, \widetilde{x}, \widetilde{k}, \widetilde{n}, \widetilde{\ell}) \mid (c_t, x_t, k_t, \ell_t, n_t) \in \widehat{\Gamma}(k_{t-1})\right\}$$

where

$$\widehat{\Gamma}(k) = \left\{ \begin{array}{c} (c, x, k', \ell, n) \mid c + x \leq F(k, n) \\ k' \leq (1-\delta)k + x \\ l + n \leq \overline{n} \\ \text{non-negativity} \end{array} \right\}$$

i.e., constraint set is a time stationary function of the "state variable" k_t.

Other Problems Like This

If don't cut the tree at period t, then (Tree–height at t)=(1 + height at $t-1$), i.e., $k_t = 1 + k_{t-1}$. Consider also $k_t = 0$ forever if you do cut the tree.

If you cut it at height k_t you get payoff $\beta^t u(k_t)$.

Let $x_t = \left\{ \begin{array}{cc} 0 & \text{if don't cut} \\ 1 & \text{if cut} \end{array} \right\}$

Then the problem can be written as:

$$\begin{aligned} &\max \sum \beta^t u(x_t, k_t) \\ &x_t \in \{0, 1\} \\ &k_{t+1} = (1 - x_t)(k_t + 1)(1 - \chi_{(k_t=0)}) \end{aligned}$$

1 Outline/Strategy for Tackling These Problems

Our strategy for solving problems like this is to use a simple fact about maximization problems over two variables (even if the second variable for us is an infinite history of all relevant variables). This property is easily described via the following.

Suppose we have an indexed family of maximization problems, one for each $x \in X$, $P(x)$. In each of these you have to pick a $y = (y_1, y_2) \in Y_1 \times Y_2$. So, $P(x)$ is given by:

$$\begin{aligned} P(x): \quad & \max_{(y_1,y_2)} u(x, y_1, y_2) \\ \text{s.t} \quad & (x, y_1, y_2) \in \Lambda(x), \quad x \text{ given.} \end{aligned}$$

Here, $\Lambda(x) \subset Y_1 \times Y_2$ is the constraint set for the problem $P(x)$. Assume that there is a solution for this problem for each $x \in X$ given by $(y_1^*(x), y_2^*(x))$ and define $V^*(x)$ to be the value of utility at the solution:

$V^*(x) = u(x, y_1^*(x), y_2^*(x))$.

This is the description of the problem in its 'raw' or sequential form.

Alternatively, for each $x \in X$, define

$\Lambda_1(x) = \{y_1 \in Y_1 | \exists y_2 \in Y_2, s.t., (x, y_1, y_2) \in \Lambda\}$

and for each $y_1 \in \Lambda_1(x)$ define

$\Lambda_2(x, y_1) = \{y_2 \in Y_2 | (x, y_1, y_2) \in \Lambda\}$.

Next, consider the following Two Step Procedure for solving $P(x)$:

Step 1: For each (x, y_1) such that $y_1 \in \Lambda_1(x)$, solve the maximization problem $P^2(x, y_1)$ given by:

$$
\begin{array}{lll}
P^2(x, y_1) & \max\limits_{y_2} & u(x, y_1, y_2) \\
\text{s.t.} & y_2 \in \Lambda_2(x, y_1) & \\
 & (x, y_1) \quad \text{fixed} &
\end{array}
$$

Assuming a solution exists for each choice of (x, y_1), this defines a function (or correspondence if there are multiple solutions), $y_2(y_1, x)$. Define $U^2(x, y_1)$ by:

$U^2(x, y_1) = u\left(x, y_1,\, y_2(x, y_1)\right)$

Step 2: For each $x \in X$ define the maximization problem $P^1(x)$ by:

$$
\begin{array}{lll}
P^1(x) & \max\limits_{y_1} & U^2(x, y_1) \\
\text{s.t.} & y_1 \in \Lambda_1(x) &
\end{array}
$$

Assuming a solution exists for each choice of x, this defines a function (or correspondence if there are multiple solutions), $y_1(x)$. Define $U^1(x)$ by:

$U^1(x) = U^2(x, y_1(x)) = u\left(x, y_1(x),\, y_2(x, y_1(x))\right).$

Then, you can show that:

1. $V^*(x) = U^1(x) = U^2(x, y_1(x)) = u\left(x, y_1(x),\, y_2(x, y_1(x))\right)$ for all $x \in X$.
2. $(y_1^*(x), y_2^*(x)) = (y_1(x), y_2(y_1(x))$ for all $x \in X$ assuming unique solutions.
3. $V^*(x) = U^1(x)$ for all $x \in X$ even if max is replaced by sup and no solution need exist.
4. Something like 2) holds even if the solution is NOT unique.

Adding More Structure

Suppose in addition that the continuation problems are also like the original problems, i.e., if each P^1 is in the class P, and that the some additional structure is placed on both the OBJ and Constraint Sets:

1. Assume that $Y_1 = X$ and $Y_2 = X \times X \times .....$ so that y_1 is an x and y_2 is an infinite string of $x's$.

2. Assume that $u(x, y_1, y_2) = h(x, y_1) + \beta u(y_1, y_2)$ for some function h.

3. Assume that there is some $\Gamma(x)$ such that $(x, y_1, y_2) \in \Lambda(x)$ if and only if $y_1 \in \Gamma(x)$ and $y_2 \in \Gamma(y_1)$.

Then, under these conditions, the problem from time 1 on, i.e., the problem that we called $P^2(x, y_1)$ above

1. does not depend on x: x enters the problem only as a constant added to the objective function and hence can be dropped (indeed the term $h(x, y_1)$ can be dropped), and

2. is equivalent to the problem $P(y_1)$.

Because of this, we can rewrite the 'result'

$$V^*(x) = U^1(x) = U^2(x, y_1(x)) = u\left(x, y_1(x),\, y_2(x, y_1(x))\right)$$

as

$$V^*(x) = h\left(x, y_1(x)\right) + \beta u(y_1, y_2(y_1)) = h\left(x, y_1(x)\right) + \beta V^*(y_1)$$

Note that our Growth Model IS of this form:

Constraint problem is already this way., i.e., $\Gamma(k_1) = \Gamma(\widehat{k})$.

Something we can do to $u(\widetilde{c}, \widetilde{\ell})$ to make this happen?

$$\begin{aligned} P(\widetilde{k}) : \max_{\{(c_t, x_t, k_t, n_t, \ell_t)\}} & \sum_0^\infty \beta^t u(c_t, \ell_t) & \\ c_t + x_t &\leq F(k_t, n_t) & t = 0, \ldots \\ k_{t+1} &\leq (1-\delta)k_t + x_t & t = 0, \ldots \\ n_t + \ell_t &\leq \overline{n} & t = 0, \ldots \\ k_0 &= \widehat{k}. & \end{aligned}$$

$0 < \beta < 1$, u increasing, concave, etc.

$$
\begin{aligned}
P(\widehat{k};\widehat{x}_0,\widehat{k}_0,\widehat{n}_0,\widehat{\ell}_0,\widehat{k}_0) \quad &: \quad \max_{c_1,\ldots,\ell_1} \sum \beta^t u(c_t,\ell_t) \\
\text{s.t.} \quad c_t + x_t \quad &\leq \quad F(k_t,\ell_t) \quad t = 1,\ldots \\
k_{t+1} \quad &\leq \quad (1-\delta)k_t + x_t \quad t = 1,\ldots \\
n_t + \ell_t \quad &\leq \quad \overline{n} \quad t = 1,\ldots \\
k_1 \quad &= \quad \widehat{k}_1 = (1-\delta)\widehat{k}_0 + \widehat{x}_0
\end{aligned}
$$

1. For any choice of $(\widehat{c}_0,\widehat{x}_0,\widehat{k}_0,\widehat{n}_0,\widehat{\ell}_0)$ This gives a new max problem.
2. Can drop $\beta^0 u(c_0,\ell_0)$ from OBJ and factor out β from remaining.
3. New max problem depends only on $\widehat{k}_1$.

This problem is identical to $P(\widehat{k}_1)$!

It is a time stationary-recursive Max problem!

Let $g(\widehat{k}) \rightarrow$new k, i.e., the k_1 from the solution to $P(\widehat{k})$. Then solution to overall problem SHOULD be $k_0 = \widehat{k},\ \ k_1 = g(\widehat{k})$, $k_2 = g(k_1) = g\left(g(\widehat{k})\right)$ etc. That is, the optimal solution SHOULD have the form *if* $(\widehat{k},k_1,k_2\ldots)$, is the solution for the problem starting from $k_0 = \widehat{k}$, *then* $(k_1,k_2\ldots)$ is the optimal solution for the problem starting from $k_0 = k_1$.

What does this say about $V(\widehat{k}) = \sup \sum \beta^t u(\quad)\ldots$? It follows that:

$$
\begin{aligned}
V(\widehat{k}) \quad &= \quad \sup u(c_0,\ell_0) + \beta V(k') \\
c_0 + x \quad &\leq \quad F(\widehat{k},\delta) \\
k' \quad &\leq \quad (1-\delta)\widehat{k} + x_0 \\
n_0 + \ell_0 \quad &\leq \quad \overline{n}
\end{aligned}
$$

In words, the last term on the RHS of the OBJ is what you get from $t = 1$ on if you have to have optimality from $t = 1$ and given that you start at k'.

2 The Canonical Form

Given the discussion above, we will examine indexed families of optimization problems of the form:

$$
\begin{aligned}
(SP) \text{ or } (SP(x_0)) \qquad & \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
\text{s.t.} \qquad & x_{t+1} \in \Gamma(x_t) \\
& x_0 \in X \quad \text{given.} \\
& \Gamma \subset X. \\
& \Gamma(x) : X \Longrightarrow X
\end{aligned}
$$

$F =$ return function

$\Gamma =$ Feasibility correspondence.. What is possible for x_{t+1} given that the state at t is x_t?

Let $V(x_0)$ denote this sup (possibly $+\infty, -\infty$), (by convention, $\sup_{x \in \emptyset} H(x) = -\infty$).

Example:

1-sector growth model, inelastic labor supply.

$$P(k_0) \;\; \max \sum \beta^t u(c_t)$$

$$\text{s.t.} \quad \begin{cases} c_t + x_t \leq F(k_t, \overline{n}) \\ k_{t+1} \leq (1-\delta)k_t + x_t. \\ k_0 \quad \text{fixed} \end{cases}$$

What is F here?

$$\begin{aligned} c_t &= F(k_t, \overline{n}) - x_t \\ &= F(k_t, \overline{n}) - (k_{t+1} - (1-\delta)k_t) \\ &= F(k_t, \overline{n}) + (1-\delta)k_t - k_{t+1} \\ &\equiv G(k_t, k_{t+1}). \quad G_1 > 0, G_2 < 0. \end{aligned}$$

What is Γ here?

$$k_{t+1} \leq F(k_t, \overline{n}) + (1-\delta)k_t$$

For this choice of F and Γ we can rewrite the 1-sector growth model in canonical form:

$$\begin{aligned} &\max \sum \beta^t u\left(G(k_t, k_{t+1})\right) \\ \text{if} \quad k_{t+1} \quad &\in \quad \Gamma(k_t) = \{k_{t+1} \mid k_{t+1} \leq F(k_t, \overline{n}) + (1-\delta)k_t\} \end{aligned}$$

Example:
Tree Cutting Problem - $P(k_0)$.

Problem - Rewrite TCP in this form.

Example

1 sector growth model, elastic labor supply.

$$P(k_0) \quad \max \sum \beta^t u(c_t, \ell_t)$$

$$\text{s.t.} \quad \begin{cases} c_t + x_t \leq F(k_t, n_t) \\ k_{t+1} \leq (1-\delta)k_t + x_t \\ n_t + \ell_t \leq \overline{n} \\ k_0 \quad \text{fixed.} \end{cases}$$

Problem: rewrite this as SP in canonical form.

Example: 1 sector Growth Model, Multiple Capital Goods, inelastic labor supply.

$$P(k_0) \quad \max \sum \beta^t u(c_t)$$

$$\text{s.t.} \begin{cases} c_t + x_{1t} + \cdots + x_{Jt} \leq F(k_{1t}, \cdots, k_{Jt}, n_t) \\ k_{jt+1} \leq (1-\delta_j)k_{jt} + x_{jt} \\ n_t + \ell t \leq \overline{n}. \\ k_{01,}, \ldots, k_{0J} \quad \text{given} \end{cases}$$

Problem: Rewrite this as SP in canonical form.

Example: Same as above, but $u(c_t, \ell_t)$.

Example: Two sector neo-classical growth model, inelastic labor supply

$$\max \sum \beta^t u(c_t)$$

$$\text{s.t.} \quad \begin{cases} c_t \leq F^c(k_{ct}, n_{ct}) \\ x_t \leq F^x(k_{xt}, n_{xt}). \\ k_{t+1} \leq (1-\delta)k_t + x_t \\ k_{ct} + k_{xt} \leq k_t \\ n_{ct} + n_{xt} \leq \overline{n} \\ k_0 \quad \text{fixed} \end{cases}$$

Problem: Rewrite in Canonical Form.

Problem: Add elastic labor supply.

Problem: Add multiple k's, one sector each.

Can't move x across sectors?

Example: 1-sector Model, adjustment costs in k, inelastic labor supply

$$\begin{aligned} & \max \sum \beta^t u(c_t) \\ \text{s.t.} \quad c_t + x_t &\leq F(k_t, \overline{n}) \\ k_{t+1} &\leq (1-\delta)k_t + g(x_t) \\ & k_0 \quad \text{fixed.} \\ g(0) &= 0, \quad \text{increasing and } \textit{strictly concave.} \end{aligned}$$

INSERT FIGURE HERE

In this example, if you try to make too big a change in k, you lose efficiency.

Problem: Write in CF.

Problem: Add, elastic ℓ, multiple sectors.

Problem: Other forms? Adjustment on n?

$$F(k_t, n_t) - g(n_t - n_{t-1}) \qquad .$$

$$k_{t+1} = (1-\delta)k_t + g\left(\frac{x_t}{k_t}\right)?$$

Example:

(k,h)–model, inelastic ℓ??

$$\max \sum \beta^t u(c_t)$$

$$\text{s.t.} \quad \begin{cases} c_t \leq F^c(k_{ct}, h_{ct}, n_{ct}) \\ x_{kt} \leq F^k(k_{kt}, h_{kt}, n_{kt}) \\ x_{ht} \leq F^h(k_{ht}, h_{ht}, n_{ht}) \\ k_{t+1} \leq (1-\delta_k)k_t + x_{kt} \\ h_{t+1} \leq (1-\delta_h)h_t + x_{ht} \\ h_0, k_0 \quad \text{fixed.} \\ n_{ct} + n_{ht} + n_{kt} \leq \overline{n} \\ k_{ct} + k_{kt} + k_{ht} \leq k_t \\ h_{ct} + h_{kt} + h_{ht} \leq h_t \end{cases}$$

Here, h is interpreted as 'knowledge' of the individual.

Problem: Write in CF

Example:

$$\max \sum \beta^t u(c_t, \ell_t)$$

$$\begin{aligned} \text{s.t.} \quad c_t &\leq F^c(k_{ct}, z_{ct}) \\ x_{kt} &\leq F^k(k_{kt}, z_{kt}) \\ x_{ht} &\leq F^h(k_{ht}, z_{ht}) \\ k_{t+1} &\leq (1-\delta_k)k_t + x_{kt} \\ h_{t+1} &\leq (1-\delta_h)h_t + x_{ht} \end{aligned}$$

Effective Labor supplies:

$$
\begin{aligned}
Z_{ct} &\leq M^{c}(n_{ct}, h_t) \\
Z_{ct} &\leq M^{x}(n_{xt}, h_t) \\
Z_{ht} &\leq M^{h}(n_{ht}, h_t) \\
k_{ct} + k_{xt} + k_{ht} &\leq k_t \\
n_{ct} + n_{xt} + n_{k+1} + \ell_t &\leq \overline{n}
\end{aligned}
$$

Example: Family Labor Supply

$$\begin{aligned}
& \sum \beta^t \left[\lambda_f \, u^f(c_{ft}, \ell_{ft}) + \lambda_m u^m(c_{mt}, \ell_{mt})\right] \\
c_{ft} + c_{mt} + x_t &\leq F(k_t, n_{ft} + n_{mt}) \\
k_{t+1} &\leq (1-\delta)k_t + x_t \\
n_{ft} + \ell_{ft} &\leq \overline{n}_f \\
n_{mt} + \ell_{ft} &\leq \overline{n}_m \\
& k_0 \quad \text{fixed.}
\end{aligned}$$

Add home good?

Example: Fertility

$$\begin{aligned}
& \sum \beta^t u(N_t, c_t/N_t) \\
\text{s.t.} \quad c_t + X_t + \theta N_{t+1} &\leq N_t F(K_t/N_t, \overline{n}) \\
K_{t+1} &\leq (1-\delta)K_t + X_t \\
& K_0 \quad \text{fixed.}
\end{aligned}$$

ETC.

Motivation for the Functional Equation Problem

Suppose that the problem bellow $(SP(k_0))$ is well defined:

$$V(k_0) \equiv \max_{c,x,l,k,n} \left\{u(c_0, l_0) + \beta u(c_1, l_1) + \beta^2 u(c_2, l_2) + ...\right\}$$
$$\text{s.t.} \begin{cases} (c_0, x_0, k_1, n_0, l_0) \in \Gamma(k_0) \\ (c_t, x_t, k_{t+1}, n_t, l_t) \in \Gamma(k_t) \ \forall \ t \geq 1 \end{cases}$$

Suppose that a solution exists: $\{c_0^*, x_0^*, l_0^*, k_0^*, n_0^*, c_1^*, k_1^*, ...\}$.

Now define

$$V(k_1^*) \equiv \max_{c,x,l,k,n} \left\{u(c_1, l_1) + \beta u(c_2, l_2) + \beta^2 u(c_3, l_3) + ...\right\}$$
$$\text{s.t.} \begin{cases} (c_1, x_1, k_2, n_1, l_1) \in \Gamma(k_1^*) \\ (c_t, x_t, k_{t+1}, n_t, l_t) \in \Gamma(k_t) \ \forall \ t \geq 2 \end{cases}$$

One should expect that $\{c_1^*, x_1^*, l_1^*, k_2^*, n_1^*, c_2^*, x_2^*, ...\}$ being a solution to the problem above. Why??

If the guess is wrong, then $\exists$ $\{\overline{c}_1, \overline{x}_1, \overline{l}_1, \overline{n}_1, \overline{k}_2, \overline{c}_2, \overline{x}_2, ...\}$ feasible starting from k_1^* s.th:

$$\sum_{t=1}^{\infty} \beta^{t-1} u(\overline{c}_t, \overline{l}_t) > \sum_{t=1}^{\infty} \beta^{t-1} u(c_t^*, l_t^*) \qquad (***)$$

Also $\{\overline{c}_1, \overline{x}_1, \overline{l}_1, \overline{n}_1, \overline{k}_2, \overline{c}_2, \overline{x}_2, ...\}$ could have been chosen starting from $t = 1$ in the the first problem above.

That means $\{c_0^*, x_0^*, l_0^*, k_1^*, n_0^*, \overline{c}_1, \overline{x}_1, \overline{l}_1, \overline{n}_1, \overline{k}_2, \overline{c}_2, \overline{x}_2, ...\}$is feasible in the first problem.

But from (***),

$$u(c_0^*, l_0^*) + \beta\left[\sum_{t=1}^{\infty}\beta^{t-1}u(\overline{c}_t, \overline{l}_t)\right] > u(c_0^*, l_0^*) + \beta\left[\sum_{t=1}^{\infty}\beta^{t-1}u(c_t^*, l_t^*)\right]$$

But this is a contradiction...

Therefore one may think that the following is true:

$$V(k_0) = u(c_0^*, l_0^*) + \beta V(k_1^*)$$

The previous reasoning give us a heuristic justification for the following functional equation problem (FEP):

$$v(k) = \sup_{k',l,c,n,k}[u(c,l) + \beta v(k')]$$

$$\text{s.t.}\begin{cases} c + x \leq F(k,n) \\ n + l \leq 1 \\ k' \leq (1-\delta)k + x \\ \text{nonnegativity} \end{cases}$$

Back To General Development

Sequence Problem for the Initial Condition $x - SP(x)$

Find $V(x)$ which is defined by:

$$V(x) = \sup_{(x_0,\ldots)}\sum_{t=0}^{\infty}\beta^t F(x_t, x_{t+1})$$

$$\text{s.t.} \quad x_{t+1} \in \Gamma(x_t).\forall t$$

$$x_0 = x \quad \text{fixed.}$$

Functional Equation Problem:

Find a function $v(x)$ satisfying:

$v(x) \equiv \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)]$

That is, this is an *identity* in x!

Fundamental Theorem of Dynamic Programming (More or less):

(a) *If* $V(x)$ solves $SP(x) \quad \forall x$, *then* $V(x)$ satisfies FEP.

(b) *If* $v(X)$ satisfies FEP then $v(x)$ solves $SP(x) \quad \forall x$.

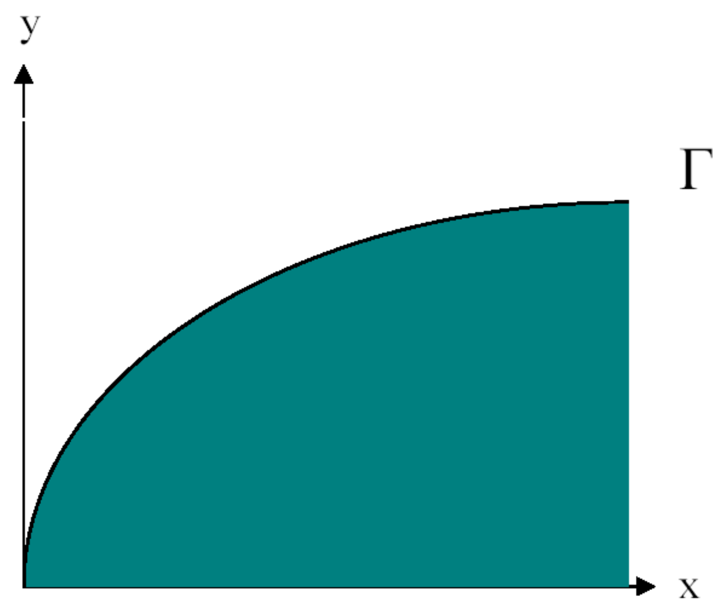
N. B. This can't be quite true as stated because:
$v(x) \equiv \ -\infty$
$v(x) \equiv \ +\infty$

always solve FEP., but won't necessarily solve SP. So, some conditions have to be added to (b).

3 The Details

Let $A = \{(x,y) \in X \times X \mid y \in \Gamma(x)\}$

This is the graph of Γ.



Example of the graph of Γ

Let

$$\pi(\overline{x}) = \{(x_0, \ldots,) \in X^\infty \mid x_{t+1} \in \Gamma(x_t)\ \forall t \geq 0, x_0 = \overline{x}\}.$$

$\pi(x_0)$ is the feasible set for $SP(x_0)$.

Assumption 4.1 $\Gamma(x) \neq \phi, \quad \forall x \in X.$

Problem: Show that Assumption 4.1 $\Longrightarrow \pi(x_0) \neq \phi \quad \forall x \in X.$
Assumption 4.2 $\forall x_0 \in X$ and all $\tilde{x} \in \pi(x_0)$,
$\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists.

N.B. We allow $+\infty, -\infty$ as possible limits, i.e., $\exists \quad a \in \bar{R} \equiv R \cup \{-\infty, +\infty\}$, such that

$$\lim_{n\to\infty} \sum_{t=0} \beta^t F(x_t, x_{t+1}) \to a.$$

INSERT GRAPH HERE.

Sufficient conditions for Assumption 4.2

SC1A4.2 $|F(x,y)| \leq M, \quad \forall (x,y) \qquad \text{and} \qquad 0 < \beta < 1.$

Problem: Prove SC1A4.2 $\Longrightarrow$ A4.2 holds.

SC2A4.2 $\forall\, x_0 \in X, \exists\ \theta, c,\ 0 < c < \infty, 0 < \theta < 1/\beta$ such that
$\tilde{x} \in \pi(x_0) \Longrightarrow F(x_t, x_{t+1}) \leq c\,\theta^t.$

Problem: Prove SC2A4.2 $\Longrightarrow$ A4.2

For each n, define $u_n : \pi(x_0) \to R$ by

$$u_n(\widetilde{x}) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}).$$

i.e., the partial sum.

And define $u(\widetilde{x}) = \lim_{n\to\infty} u_n(x)$.

By A.4.2, $u : \pi(x_0) \to \bar{R}$.

Finally, define $V^* : X \to \bar{R}$ by:

$$V^*(x) = \sup_{\overline{x}\in\pi(x_0)} u(\overline{x}).$$

What it means for V^* to solve SP:

Then V^* is a well-defined function satisfying:

a. If $|V^*(x_0)| < \infty$ then

$$V^*(x_0) \geq u(\tilde{x}) \qquad \forall \tilde{x} \in \pi(x_0) \tag{2}$$

and

$$\begin{aligned} \forall \varepsilon &> 0, \quad \exists \tilde{x} \in \pi(x_0) \Longrightarrow \\ u(\tilde{x}) &\geq V^*(x_0) - \epsilon. \end{aligned} \tag{1}$$

b. If

$$\begin{aligned} V^*(x_0) &= +\infty, \quad \exists\, \tilde{x}^k \in \pi(x_0) \Rightarrow \\ u\left(\tilde{x}^k\right) &\rightarrow \infty. \end{aligned}$$

c. If $V^*(x_0) = -\infty$ then $u(\tilde{x}) = -\infty\ \forall \tilde{x} \in \pi(x_0)$.

That is V^* divides X into 3 mutually exclusive and exhaustive subsets.

$$\begin{aligned} X &= A \cup B \cup C \\ |V^*(x)| &< \infty \rightarrow \qquad x \in A \\ V^*(x) &= +\infty \rightarrow \qquad x \in B \\ V^*(x) &= -\infty \rightarrow \qquad x \in C \end{aligned}$$

What it means for v^* to solve FE.

a. If $|v^*(x_0)| < \infty$ then

$$v^*(x_0) \geq F(x_0, y) + \beta v^*(y) \qquad \forall\ y\ \Gamma(x_0) \tag{4}$$

and $\forall \varepsilon > 0, \quad \exists\, y \in \Gamma(x_0)$ such that

$$v^*(x_0) \leq F(x_0, y) + \beta v^*(y) + \varepsilon. \tag{5}$$

b. If $x^*(x_0) = +\infty, \quad \exists\, y^k \in \Gamma(x_0)$ such that

$$\lim_{k \to \infty} \{F\left(x_0, y^k\right) + \beta v^*\left(y^k\right)\} = \infty. \tag{6}$$

c. If $v^*(x_0) = -\infty$ then $F(x_0, y) + \beta v^*(y) = -\infty$, $\forall y \in \Gamma(x_0)$.

Thus, as above, if v^* is a solution to FE it divides $X = \hat{A} \cup \hat{B} \cup \hat{C}$ such that:

$$\begin{aligned} |v^*(x)| &< \infty \rightarrow x \in \hat{A} \\ v^*(x) &= +\infty \rightarrow x \in \hat{B} \\ v^*(x) &= -\infty \rightarrow x \in \hat{C}. \end{aligned}$$

Goal

Theorem A. If V^* solves SP then V^* solves FE.

Theorem B. If v^* solves FE then v^* solves SP.

That is – $A = \hat{A}$, $B = \hat{B}$ and $C = \hat{C}$.

Most of the difficulties are with sup instead of max, and the possibility that V^* and/or $v^* = \pm\infty$ for some/all x_0's.

Lemma 4.1 Suppose A4.2. Then $\forall x_0 \in X$ and $\forall \tilde{x} \in \pi(x_0)$

$$u(\tilde{x}) = F(x_0, x_1) + \beta u\left(\tilde{x}^1\right)$$

where $x^1 = (x_1, \ldots)$.
Proof. Under 4.2. $\forall x_0 \in X$, $\forall \tilde{x} \in \pi(x_0)$,

$$\begin{aligned} u(\tilde{x}) &= \lim_{n\to\infty} \sum_0^n \beta^t F(x_t, x_{t+1}), \\ &= \lim_{n\to\infty} F(x_0, x_1) + \beta \lim_{n\to\infty} \sum_{t=0}^n \beta^t F(x_{t+1}, x_{t+2}) \\ &= F(x_0, x_1) + \beta \cdot u\left(\tilde{x}^1\right) \ . \end{aligned}$$

Where the last equality comes from the definition of $u(\tilde{x}^1)$.
Theorem 4.2 Under A.4.1, and A.4.2 if V^* solves SP then V^* solves FE. That is:

$$\forall x \in X, \quad V^*(x) = \sup_{y \in \Gamma(x)} F(x, y) + \beta V^*(y).$$

Intuition/Discussion of Proof:

First, for this first part it's useful to do an intuitive version of the proof and then go through the technical details. YOU SHOULD DO THIS YOURSELF FOR THE OTHER PARTS OF THE PROOF!

To start, suppose $|V^*(x)| < \infty$ for a particular x. For example, $V^*(x) = 7.218$. And what we want to show is – at this same x, the valued 7.218 solves the FE. In other words,

$$7.218 = \sup\nolimits_{y \in \Gamma(x)} F(x, y) + \beta V^*(y).$$

What would it mean to show this? First, it means that 7.218 is an upper bound for the RHS of this equation. Second it means that there is no other, smaller upper bound for the RHS.

To see that 7.218 is an upper bound for the RHS, proceed by contradiction. That is, there is some $y^* \in \Gamma(x)$ with:

$$F(x, y^*) + V^*(y^*) > 7.218.$$

From here, we proceed to construct a feasible plan beginning from x, $\tilde{x}$, for which $u(\tilde{x}) > 7.218$. To do this, first construct a plan from y^*, $\tilde{y}$ which is feasible ($\tilde{y} \in \pi(y^*)$) and gets really close to $V^*(y^*)$. Then, it can be checked that $\tilde{x} = (x, y^*, \tilde{y}) \in \pi(x)$ and by construction

$$u(x, y^*, \tilde{y}) = u(\tilde{x}) \text{ is really really close to } F(x, y^*) + V^*(y^*) > 7.218.$$

But this is a contradiction that 7.218 is an upper bound for the problem $SP(x)$.

The rest of the proofs are similar intuitively. And all that is left is to fill in the $\varepsilon' s$ and $\delta' s$.

Back to proof:

Suppose $|V^*(x)| < \infty$ for a particular x.

Need to show

$$V^*(x) = \sup_{y \in \Gamma(x)} F(x,y) + \beta V^*(y)$$

since $V^*(x) < \infty$. This is the same as showing

$$V^*(x) \geq F(x,y) + \beta V^*(y) \qquad \forall y \in \Gamma(x) \tag{4A}$$

and $\forall \varepsilon > 0$, $\exists y \in \Gamma(x) \Rightarrow$

$$V^*(x) - \varepsilon < F(x,y) + \beta V^*(y) \tag{5A}$$

To show (4A):
Let $x_1 \in \Gamma(x)$ and choose $\varepsilon > 0$. By the definition of $V^*(x_1)$, $\exists \tilde{x}^1 = (x_1, \ldots) \in \pi(x_1)$ such that
$$u(\tilde{x}^1) \geq V^*(x_1) - \varepsilon.$$
Since $x_1 \in \Gamma(x)$ and $\tilde{x}^1 \in \pi(x_1)$, it follows that $(x_1, \tilde{x}^1) \in \pi(x)$. Thus, from (2) and Lemma 4.1

$$\begin{aligned} V^*(x) &\geq u\left(x_1, \tilde{x}^1\right) = F(x, x_1) + \beta u\left(\tilde{x}^1\right) \\ &\geq F(x, x_1) + \beta V^*(x_1) - \beta\varepsilon. \end{aligned}$$

Since ε was arbitrary,

$$V^*(x) \geq F(x, x_1) + \beta V^*(x_1) \qquad \forall x_1 \in \Gamma(x)$$

follows.
Note—implicit: If $|V^*(x)| < \infty$ and $y \in \Gamma(x)$ then $|V^*(y)| < \infty$—show this.

To show that (5A) holds at x, choose $\varepsilon > 0$. From (3) $\exists \tilde{x} \in \pi(x)$, $\tilde{x} = (x, x_{1,\ldots})$ such that

$$\begin{aligned} V^*(x) &\leq u(\tilde{x}) + \varepsilon \\ &= F(x, x_1) + \beta u\left(\tilde{x}^1\right) + \varepsilon \end{aligned}$$

where $\tilde{x}^1 = (x_1, \ldots)$. The equality comes from Lemma 4.1.

But $\tilde{x} \in \pi(x) \Rightarrow x_1 \in \Gamma(x)$ and by the definition of $V^*(x_1)$, it follows that

$$\begin{aligned} V^*(x) &\leq F(x, x_1) + \beta u\left(\tilde{x}^1\right) + \varepsilon \\ &\leq F(x, x_1) + \beta V^*(x_1) + \varepsilon \end{aligned}$$

That is $x_1 \in \Gamma(x)$ is a choice of y that will work in (5A).
If $V^*(x_0) = \infty$ then

$\exists \tilde{x}^k \in \pi(x_0)$ such that $u\left(\tilde{x}^k\right) \to \infty$. Since $x_1^k \in \Gamma(x_0), \forall k$, and $u\left(\tilde{x}^k\right) \to \infty$,

$u\left(\tilde{x}^k\right) = F\left(x_0, x_1^k\right) + \beta u\left(\tilde{x}^k\right) \leq F\left(x_0, x_1^k\right) + \beta V^*\left(x_1^k\right)$.

It follows that (6) holds for the sequence $y^k = x_1^k$, and $x_1^k \in \Gamma(x_0), \forall k$.
If $V^*(x_0) = -\infty$ then

(7) $\quad u(\tilde{x}) = F(x_{0,}x_1) + \beta u(\tilde{x}^1) = -\infty \; , \forall \tilde{x} \in \pi(x_0)$.

Since $F(x, y) \in R$, $\forall (x, y)$, it follows that $u(\tilde{x}^1) = -\infty$, $\forall x_1 \in \Gamma(x_0)$, $\forall \tilde{x}^1 \in \pi(x_0)$.

Hence, $V^*(x_1) = -\infty \; \forall x_1 \in \Gamma(x_0)$.

But, since F is real valued and $\beta > 0$, (7) follows from this. This completes the proof.

Theorem 4.3 (Theorem B) Suppose A.4.1, A.4.2 hold. If v^* is a solution to FE

AND

(8) $\quad \lim_{n\to\infty} \beta^n v(x_n) = 0 \qquad \forall \tilde{x} \in \pi(x_0) \quad \forall x_0 \in X$.

Then $v^* = V^*$.

What does it mean to show this?

Proof.

1. If $v^*(x_0) < \infty$, then (4) and (5) hold. It's enough to show that (2) and (3) hold.

First (2):
$\tilde{x} \in \pi(x_0) \Longrightarrow x_1 \in \Gamma(x_0)$ so (4) $\Longrightarrow \forall x \in \pi(x_0)$,

$$\begin{aligned}
v^*(x_0) &\geq F(x_0, x_1) + \beta v^*(x_1) \\
&\geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 v^*(x_2) \\
&\vdots \\
&\geq u_n(\tilde{x}) + \beta^{n+1} v^*(x_{n+1})
\end{aligned}$$

so

$$v^*(x_0) \geq \lim u_n(\tilde{x}) + \lim \beta^{n+1} v^*(x_{n+1}).$$

So from (8),
$v^*(x_0) \geq u(\tilde{x})$,$\forall \tilde{x} \in \pi(x_0)$,
i.e. (2) holds.
To see that (3) holds, fix $\varepsilon > 0$. We want to find an $\tilde{x} \in \pi(x_0)$ such that $u(x) \geq V^*(x) - \varepsilon$.

Choose $\delta_t \in R$ such that $\sum_{t=1}^{\infty} \beta^{t-1} \delta_t \leq \varepsilon/2$.

Since (5) holds (at all x?), [show if it holds at x_0, it must hold at x_1?], we can find $x_1 \in \Gamma(x_0)$, $x_2 \in \Gamma(x_1), \ldots$ so that $v^*(x_t) \leq F(x_t, x_{t+1}) + \beta v^*(x_{t+1}) + \delta_{t+1}$. Then, $(x_0, x_1, \ldots) \in \pi(x_0)$ by construction, and

$$\begin{aligned}
v^*(x_0) &\leq \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} v^*(x_{n+1}) + (\delta_1 + \cdots + \beta^n \delta_{n+1}) \\
&\leq u_n(x) + \beta^{n+1} v^*(x_{n+1}) + \varepsilon/2.
\end{aligned}$$

Thus, using (8),

$$v^* (x_0) \leq u_n (\tilde{x}) + \varepsilon$$

for all n sufficiently large (large enough so that $\beta^{n+1} v^* (x_{n+1}) < \varepsilon/2$). Taking limits gives

$$v^* (x_0) \leq u (\tilde{x}) + \varepsilon,$$

i.e. (3) holds for this $\tilde{x}$.

$v^* (x_0) = -\infty$ case, not possible by (7) and (8). Why?

If $v^*(x_0) = \infty$, i.e. (6) holds, want to show $\exists x^k \in \pi (x_1)$ such that $u \left(\tilde{x}^k\right) \to \infty$.

As a first step, we establish the following Claim:

Claim. There exists a n, $\infty > n \geq 0$, and $(x_0, \ldots, x_n)$ such that:

i. $x_t \in \Gamma (x_{t-1}), \quad \forall t = 1, \ldots, n$

ii. $v^* (x_t) = \infty, \quad \forall t = 0, \ldots n$

iii. $v^* (x_{n+1}) < \infty, \quad \forall x_{n+1} \in \Gamma (x_n)$

Proof. Suppose not. I.e., suppose that for $\forall n$, and $\forall (x_0, x_1^n, \ldots, x_n^n)$ such that $x_t^n \in \Gamma \left(x_{t+1}^n\right) \forall t$, $\exists x_{n+1}^n \in \Gamma (x_n^n)$ with $v^* \left(x_{n+1}^n\right) = \infty$. Then consider the sequence $\tilde{x} = (x_0, x_1^0, x_2^1, \ldots)$. By construction, $x_{n+1}^n \in \Gamma \left(x_n^{n-1}\right), \forall n$ and $v^* \left(x_{n+1}^n\right) = \infty, \forall n$. But then $\beta^n v^* \left(x_{n+1}^n\right) \nrightarrow 0$. Contradicting (8).
So, choose such an n and such a sequence $x_n \in \Gamma (x_{n-1})$. Fix an $A > 0$. Since $v^* (x_n) = \infty$, by (6) we can choose $x_{n+1}^A \in \Gamma (x_n)$ such that

$$F \left(x_n, x_{n+1}^A\right) + \beta v^* \left(x_{n+1}^A\right) \geq \beta^{-n} \left[A + 1 - \sum_{t=0}^{n-1} \beta^t F (x_t, x_{t+1})\right]. \qquad (*)$$

Also, since $v^* \left(x_{n+1}^A\right) < \infty$, we can find $\tilde{x}_{n+1}^A$ such that

i.

$$\tilde{x}_{n+1}^A \in \pi \left(x_{n+1}^A\right)$$

ii.

$$u\left(\tilde{x}^A_{n+1}\right) \geq v^*\left(x^A_{n+1}\right) - \beta^{-(n+1)}. \qquad (**)$$

Then, by construction, $\tilde{x}^A \equiv \left(x_0, \ldots, x_n, \tilde{x}^A_{n+1}\right) \in \pi(x_0)$, and

$$\begin{aligned} u\left(\tilde{x}^A\right) &= \sum_{t=0}^{n-1} \beta^t F(x_t, x_{t+1}) + \beta^n F\left(x_n, x^A_{n+1}\right) + \beta^{n+1} u\left(\tilde{x}^A_{n+1}\right) \\ &\geq \sum_{t=0}^{n-1} \beta^t F(x_t, x_{t+1}) + \beta^n \beta^{-n}\left[A + 1 - \sum_{t=0}^{n-1} \beta^t F(x_t, x_{t+1})\right]. \end{aligned}$$

From * and **

$$\beta^{n+1} v^*\left(x^A_{n+1}\right) + \beta^{n+1}\left[v^*\left(x^A_{n+1}\right) - \beta^{-(n+1)}\right] = A.$$

Thus $V^*(x_0) \geq u\left(\tilde{x}^A\right) \geq A$,$\forall A \Rightarrow V^*(x_0) = \infty$.

This completes the proof of the Theorem.

It follows that there can be AT MOST one solution to FE satisfying (7) since by Theorem 4.3, every solution satisfying (7), v^* satisfies $v^*(x) = V^*(x)$.

FE is called Bellman's Equation. Theorems 4.2 and 4.3 $\leftrightarrow$ "Principle of Optimality".

Problem. Show (8) is necessary.

Problem. Suppose that

1. $\forall x \in X$, $\exists x^*(x)$ such that $V^*(x) = u(x^*(x)) < \infty$, i.e., there is a solution and u is finite at the solution—max = sup.
2. $\forall x \in X$,$\exists y^*(x)$ solving $\max_{y\in\Gamma(x)}\left[F(x,y) + \beta V^*(x)\right]$.

Prove Theorem 4.2 in this case.

Problem. Suppose that v^* satisfies FE, that (1) holds, (2) holds for v^* and (3) v^* is bounded. Prove Theorem 4.3 in this case.

Problem. Suppose $T < \infty$.

$$\text{(SP)} \qquad \max \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$$
$$\text{s.t.} \qquad x_{t+1} \in \Gamma(x_t) \qquad t = 0, \dots \quad (x_{t+1} = \text{"final rate"}).$$

Define $V^* = \sup$ as before. And define the FE as before—Are Theorems (4.2), (4.3) satisfied? If yes prove it, if no show where proofs go wrong.

Plans, Optimal Plans, Policy Rules and Policy Functions.
$\tilde{x} \in \pi(x_0)$ is a *Feasible Plan* (from x_0).
$\tilde{x}^* \in \pi(x_0)$ is an *Optimal Plan* (from x_0) if and only if $V^*(x_0) = u(\tilde{x}^*)$.
That is, the sup is attained at the plan x^*.
Note: There may be more than 1 given our assumptions so far.

Problem. Give an example with multiple optimal plans.

Optimal plans satisfy BE.

Theorem 4.4. Under A.4.1, A.4.2, if $\tilde{x}^* \in \pi(x_0)$ is an OP, $x^* = (x_0^*, x_1^*, \dots)$. Then

1. $$V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*) \qquad t = 0, \dots \tag{9}$$

2. $(x_{t+1}^*, \dots)$ is an OP from x_t^*.

Proof. Since x^* is an OP, $x_1^* \in \Gamma(x_0)$ and

$$\begin{aligned} V^*(x_0) &= u(\tilde{x}^*) = F(x_0, x^*) + \beta u(\tilde{x}^{*\prime}) \\ &\geq u(\tilde{x}) = F(x_0, x_1) + \beta u(\tilde{x}') \qquad \forall x \in \pi(x_0). \end{aligned} \tag{2}$$

In particular, this holds for all feasible plans with $x_1 = x_1^*$. Now, $(x_1^*, x_2, \dots) \in \pi(x_1^*) \Rightarrow (x_0, x_1^*, \dots) \in \pi(x_0)$

(since $x_1^* \in \Gamma(x_0)$).

Thus, from (10), $\forall (x_1^*, x_2, \ldots) \in \pi(x_1^*)$,

$$F(x_0, x_1^*) + \beta u(\tilde{x}^{*\prime}) \geq F(x_0, x_1^*) + \beta u(x_1^*, x_2, \ldots)$$

so

$$u(\tilde{x}^{*\prime}) \geq u(x_1^*, x_2, \ldots) \ \forall (x_1^*, x_2, \ldots) \in \pi(x_0).$$

Thus, $u(\tilde{x}^{*1}) = V^*(x_1^*)$ and $(x_1^*, x_2^*, \ldots)$ is an OP from x_1^* (since we just showed that it attains the sup.

This proves (1) and (2) for $t = 0$. Now proceed exactly the same using induction.

Converse

Theorem 4.5. Under A.4.1, A.4.2. If $x^* \in \pi(x_0)$ and

$$\limsup_{t\to\infty} \beta^t V^*(x_t^*) \leq 0 \tag{11}$$

and (9) holds at this $\tilde{x}^*$. Then $V^*(x_0) = u(\tilde{x}^*)$ i.e., x^* attains the sup.

Proof. Suppose $\tilde{x}^*$ satisfies (9) and (11). Then,

$$\begin{aligned}
V^*(x_0) &= F(x_0, x_1^*) + \beta V^*(x_1^*) \\
&= F(x_0, x_1^*) + \beta\left[F(x_1^*, x_2^*) + \beta V^*(x_2^*)\right] \\
&= u_1(\tilde{x}^*) + \beta^* V^*(x_2^*) \\
&\ \ \vdots \\
&= u_n(\tilde{x}^*) + \beta^{n+1} V^*\left(x_{n+1}^*\right).
\end{aligned}$$

Taking limits then and using (11) we get

$$V^*(x_0) = \lim u_n(\tilde{x}^*) = u(\tilde{x}^*)$$

i.e. x^* attains the sup so x^* is an OP.

Let $G : X \Longrightarrow X$ satisfy $G(x) \subset \Gamma(x)$,$\forall x$.
G is called a "policy correspondence". It is a subset of feasible actions at x.

$g : X \to X$

is a "policy function" if g is a "policy correspondence" AND $g(x)$ is a single point for all $x \in X$.
IF $\tilde{x} = (x_0, \ldots)$ satisfies $x_{t+1} \in G(x_t)$,$\forall t$, then $\tilde{x}$ is said to be *generated from* x_0 by G. It's a possible path if you always follow the "policy" G.

Finally, G^* = optimal policy correspondence:

$$G^*(x) = \{y \in \Gamma(x)\, V^*(x) = F(x, y) + \beta V^*(y)\}.$$

Then from Theorem 4.2 and Theorem 4.4:

If x^* is an OP from x_0, then $x^*_{t+1} \in G^*(x^*_t)\ \forall t$.

Conversely

If

$\tilde{x}^* \in \pi(x_0)$ AND

$x^*_{t+1} \in G(x^*_t)\ \forall t$ AND

(11),

then $\tilde{x}^*$ is an OP from x_0.

So find G^*, then $x^*_{t+1} \in G^*(x^*_t)$ defines the time series of the solution.

Thus, to solve the SP we have the following outline:

1. Find V^*.

2. Given V^*, find G^*

3. Check that (11) is satisfied.

This shows that G^* = OP.

In principle one could do this using either SP or FE for (1). But, in practice it's easier to do it using FE and this solves (2) at the same time.

Algorithm:

1. Guess $V^0(x)$.

2. Solve $\sup_{y \in \Gamma(x)} F(x, y) + \beta V^0(y)$.

3. For $\forall x$, define $G^0(x) = \arg\max \{y \in \Gamma(x) | F(x, y) + \beta V^0(y)\}$

4. For all x, define V^1 by $V^1(x) = F(x, G^0(x)) + \beta V^0(G^0(x))$.

5. Put V^1 into step (1) and iterate. $\to G^1, V^2, \ldots$

Suppose for some V^0 and some T we find

$$V^{T+1}(x) \equiv F\left(x, G^T(x)\right) + \beta V^T(G(x)) \equiv V^T(x).$$

Then, we see that V^T solves FE, i.e., $V^T = V^*$! (And $G^T = G^*$ as well).

Questions.
What if $V^T \to V$? Will it still work? (yes)
When will V^T converge at all? Does it depend on V_0? (under Blackwell's sufficient condition, $V^T \to V^*$ independent of starting place).

Next order of business:

1. Make sure this procedure works.

2. Get some properties of V^*, G^* going for us.

A.4.3. $X \subset R^\ell$ is convex. Γ non-empty, compact valued and continuous.
Γ is l.h.c. if $\forall x^*, \forall y^* \in \Gamma(x^*), \forall x^n \to x^*, \exists y^n \in \Gamma(x^n)$ such that $y^n \to y^*$.
Γ is u.h.c. if $\forall x^n, \forall y^n$ such that $y^n \in \Gamma(x^n)$ and $(x^n, y^n) \to (x^*, y^*)$ then $y^* \in \Gamma(x^*)$.
Γ is continuous if both of these hold.

Examples: Enter Graphs Here.

A.4.4. F is bounded and continuous, $0 < \beta < 1$.

Problem. Show that if A.4.3 and A.4.4 then A.4.1 and A.4.2.

1. So SP is well defined.

2. Solutions (V^*, G^*) of SP and FE are the same. (V^* is bounded.)

If $|F(x,y)| \leq B \quad \forall \quad x, y \in A$, then $|V^*(x)| \leq \frac{B}{1-\beta} \quad \forall\, x$.

Let $C(X) = \{f : F \Longrightarrow R, \text{ continuous, bounded}\}$. Clearly, *if* V^* is continuous, it is in $C(X)$.

Consider:

$$[1] \qquad v(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta v(y).$$

For any $v \in C(X)$, the RHS of [1] has a solution (maximize a continuous function on the compact set $\Gamma(x)$) and it this maximized value is continuous. Accordingly define the function $T : C(X) \to C(X)$

by if $f(x) \in C(X)$, then

$$(T(f))(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta v(y).$$

Thus [1] is $T(v) \equiv v$, i.e., v is a fixed point of T.

Let $d(f,g) = \sup_x |f(x) - g(x)|$.

It can be shown that under metric d, C is a complete metric space.

Theorem 4.6. If 4.3, and 4.4, then,

1. T has a unique fixed point (which must be V^*).

2. And for all $V_0 \in C(X)$,

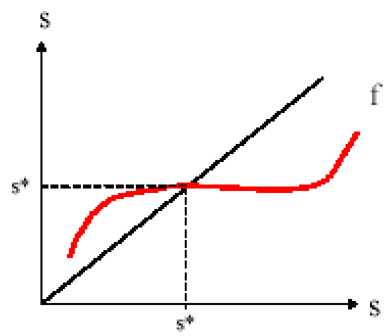
$$\|T^n(V_0) - V^*\| \leq \beta^n \|V_0 - V^*\| \quad (\to 0).$$

3. The opt. policy corres, $G^* \equiv \arg\max_{y \in \Gamma(x)} (F(x,y) + V^*(y))$ is non-empty compact valued and u.h.c.

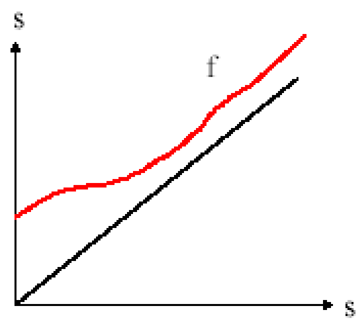
3.1 Some Math We Need Before the Proof

Fixed Points

S a set, $f : S \to S$ a function. S^* is called a fixed point for f if $f(s^*) = s^*$ i.e., f leaves s^* "fixed." Some f's have fixed points and some f's don't.



One fixed point



No fixed point

Brouwer's Theorem

Let $S \subset R^m$ be the closed disk with interior, i.e. $S = \{x \in R^m \mid \|x\| \le 1\}$. Then *every* continuous fct $f : S \to S$ has at least one fixed point.

Results like this are rare! Not true for $C(X)$.

$T : C \to C \cdot f \to f + 1$, i.e., $Tf(x) = f(x) + 1 \quad \forall x$.

T is a very nice mapping but has NO fixed points.

To get a FP, in general you need strong assumptions.

Contractions and Contraction Mapping Theorem

Theorem 3.2 Contraction Mapping Theorem

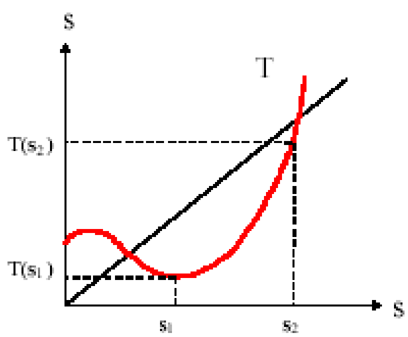
If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction of modulus β, that is, $\rho(T(x), T(y)) \leq \beta\rho(x, y) \forall\, x, y \in S$,
Then,

1. T has exactly one fixed point, s^{*}.

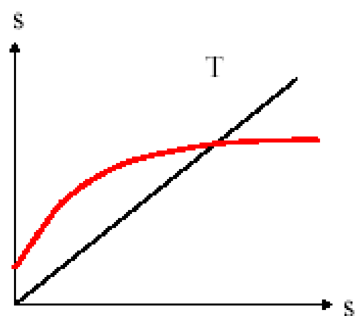
2. $\rho\left(T^n(x), s^*\right) \leq \beta^n \rho(x, s^*). \quad \forall\, n, x \in S.$

Note: T^n defines a difference equation on S, i.e.,
$s, T(s), T(T(s)) = T^2(s), \quad s_0, s_1, s_{2,} \ldots$

We are asking a hard question. When is it true that $T^n(s_0) \to s^* \quad \forall\, s_0$?



Not a contraction



Contraction

Completeness

Let $X \subset R^m$, and define $C(X)$ = bounded continuous functions from $X \to R$.

Define $\|f\| = \sup_{x \in X} |f(x)|$.
This is known as the supnorm of f.

Define $d(f,g) = \|f - g\| = \sup_{x \in X} |f(x) - g(x)|$. It can be shown that d is a metric – that is:

$$\begin{aligned} d &\geq 0 \\ d(f,g) &= d(g,f) \\ d(f,g) &\leq d(f,\widehat{f}) + d(\widehat{f},g) \quad \forall\, f,g,\widehat{f} \end{aligned}$$

Theorem 3.1 $(C(X), \|\cdot\|)$ is a complete metric space.

Theorem 3.2 If (S,ρ) is complete and $\widehat{S} \subset S$ is closed, then $(\widehat{S},\rho)$ is complete also.

Blackwell's Theorem

Theorem 3.3 Let $x \subset R^\ell$, $B(x)$ be a space of bounded real valued functions with:

$$d(f,g) \equiv \sup_{x \in X} |f(x) - g(x)| \equiv \|f - g\|$$

Let $T : B \to B$ satisfy:

(a) $\forall\, f,g \in B$, such that $f(x) \leq g(x) \quad \forall\, x$, then $Tf(x) \leq Tg(x) \quad \forall\, x$, and

(b) $\exists\, \beta \in (0,1)$ such that, $\forall\, a \geq 0, \quad \forall\, f \in B, \quad (T(f+a))(x) \leq Tf(x) + \beta a \quad \forall x \in X$.

THEN, T is a contraction of modulus β.

Theorem of the Maximum:

Theorem 3.6 Let $x \subset R^{\ell}, y \subset R^m, f : X \times Y \rightarrow R$ is continuous and $\Gamma : X \rightarrow Y$ is compact valued and continuous.
Then,

(a) $h(x) = \max_{y \in \Gamma(x)} f(x, y)$ is continuous.

(b) $G(x) = \arg\max_{y \in \Gamma(x)} f(x, y)$ is non-empty, compact valued and u.h.c.

3.2 Back to the Proof of the Theorem

Recall what we want to show:

Theorem 4.6. If 4.3, and 4.4, then,

1. T has a unique fixed point (which must be V^*).
2. And for all $V_0 \in C(X)$,

$$\|T^n(V_0) - V^*\| \leq \beta^n \|V_0 - V^*\| \quad (\rightarrow 0).$$

3. The optimal policy correspondence, $G^* \equiv \arg\max_{y \in \Gamma(x)} (F(x, y) + V^*(y))$ is non-empty compact valued and u.h.c.

Proof: Given any $V_0 \in C$ it follows that

$$P(X) : \max_{y \in \Gamma(x)} F(x, y) + \beta V_0(y)$$

has a continuous objective function and a compact feasible set. So,

(a) $G_{V_0}(x) = \arg\max(\quad)$ is non-empty and compact valued.

(b) $G_{V_0}(x)$ is u.h.c. (Theorem of the Maximum).

(c) $V(x) = F(x, G_{V_0}(x)) + \beta V_0(G_{V_0}(x))$ is bounded and continuous.

Thus, $T : C \to C$ from (c), and (3) follows from (a) and (b) at any fixed point. Thus we need show (1) and (2). These will follow from the Contraction Mapping Theorem once we show that Blackwell's sufficient conditions are satisfied by T.

If $f(x) \leq g(x) \quad \forall\, x,\ f, g \in C$

$$\begin{aligned} T(f)(x) &\equiv \max_{y \in \Gamma(x)} [F(x,y) + \beta f(y)] \\ &\leq \max_{y \in \Gamma(x)} [F(x,y) + \beta g(y)] \\ &\equiv (Tg)(x). \end{aligned}$$

(since it is true pointwise, and F is the same).

If $f \in C$, $a \geq 0$, then

$$\begin{aligned} T(f)(x) &\equiv \max_{y \in \Gamma(x)} [F(x,y) + \beta (f + a)(y)] \\ &= \max_{y \in \Gamma(x)} [F(x,y) + \beta f(y) + \beta a] \\ &\equiv (Tf)(x) + \beta a. \end{aligned}$$

i.e., BSC are satisfied so T is a contraction, so (1) and (2) hold.

Summary Then:
Theorems 4.3 and 4.6 $\Rightarrow V^*$ is bounded and continuous.

Theorems 4.5 and 4.6 $\Rightarrow \exists$ at least one optimal plan...any plan generated by G^* (since $G^* \neq \varnothing$)) implied.

Problem: Show that F bounded is necessary for this. How is this true?

4 Properties of V^*, G^*

A4.5 $\forall\, y$, $F(x, y)$ is strictly increasing in x (but not necessarily in y.)

A4.6 $x \leq x'$ (vector sense) $\Rightarrow \Gamma(x) \subseteq \Gamma(x')$.

Theorem 4.7 *If* A4.3–4.6 hold and V^* is unique solution to:

$$[1] \qquad v^*(x) \equiv \max_{y \in \Gamma(x)} [F(x, y) + \beta v^*(y)],$$

then, V^* is strictly increasing.

Proof: Let $C'(x)$ be the set bounded increasing functions and let C'' be those that are strictly increasing. $C^{\prime'}$ is a closed subset of C and hence, it is also complete under the sup norm. By A4.5 and A4.6, if $v \in C'(x) \Rightarrow T(v) \in C''(x)$, i.e., $T = C' \to C''$. Thus, the unique F.P. of T is in C''. To see this, pick any $V \in C'$, and consider $T^n(V) \in C''$. From above, $T^n(V) \to V^*$– the F.P. of T. Thus, since C' is closed, $V^* \in C'$. But $V^* = T(V^*)$ and hence, $V^* = T(V^*) \in C''$ (since $T(V) \in C'' \quad \forall\, V \in C'$).

A4.7 F is strictly concave.

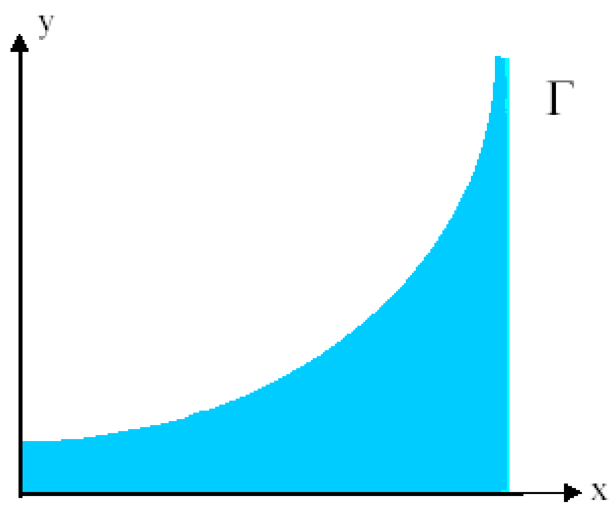
$$\begin{aligned} F\left(\theta(x, y) + (1 - \theta)(x', y')\right) &\geq \theta F(x, y) + (1 - \theta) F(x', y') \\ \forall\, (x, y), (x', y') &\in A, \quad \forall\, \theta \in (0, 1). \end{aligned}$$

Moreover, the inequality is *strict* if $x \neq x$.

A4.8 Γ is convex–(really graph of Γ is convex).

$$\begin{aligned} \forall\, \theta \in \ & [0, 1], \quad \forall\, x, x', y', y' \ni y \in \Gamma(x), y' \in \Gamma(x') \Rightarrow \\ & \theta y + (1 - \theta) y' \in \Gamma\left(\theta x + (1 - \theta)\, x'\right). \end{aligned}$$

Note: This rules out IRS *across* x. e.g., $x \in R, \quad \Gamma(x) = \{y \mid 0 \leq y \leq f(x)\}$. $\Gamma(x)$ is convex $\forall x$, but A is not if f is IRS!



Correspondence convex-valued, but graph not convex

Theorem 4.8 If A4.3, A4.4, A4.7 and A4.8 are satisfied, then V^* is strictly concave and $G^*$is a continuous function.

Proof: Let C' = bounded, continuous, weakly concave functions and let C'' = those that are strictly concave. C' is closed in C. We will show $T(C') \subset C''$.

Suppose $V \in C'$ and $x_0 \neq x, \theta \in (0,1), x_\theta = \theta x_0 + (1-\theta)x_1$.Let $y_i \in G(x_i)$, $i = 0, 1$, and define $y_\theta = \theta y_0 + (1-\theta)y_1$.

Then by 4.8, $y_\theta \in \Gamma(x_\theta)$ for all θ.

Thus,

$$
\begin{aligned}
TV(x_\theta) &\geq F(x_\theta, y_\theta) + \beta V(y_\theta) \quad \text{(since } y_\theta \in \Gamma(x_\theta)\text{)} \\
&> \theta \left[F(x_0, y_0) + \beta V(y_0)\right] + (1-\theta)\left[F(x_1, y_1) + \beta V(y_1)\right] \\
&\quad \text{(strict concavity of } F\text{, A.4.7, weak concavity of } V\text{)} \\
&= \theta TV(x_0) + (1-\theta) TV(x_1) \quad \text{as desired.}
\end{aligned}
$$

i.e., $T(C') \subset C''$, since C' is closed, it follows that the unique FP of $T \in C''$. Since V^*, F are strictly concave, it follows that $\forall\ x$ there is a unique solution to:

$$
\max_{y \in \Gamma(x)} F(x, y) + \beta V^*(y)
$$

that is, $G^*(x)$ is a function. Since it is uhc, it is continuous.

4.1 Other Related Results

Convergence of Approximating Policy Functions:

Theorem 4.9 Suppose V_0 is bounded continuous and concave. Define V_n and g_n by

$$
\begin{aligned}
V_{n+1} &= TV_n \\
g_n &= \arg\max_{y \in \Gamma(x)} F(x, y) + \beta V_n(y)
\end{aligned}
$$

Then,

1. $g_n(x) \to g(x)\ \forall x$
2. if X is compact, $\|g_n - g\| \to 0$.

Differentiability of V^*:

Theorem 4.11 A4.3–4.4, 4.7, 4.8 and F is C^1 on $int(A)$, if $x_0 \in int(X)$, and $g(x_0) \in int(\Gamma(x_0))$, then V is continuously differentiable at x_0, and

$$\frac{\partial V}{\partial x_0}|_{x_0} = \frac{\partial F}{\partial x_i}|_{(x_0, g,(x_0))}.$$

*Proof***:** Choose a neighborhood of x_0, U, such that, $g(x_0) \in \text{int}(\Gamma(x))$ for all $x \in U$. Notice we can do this since $g(x_0) \in \Gamma(x_0)$ and $\Gamma(\cdot)$ is continuous.

Define a function W on U:

$W(x) = F(x, g(x_0)) + \beta V(g(x_0)) \ \forall\, x \in U.$

Then, we have the following subsidiary points:

1. $W(x) \leq V(x) \quad \forall\, x \in U$.
2. $W(x_0) = V(x_0)$.
3. W is concave and C^1, since F is C^1 and $\beta V(g(x_0))$ is a constant.

The first property above follows from the fact that $g(x_0) \in \Gamma(x)$ for all $x \in U$:

$$\begin{aligned} V(x) &= F(x, g(x)) + \beta V(g(x)) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \\ &\geq F(x, g(x_0)) + \beta V(g(x_0)) = W(x) \end{aligned}$$

Next we need the following definition:

A supergradient q of a funtion $H(\cdot) : U \to R$ at x_0 satisfies $H(x) \leq H(x_0) + q(x - x_0) \ \forall\, x \in U$.

Notice that any concave function has at least one supergradient since its hypograph is a convex set. Then the existence of such a q follows from the separating hyperplane theorem.

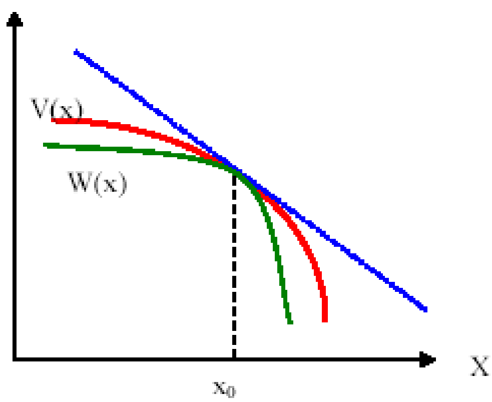
Using 2 and 3 above, it follows that for some supergradient q of $v(\cdot)$ we have

$$W(x) - W(x_0) \leq V(x) - V(x_0) \leq q(x - x_0) \tag{3}$$

Then it follows that q is also a supergradient of $W(\cdot)$ at x_0.

We also have the following result from Convex Analysis: $W(\cdot)$ differentiable $\Longrightarrow q$ unique. Furthermore, any concave function (hence particularly true for $W(\cdot)$) with a unique supergradient at an interior point of its domain is differentiable and $DW(x_0) = q$.

Then using the inequality (3) above and a usual directional limit we get that $DW(x_0) = DV(x_0)$.



Thus V is differentiable at x_0 and

$$\left.\frac{\partial V}{\partial x_i}\right|_{x_0} = \left.\frac{\partial W}{\partial x_0}\right|_{(x_0)} = \left.\frac{\partial F}{\partial x_i}\right|_{(x_0,g,(x_0))}$$

5 Examples

Examples of closed form solutions are rare. (Well, there are 2 or 3).

Example 1 **Full depreciation, Log/Cobb-Douglas.**

$$
\begin{aligned}
& u = \sum \beta^t \log c_t \\
\text{s.t.} \quad & c_t + k_{t+1} \leq A k_t^\alpha .
\end{aligned}
$$

Then,

$$
\begin{aligned}
V^*(k) \; = \; & \\
& \frac{\log A}{(1-\beta)(1-\alpha\beta)} + \frac{1}{1-\beta}\left[\log(1-\alpha\beta) + \frac{\alpha\beta}{(1-\alpha\beta)}\log(\alpha\beta)\right] \\
& + \frac{\alpha}{(1-\alpha\beta)}\log k
\end{aligned}
$$

Proof Just show that V^* is an fixed point for T!

This does not give a lot of insight however.

Alternative:
Guess that $k' = g_k(k)$ is given by $k' = \varphi f(k)$, (constant, savings rate– you might guess this because $k \uparrow \quad r \downarrow$, but under log, c/W independent of r for some φ.)
If correct, this implies that

$$k_{t+1} = \varphi f(k_t) = \varphi A k_t^\alpha \;\; \forall t,$$

and

$$c_t = (1-\varphi)f(k_t) = (1-\varphi)Ak_t^{\alpha}.$$

Thus,

$$\begin{aligned}
k_1 &= \varphi A k_0^{\alpha},\ k_2 = \varphi A k_1^2 = \varphi A(\varphi A k_0^{\alpha})^{\alpha} = (\varphi A)^{1+\alpha} k_0^{\alpha^2} \\
k_3 &= \varphi\ A k_2^{\alpha} = \varphi A\left[(\varphi A)^{1+\alpha} k_0^{\alpha^2}\right]^{\alpha} = (\varphi A)^{1+\alpha+\alpha^2} k_0^{\alpha^3} \\
k_t &= (\varphi A)^{1+\alpha+\ldots\alpha^{t-1}} k_0^{\alpha^t}
\end{aligned}$$

Hence,

$$\begin{aligned}
c_t &= (1-\varphi)Ak_t^{\alpha} = (1-\varphi)A\left[(\varphi A)^{1+\alpha+\ldots+\alpha^{t-1}} k_0^{\alpha^t}\right]^{\alpha} \\
&= (1-\varphi)A[\varphi A]^{\alpha+\alpha^2+\ldots\alpha^t} k_0^{\alpha^{t+1}}
\end{aligned}$$

Thus,

$$\begin{aligned}
u(\varphi) &= \sum \beta^t \log c_t = \sum \beta^t \left(\log\left[(1-\varphi)A \cdot [\varphi A]^{\alpha+\ldots+\alpha^t} k_0^{\alpha^{t+1}}\right]\right) \\
&= \sum \beta^t \left\{\log\left[(1-\varphi)A\right] + (\alpha+\ldots+\alpha^t)\log[\varphi A] + \alpha^{t+1}\log(k_0)\right\} \\
&= \frac{\log\left[(1-\varphi)A\right]}{1-\beta} + \log[\varphi A]\sum_{t=0}^{\infty}\beta^t\sum_{s=1}^{t}\alpha^s \\
&\quad +\alpha\log(k_0)\sum_{t=0}^{\infty}(\beta\alpha)^t
\end{aligned}$$

This uses:

$$\begin{aligned}
(\alpha+\ldots+\alpha^t)(1-\alpha) &= \alpha+\ldots+\alpha^t-\alpha^2-\alpha^3-\ldots-\alpha^{t+1} \\
&= \alpha-\alpha^{t+1} = \alpha(1-\alpha^t) \quad \text{so} \\
\alpha+\alpha^2+\ldots+\alpha^t &= \frac{\alpha(1-\alpha)}{(1-\alpha)}
\end{aligned}$$

Hence,

$$\begin{aligned}
u(\varphi) &= \frac{\log(1-\varphi)}{(1-\beta)} + \frac{\log A}{(1-\beta)} + \frac{\log(\varphi A)}{(1-\alpha)} \sum_{0}^{\infty t} \beta^t \alpha(1-\alpha^t) + \frac{\alpha}{1-\alpha\beta} \log k_0 \\
&= \frac{\log(1-\varphi)}{(1-\beta)} + \frac{\log A}{(1-\beta)} + \frac{\alpha}{(1-\alpha)} \log(\varphi A) \left[\sum_{0}^{\infty} \beta^t - \sum_{0}^{\infty} (\beta\alpha)^t \right] \\
&\quad + \frac{\alpha}{1-\alpha\beta} \log k_0 \\
&= \frac{\log(1-\varphi)}{(1-\beta)} + \frac{\log A}{(1-\beta)} + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)} \log \varphi \\
&\quad + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)} \log A + \frac{\alpha}{(1-\alpha\beta)} \log k_0
\end{aligned}$$

What is the optimal choice of φ?

$$\begin{aligned}
& \max_{\varphi \in (0,1)} u(\varphi) \\
& \Leftrightarrow \\
\max_{\varphi} \widehat{u}(\varphi) \quad \text{where} \quad \widehat{u}(\varphi) &= \log(1-\varphi) + \frac{\alpha\beta}{(1-\alpha\beta)} \log \varphi
\end{aligned}$$

The rest is constants (Note, it had to end up independent of k_0, if this guess is correct otherwise the optimal φ would end up depending on k, $\Rightarrow$ constant φ would have to be wrong!)

FOC

$$\begin{aligned}
\frac{1}{1-\varphi} &= \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{\varphi} \\
1-\varphi &= \frac{1-\alpha\beta}{\alpha\beta}\cdot\varphi \\
1 &= \varphi\left[\frac{1-\alpha\beta}{\alpha\beta}+1\right] = \varphi\left[\frac{1-\alpha\beta+\alpha\beta}{\alpha\beta}\right] = \frac{1}{\alpha\beta}\varphi
\end{aligned}$$

That is

$$\varphi^* = \alpha\beta.$$

So, *if* a policy of this firm *is* optional *then* $\varphi = \alpha\beta$.

To show that this is in fact optimal substitute φ^* into $u(\varphi)$ to get:

$$\begin{aligned}
u(\varphi^*) &= \frac{\log(1-\alpha\beta)}{(1-\beta)} + \frac{\log A}{(1-\beta)}\left[1+\frac{\alpha\beta}{1-\alpha\beta}\right] \\
&\quad + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)}\log(\alpha\beta) + \frac{\alpha}{1-\alpha\beta}\log(k_0).
\end{aligned}$$

Thus, if our guess is correct,

$$\begin{aligned}
V^*(k) &= \frac{\log A}{(1-\beta)(1-\alpha\beta)} + \frac{1}{(1-\beta)}\left[\log(1-\alpha\beta)+\frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)\right] \\
&\quad + \frac{\alpha}{1-\alpha\beta}\log k
\end{aligned}$$

and $g^*(k) = \alpha\beta A k^{\alpha}$.

To show that this is correct, it is necessary and sufficient to verify that V^* defined this way is an fixed point of T, i.e.

$$V^*(k) = \left[\log\left((1-\alpha\beta)Ak^\alpha\right) + \beta V^*\left(\alpha\beta Ak^\alpha\right)\right]$$

or equivalently $g(k) = \alpha\beta Ak^\alpha$ solves

$$\max_{0 \leq y \leq Ak^\alpha} \left[\log\left(Ak^\alpha - y\right) + \beta V^*(y)\right].$$

Problem. Do this.

Alternative Guess and Verify Strategy:

1. Guess that $V^*(k) = D_0 + D_1 \log k$ for some choices of D_0, D_1.
2. For each D_0, D_1 find
$$g_{D_0,D_1}(k) = \arg\max_{0 \leq y \leq Ak^\alpha} \left[\log\left(Ak^\alpha - y\right) + \beta\left[D_0 + D_1 \log y\right]\right].$$
3. Use (2) to find
$$V_{D_0,D_1}(k) \equiv \log\left(Ak^\alpha - g_{D_0,D_1}(k)\right) + \beta\left[D_0 + D_1 \log\left(g_{D_0,D_1}(k)\right)\right].$$
4. Find D_0^*, D_1^* so that $V_{D_0^*,D_1^*}(k) = D_0^* + D_1^* \log k$.

   I.e., use this procedure to form an Educated Guess for V^*.
5. Verify by showing that $V_{D_0^*,D_1^*}$ is a FP of T.

Example 2:L-Q Problems

Example 3: Ak Models

Problems

1.

$$\max \sum \beta^t \log c_t$$
$$\text{s.t.} \qquad c_t + k_{t+1} \leq A k_t^\alpha + (1-\delta) k_t \qquad \delta < 1.$$

Guess that $V^* (k) = B_0 + B_1 \log k$ for some B_0, B_1. What happens when you "Smart Guess"?

2.

$$\max \sum \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \qquad \sigma > 0, \quad \sigma \neq 1$$
$$\text{s.t.} \qquad c_t + k_{t+1} \leq A k_t^\alpha$$

as above.

3.

$$\begin{aligned} &\max \sum \beta^t \log c_t \\ &\text{s.t. } c_t + k_{t+1} \leq A \left[\alpha k_t^\rho + (1-\alpha) 1^\rho\right]^{\frac{1}{\rho}} \\ &\rho \leq 1, \qquad \rho \neq 0. \end{aligned}$$

(Note: this comes from $\bar{n}_t = 1 \ \forall t$, $F(k,n) = u\left[\alpha k^\rho, (1-\alpha) n^\rho\right]^{\frac{1}{\rho}}$, $u(c,\ell) = \log c + 0 \cdot \log \ell$.

6 Applying the Methods

Growth Model with Inelastic Labor Supply

$$\max \sum \beta^t u(c_t) \qquad \text{(SP)}$$
$$\text{s.t. } c_t + k_{t+1} \leq F(k_t, 1) + (1-\delta) k_t$$
$$x_t \geq 0 \qquad k_{t+1} \geq (1-\delta) k_t \qquad c_t \geq 0.$$

Assume that non-negativity is not binding and let:

$$f(k_t) = F(k_t, 1) + (1-\delta) k_t.$$

$$\begin{aligned} &\max \sum \beta^t u(c_t) \qquad\qquad\qquad\qquad (SP) \\ &c_t + k_{t+1} \leq f(k_t). \end{aligned}$$

So,

$$\begin{array}{ll} & \max \sum \beta^t u(f(k_t) - k_{t+1}) \\ \text{s.t.} & 0 \leq k_{t+1} \leq f(k_t) \end{array}$$

where

$$\begin{aligned} F &= u(f(k_t) - k_{t+1}) \\ \Gamma(k_t) &= [0, f(k_t)]. \end{aligned}$$

States: What is x ? $x_t = k_t$ or $x_{t+1} = (k_{t+1}, \ell_t)$.

6.1 Assumptions

Utility

u1. $0 < \beta < 1$;

u2. u is continuous;

u3. u is strictly increasing;

u4. u is strictly concave;

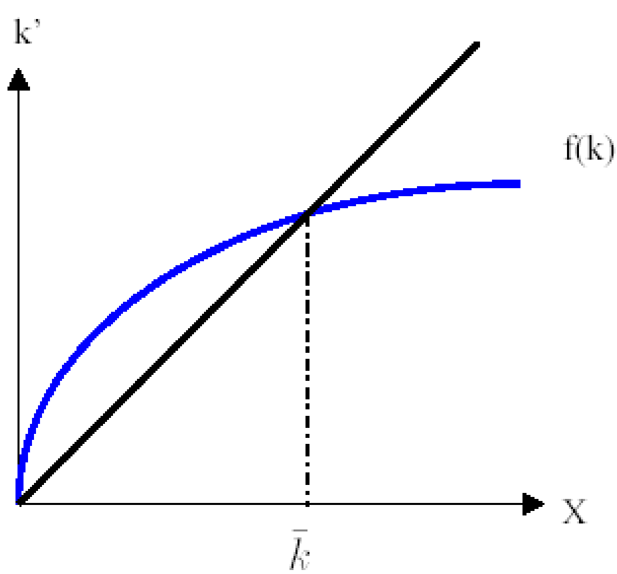
u5. u is C^1.

Technology

t1. f is continuous;

t2. $f(0) = 0$, $\exists\ \bar{k} > 0$ such that

$$\left.\begin{array}{rl} \bar{k} \geq f(k) \geq k & \forall k \in \left[0, \bar{k}\right]. \\ f(k) < k & \forall k \in \left(\bar{k}, \infty\right). \end{array}\right] \bar{k} = \max \text{ sustainable capital stock.}$$



(Feasibility implies k must fail if $k_0 > \bar{k}$ under this condition.)

t3. f is strictly increasing;

t4. f is weakly concave;

t5. f is C^1.

FE

$$V^*(k) \equiv \max_{0 \le y \le f(k)} \left[u(f(k) - y) + \beta V^*(y) \right].$$

Let $X = \left[0, \bar{k}\right]$.

6.2 Results

Then the results from the general case imply under these assumptions: (Not all assumptions are necessary for all parts).

a) SP $\Longleftrightarrow$ FE.

b) There is a unique bounded continuous function solving FE, V^*, and G^* is non-empty and u.h.c. Thus, $\forall k_0 \in \left[0, \bar{k}\right], \exists (k_0^*, k_1^*, \ldots)$ solving (SP).

c) V^* is strictly increasing.

d) V^* is strictly concave, $G^* = g^*$ is a function that is continuous.

e) If $g^*(k) \in (0, f(k))$, then V^* is differentiable at $k \in \left(0, \bar{k}\right)$ and $V^{*\prime}(k) = U'(f(k) - g^*(k)) f'(k)$.

f) If $f'(0) = \infty$, $U'(0) = \infty$, then $0 < g^*(k) < f(k)$ $\forall k \in \left[0, \bar{k}\right]$. (Inada Conditions)

Characterizing g^* :

Recall that g^* solves

$$\max_{0\leq y\leq f(k)} U\left(f\left(k\right)-y\right)+\beta V^{*}\left(y\right).$$

FOC and ENV are:

$$U'\left(f\left(k\right)-g^{*}\left(k\right)\right)=\beta V^{*\prime}\left(g^{*}\left(k\right)\right). \tag{FOC}$$

$$V^{*\prime}\left(k\right)=U'\left(f\left(k\right)-g^{*}\left(k\right)\right)f'\left(k\right). \tag{Env}$$

g) From FOC suppose k is increased. $k_0\to k_1$ with $k_1>k_0$. If $g^{*}\left(k_1\right)\leq g^{*}\left(k_0\right)$ then $f\left(k_0\right)-g^{*}\left(k_0\right)<f\left(k_1\right)-g^{*}\left(k_1\right)$ (since $f(k_1)>f(k_0)$). Thus, from the concavity of U:

$$U'\left(f\left(k_0\right)-g^{*}\left(k_0\right)\right)>U'\left(f\left(k_1\right)-g^{*}\left(k_1\right)\right).$$

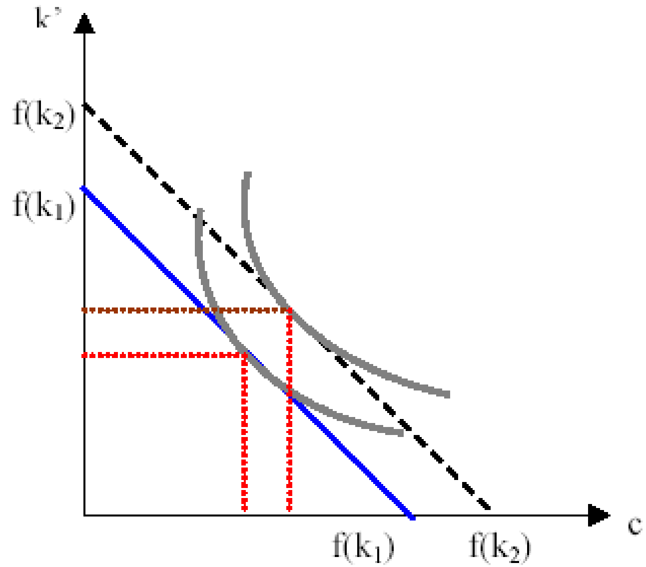
Thus, using the FOC

$$V^{*\prime}\left(g^{*}\left(k_0\right)\right)>V^{*\prime}\left(g^{*}\left(k_1\right)\right).$$

Thus, since V^{*} strictly concave, $g^{*}\left(k_0\right)<g^{*}\left(k_1\right)$, contradiction. Thus, g^{*} is strictly increasing.

h) Since V^{*} concave,

$$\begin{aligned} k &\uparrow \; g^{*}\left(k\right)\uparrow \\ &\Rightarrow V^{*\prime}\left(g^{*}\left(k\right)\right)\downarrow \\ &\Rightarrow U'\left(f\left(k\right)-g^{*}\left(k\right)\right)\downarrow \\ &\Rightarrow \left(f\left(k\right)-g^{*}\left(k\right)\right)\uparrow (U \text{ is concave}), \\ i.e., c^{*}\left(k\right) &= f\left(k\right)-g^{*}\left(k\right) \text{ is increasing in k too!} \end{aligned}$$



This result is a bit of an oddity... this is just a 2 variable budget problem, and we've just shown that both demand curves are increasing in Wealth. Why can't either c or k' be an inferior good?

Steady States i.e. $g(k^*) = k^*$

i) $g(0) = 0$ — feasibility

j) If $k^* = g(k^*)$ use FOC and the ENV to get

$$U'(f(k^*) - k^*) = \beta U'(f(k^*) - k^*) f'\left(k^t\right)$$

so

$$\frac{1}{\beta} = f'(k^*) .$$

k) If $f'(0) = \infty$, $f'(\infty) < \frac{1}{\beta}$ (e.g. $f'(\infty) = 0$). There is at least one strictly positive solution to this. If f is *strictly* concave, there is *exactly* one.

1) *If* $f'(k^*) = \frac{1}{\beta}$ *then* $g(k^*) = k^*$.

From Env:

$$V^{*\prime}(k^*) = U'(f(k^*) - g^*(k^*)) f'(k^*)$$

i.e.

$$\beta V^{*'}(k^*) = U'(f(k^*) - g^*(k^*)).$$

From FOC:

$$U'(f(k^*) - g^*(k^*)) = \beta V^{*\prime}(g^*(k)).$$

Thus

$$\begin{aligned} \beta V^{*\prime}(k^*) &= \beta V^{*\prime}(g^*(k^*)) \\ \Rightarrow k^* &= g^*(k^*) \end{aligned}$$

since V^* is strictly increasing.

Global Dynamics:

First a little Math Result:

If $W(z)$ is strictly concave and differentiable then $(W'(z) - W'(\hat{z}))(z - \hat{z}) \leq 0$ with equality $\Leftrightarrow z = \hat{z}$.

Proof.

$$\begin{aligned} z &< \hat{z} \Rightarrow (W'(z) - W'(\hat{z})) > 0, \qquad (z - \hat{z}) < 0. \\ z &> \hat{z} \Rightarrow (W'(z) - W'(\hat{z})) < 0, \qquad (z - \hat{z}) > 0. \\ z &= \hat{z} \rightarrow 0. \end{aligned}$$

Thus, since V^* is strictly concave and differentiable on $\left(0, \bar{k}\right]$, $z = k$, $\hat{z} = g(k)$ gives

$$[V^{*\prime}(k) - V^{*\prime}(g^*(k))][k - g^*(k)] \leq 0 \qquad \forall k \in \left(0, \hat{k}\right] \qquad (*)$$

with equality iff $k = g(k)$.

From ENV

$$V^{*\prime}(k) = U'(f(k) - g^*(k)) f'(u).$$

From FOC

$$V^{*\prime}(g(k)) = \frac{1}{\beta} U'(f(k) - g^*(k)).$$

Thus * is

$$\left[U'(f(k) - g^*(k)) f'(k) - \frac{1}{\beta} U'(f(k) - g^*(k)\right][k - g^*(k)] \leq 0$$
$$equality \Leftrightarrow k = g(k).$$

or

$$\begin{aligned} \left[f'(k) - \frac{1}{\beta}\right][k - g^*(k)] &\leq 0 \\ equality &\Leftrightarrow k = g(k). \end{aligned}$$

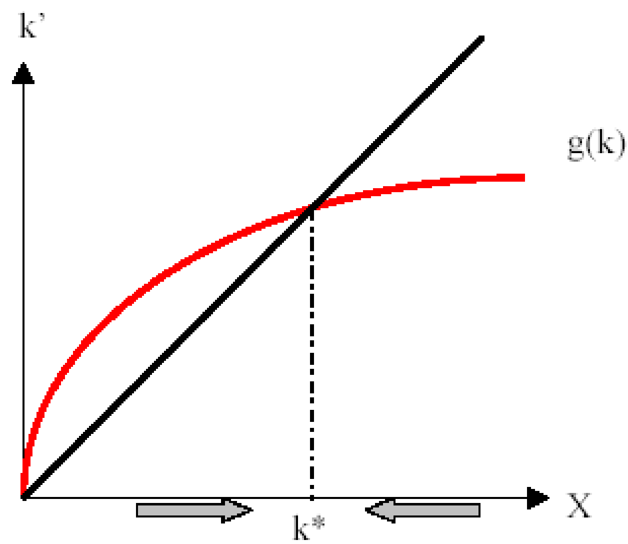
Thus, since $f'(k^*) = \frac{1}{\beta} \Rightarrow k^* = g(k^*)$ as desired. But we already showed that $\exists!$ positive stationary point so that $k \neq g(k)$ for $k \neq k^*$, $k > 0$. Thus

$$\left[f'(k) - \frac{1}{\beta}\right][k - g^*(k)] < 0$$

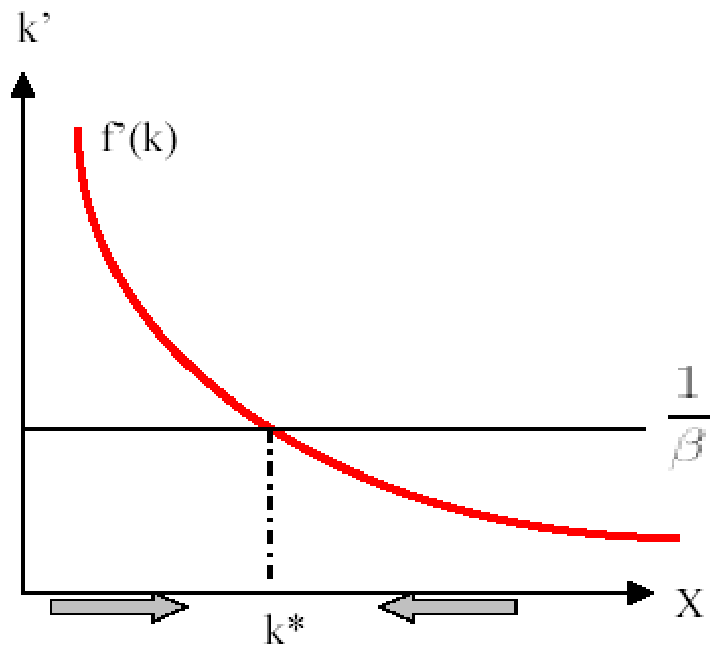
if $k \neq k^*$.

1. If $k < k^* \Rightarrow f'(k) > \frac{1}{\beta} \Rightarrow \left[f'(k) - \frac{1}{\beta}\right] > 0 \Rightarrow (k - g^*(k)) < 0 \Rightarrow k < g^*(k)$, and since g is monotone, $k < k^* \Rightarrow k < g(k) < g(k^*) = k$, $g(k) \in (k, k^*)$.

2. If $k > k^* \Rightarrow f'(k) < \dfrac{1}{\beta} \Rightarrow \left[f'(k) - \dfrac{1}{\beta}\right] < 0 \Rightarrow (k - g^*(k)) > 0 \Rightarrow k > g^*(k)$, and since g is monotone $k > k^*$, $k > g(k) > g(k^*) = k^*$, $g(k) \in (k^*, k)$.



Proposition. In the growth model, there are 2 steady states $k = 0$, $k = k^*$ $\left(f(k^*) = \dfrac{1}{\beta}\right)$. If $k_0 > 0$, $k_0 < k^*$, $k^*_{t+1} > k^*_t$ $\forall t$ and $k^*_t \to k^*$. If $k^* < k_0 < \bar{k}$.... $k^*_{t+1} < k^*_t$ and $k^*_t \to k^*$.



Does it always work this nice? NO!

Theorem 6.1 (Boldrin & Montrucchio) Let $X \subset R$ be compact, $g : X \rightarrow X$, C^2. Then $\exists F, \beta$, and Γ such that $g^* = g$ (and $\Gamma(x) \equiv X$).

An application of Theorems 4.9 and 4.11:

We can think the FEP as a function of the parameter β: different problems using different β's potentially may yield different fixed points v's and different policy functions.

The idea here is to show that $g^*(k; \beta)$ is increasing in β.

More precisely, let $C'(X)$ be the set of concave, bounded and continuous functions $h : X \rightarrow \mathbb{R}$.

Fix $\beta' > \beta$. Let T be the operator using β and consider the operator $T' : C'(X) \to C'(X)$ be the operator using β':

$$T'(h)(k) = \max_{k' \in \Gamma(k)} \{u(f(k) - k') + \beta' h(k')\} \quad \text{for any } h \in C'(X)$$

Let $v_k(\cdot;\beta')$ be the k^{th} iteration using the operator T' and $g_k(\cdot;\beta')$ be the corresponding policy function.

By the contraction mapping Theorem, a fixed point for operator T' will exist and it can be found starting from any initial guess $v_0(\cdot;\beta') \in C'(X)$.

Next use $v^*(\cdot;\beta)$ as the initial guess for the operator T', where $v^*(\cdot;\beta)$ is the fixed point for the operator T:

$$v^*(k;\beta) = \max_{k' \in \Gamma(k)} \{u(f(k) - k') + \beta v^*(k';\beta)\}$$

Applying T' to $v^*(k;\beta)$ we get $v_1(k;\beta')$ and $g_1(k;\beta')$. The function $g_1(k;\beta')$ satisfies:

$$u'\left[f(k) - g_1(k;\beta')\right] = \beta' v^{*'}(g_1(k;\beta');\beta)$$

Now compare the expression above with

$$u'\left[f(k) - g^*(k;\beta)\right] = \beta v^{*'}(g^*(k;\beta);\beta)$$

Claim 1:$g_1(k;\beta') > g^*(k;\beta)$.

Proof:
Suppose not.
Case 1: If $g_1(k;\beta') = g^*(k;\beta)$, then from $\beta' > \beta$ we get $\beta' v^{*'}(g_1(k;\beta');\beta) > \beta v^{*'}(g^*(k;\beta);\beta)$. But it contradicts $u'\left[f(k) - g^*(k;\beta)\right] = u'\left[f(k) - g_1(k;\beta')\right]$.

Case 2: If $g_1(k;\beta') < g^*(k;\beta)$, then by concavity of v^* we have that

$$\beta' v^{*'}(g_1(k;\beta');\beta) > \beta v^{*'}(g^*(k;\beta);\beta).$$

But then by concavity of $u(\cdot)$:

$$[u'[f(k) - g^*(k;\beta)] > [u'[f(k) - g_1(k;\beta')]$$

and therefore $\beta' v^{*'}(g_1(k;\beta')) < \beta v^{*'}(g^*(k;\beta);\beta)$, a contradiction.

Claim 2:$v_1'(k;\beta') > v^{*'}(k;\beta)$ for all $k \in X$.

Proof:
Use Th. 4.11 for the first iteration of T'. Then:

$$v_1'(k;\beta') = u'[f(k) - g_1(k;\beta')]f(k') > u'[f(k) - g^*(k;\beta)]f(k') = v^{*'}(k;\beta)$$

since $g_1 > g^*$.

Now the reasoning goes by induction: we have proved that $v_1'(k;\beta') > v^{*'}(k;\beta) = v_0'(k;\beta')$; therefore suppose $v_n'(k;\beta') > v_{n-1}'(k;\beta')$.

Claim 3: $v_{n+1}'(k;\beta') > v_n'(k;\beta')$ and $g_{n+1}(k;\beta') > g_n(k;\beta')$

The proof for this claim is extremely similar to the proof for claim 1 and 2, and we left it as an exercise.

Therefore we have shown that $g_n(k;\beta')$ is increasing in n and that

$$g_n(k;\beta') > g^*(k;\beta) \ \forall \ n$$

Therefore it follows from Th. 4.9 that $g^*(k;\beta') > g^*(k;\beta)$.