Econ 8106 MACROECONOMIC THEORY

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1 A First Cut Model of Aggregate Time Series

In this section, we lay out the basics of the most common approach to modelling in Economics, Competitive Theory.

Here, we want to build a version of a model that generates output that can be compared with aggregate series from data like the time series of US output, consumption, investment and hours that we have seen in class.

Thus, at a minimum we need a model that has choices of consumption, investment, hours, output, etc. One could make the case for including more, much more, for the sake of realism. This is not what 'modeling' is however.

Models are ALWAYS imperfect representations of the reality that they are supposed to help us understand, by necessity.

Thus, our first cut model will include only two types of agents, firms and 'households.' Since we want to include both consumption and investment, we will include two types of firms.

The basics are:

(A) Two types of firms

- Investment firms (use capital and labor to produce investment goods, goes to HH for investment use) : $j = 1, 2, ..., J_x$
- Each Investment firm has a production function: $F_{xt}^{j}\left(k_{xt}^{j}, n_{xt}^{j}\right)$, $j = 1, 2, ..., J_x$
- Consumption firms (use capital and labor to produce consumption goods, goes to HH for consumption use) : $j = 1, 2, ..., J_c$
- Each Consumption firm has a production function: $F_{ct}^{j}\left(k_{ct}^{j}, n_{ct}^{j}\right)$, $j = 1, 2, ..., J_c$

- Firms are owned by the households: θ_{ij}^{c}, θ_{ij}^{x} : consumer i's ownership of firm j:

$0 \leq \theta_{ij}^{c} \leq 1$ and $0 \leq \theta_{ij}^{x} \leq 1$,

$$\sum_{i=1}^{I} \theta_{ij}^{c} = 1$$

$$\sum_{i=1}^{I} \theta_{ij}^{x} = 1$$

(B) Utility Maximizing Households (HH)

- Indexed by $i = 1, 2, ..., I$
- Infinitely lived agents
- Decide about their own consumption, labor supply, leisure, investment, etc.
- Have endowments of firm ownership (see above) leisure, $\bar{n}_t^i$, and initial capital stocks, k_0^i.
- Have Utility Functions: $U^i(\cdot)$, a function of $(c_t^i, \ell_t^i)_{t=0}^{\infty}$ assumed increasing in c and in ℓ.

A *Competitive Equilibrium* is:

Sequences of prices for consumption (c), investment (x), capital rental (k), and labor wage (n)

$$(p_{ct}, p_{xt}, r_t, w_t) \quad t = 0, 1, ...$$

and sequences of quantities:

(A) Household quantities–$(c_t^i, x_t^i, k_t^i, n_t^i, \ell_t^i)$ $t = 0, 1, ...$ $i = 1, 2, ..., I$

(B) (i) Investment firms quantities– $\left(k_{xt}^j, n_{xt}^j, x_t^j\right)$ $t = 0, 1, ...j = 1, 2, ..., J_x$

(ii) Consumption firms quantities– $\left(k_{ct}^j, n_{ct}^j, c_t^j\right)$ $t = 0, 1, ...j = 1, 2, ..., J_c$

(C) Profits for each household π^i, $i = 1, .., I$

Such That:

< Maximizing Behavior >

(A) For all i $(c_t^i, x_t^i, k_t^i, n_t^i, \ell_t^i)_{t=0}^{\infty}$ solves

$Max \quad U^i(\cdot)$

s.t. $\sum_{t=0}^{\infty} (p_{ct} c_t^i + p_{xt} x_t^i) \leq \sum_{t=0}^{\infty} (w_t n_t^i + r_t k_t^i) + \pi^i$

$k_{t+1}^i \leq (1-\delta) k_t^i + x_t^i$

$\ell_t^i + n_t^i \leq \bar{n}_t^i$ for all t

k_0^i given (constraint on capital formulation by household)

.... non-negativity of all variables in all time periods.

(B1) For all j, $j = 1, 2, ..., J_x$, for all t $\left(k^j_{xt}, n^j_{xt}, x^j_t\right)^\infty_{t=0}$ solves

$Max \quad \sum_{t=0}^{\infty}(p_{xt}x^j_t - w_t n^j_{xt} - r_t k^j_{xt})$

s.t. $x^j_t \leq F^j_{xt}\left(k^j_{xt}, n^j_{xt}\right) \quad \forall\ t$

.......non-negativity of all variables in all time periods.

(B2) For all j, $j = 1, 2, ..., J_c$, $\left(k^j_{ct}, n^j_{ct}, c^j_t\right)^\infty_{t=0}$ solves

$Max \quad \sum_{t=0}^{\infty}(p_{ct}c^j_t - w_t n^j_{ct} - r_t k^j_{ct})$

s.t. $c^j_t \leq F^j_{ct}\left(k^j_{ct}, n^j_{ct}\right) \quad \forall\ t$

...... non-negativity of all variables in all time periods.

<Accounting> (Supply = Demand)

(i) $\sum_{i=1}^{I} c^i_t = \sum_{j=1}^{J_c} c^j_t \quad \forall\ t$

(ii) $\sum_{i=1}^{I} x^i_t = \sum_{j=1}^{J_x} x^j_t \quad \forall\ t$

(iii) $\sum_{i=1}^{I} n^i_t = \sum_{j=1}^{J_x} n^j_{xt} + \sum_{j=1}^{J_c} n^j_{ct} \quad \forall\ t$

(iv) $\sum_{i=1}^{I} k^i_t = \sum_{j=1}^{J_x} k^j_{xt} + \sum_{j=1}^{J_c} k^j_{ct} \quad \forall\ t$

<Profits> (are correct)

$$\sum_{i=1}^{I} \pi^i = \sum_{j=1}^{J_x}\sum_{t=0}^{\infty}\left(p_{xt}x^j_t - w_t n^j_{xt} - r_t k^j_{xt}\right) + \sum_{j=1}^{J_c}\sum_{t=0}^{\infty}\left(p_{ct}c^j_t - w_t n^j_{ct} - r_t k^j_{ct}\right)$$

Note: This is sometimes also called a *Walrasian Equilibrium*, or a *Competitive Equilibrium*, or a *Perfectly Competitive Equilibrium* or an *Arrow-Debreu Equilibrium*. The essence of this is price taking behavior by all agents. This was first formally described/formalized by Walras.

A CE implicitly generates a whole time-series of individual household's allocations and outputs of the firms as well as prices and interest rates as well as the appropriate aggregate quantities for the agents taken as a group.

This model is at the same time both too complex and too simple. It is too complex to just solve as it is – i.e., What time series would it generate, etc.? And it is too simple in that it implicitly (or explicitly) rules out many things. A partial list is:

- Government, both taxing and spending,
- Many types of c or x or n,
- Price-Setting Firms, or Firms with 'market power',
- Production taking time to complete,
- External Effects,
- Banks/Loans/Stock Trading/ more or less all kinds of Finance topics,

- Home Production,
- Uncertainty,
- Money,
- etc.....

Most of these omissions are done to simplify the presentation and some of them will be considered explicitly as we go on.

1.1 Some Properties of CE

Problem: Define a CE when there is only one period, no production (endowment economy) no k or x, but there are endowments of c as well.

Problem: Suppose that $\bar{n}^i = 0$, in the economy above. What is the CE? Is this still true when $\bar{n}^i > 0$? What if $U_\ell = 0$?

Problem: Define a CE when there is only one period, no production (endowment economy) no k or x, but there are two consumption goods.

Define $\pi^j_{xt} = (p_{xt}x^j_t - w_t n^j_{xt} - r_t k^j_{xt})$, $\pi^j_x = \sum_{t=0}^{\infty} \pi^j_{xt}$, and $\pi^j_{ct} = (p_{ct}c^j_t - w_t n^j_{ct} - r_t k^j_{ct})$, $\pi^j_c = \sum_{t=0}^{\infty} \pi^j_{ct}$, these are the profits for each firm in each period and overall respectively.

Problem: Show that if $\left(k^j_{xt}, n^j_{xt}, x^j_t\right)_{t=0}^{\infty}$ solves

$Max \quad \sum_{t=0}^{\infty}(p_{xt}x^j_t - w_t n^j_{xt} - r_t k^j_{xt})$

s.t. $x^j_t \leq F^j_{xt}\left(k^j_{xt}, n^j_{xt}\right) \quad \forall\ t$

.......non-negativity of all variables in all time periods,

Then, in each period,

$\left(k^j_{xt}, n^j_{xt}, x^j_t\right)$ solves

$Max \quad \pi^j_{xt} = (p_{xt}x^j_t - w_t n^j_{xt} - r_t k^j_{xt})$

s.t. $x^j_t \leq F^j_{xt}\left(k^j_{xt}, n^j_{xt}\right)$

.......non-negativity

And that the same holds for consumption firms.

Problem: Is the converse to this problem also true?

Problem: Suppose that there was just one firm which could allocate it's purchases of k and n across all the production functions of all the firms in the economy (how would you formulate this exactly?) Show that it would choose the same total purchases as the individual firms for any sequence of prices. It follows that CE maximizes TOTAL profits as well.

Problem: Show that firm maximizing plans, given prices are also 'cost minimizing.' In every period for every firm, there is no way of choosing an alternative set of inputs that produces the same outputs at lower 'total cost.'

Problem: Show that CE minimizes the total cost of producing the output of the two sectors.

Problem: Consider the alternative version of the Household Maximization Problem:

$Max \quad U^i(\cdot)$

s.t. $\sum_{t=0}^{\infty}(p_{ct}c_t^i + p_{xt}x_t^i + w_t\ell_t^i) \leq \sum_{t=0}^{\infty}(w_t\bar{n}_t^i + r_tk_t^i) + \pi^i$

$k_{t+1}^i \leq (1-\delta)k_t^i + x_t^i$

$\ell_t^i + n_t^i \leq \bar{n}_t^i$ for all t

k_0^i given (constraint on capital formulation by household)

.... non-negativity of all variables in all time periods.

That is, the household 'sells' all of it's hours to the market, and 'buys' back the leisure that it wants. Show that this version gives rise to the same decisions.

1.2 Existence of Equilibrium:

The question of the existence of a CE in settings like this is studied in Bewley (JET-1972). Where it is established under fairly standard assumptions (convexity and monotonicity assumptions) and in a lot more generality than what we have here. For the finite horizon case it was extensively studied by Arrow, Debreu and McKenzie in the 1950's and 1960's. The standard reference that summarizes much of this work is: Debreu, Theory of Value.

1.3 Some Assumptions and Results:

F1) F is continuous, strictly increasing, strictly quasi-concave, weakly concave and $F(0) = 0$ for all j and t.

F2) F is 'constant returns to scale' (CRS), $F(\lambda k, \lambda n) = \lambda F(k, n)$ for all $\lambda \geq 0$, for all (k, n), for all j and for all t.

HH1) U is continuous, strictly increasing and strictly concave.

HH2) Endowments are positive: $k_0^i > 0$ and $\bar{n}_t^i >> 0$ (i.e., $\exists \bar{n} > 0$ such that $\bar{n}_t^i \geq \bar{n}$ for all t).

For the most part, the reasons behind these assumptions are clear. They do things like guarantee the equivalence of cost minimization and utility maximization, the uniqueness of the solution to household maximization problems, etc. Typically, they can be weakened in the results presented below, but this is usually at the cost of some mess.

The reason for NOT assuming the strict concavitiy of the $F's$ is to allow for F2), i.e., CRS production functions are NOT strictly concave, and we want to allow for CRS. (Indeed, we will typically ASSUME CRS!)

Question: What does it mean for the Utility function to be continuous?

Claim: Under A1, $\pi^j_{ct} \geq 0$ for all t, $\pi^j_{xt} \geq 0$ for all t. (Since $F(0) = 0$.)

Problem: Show this.

Claim: Under F1 and F2, $\pi^j_{ct} = 0$ for all t, $\pi^j_{xt} = 0$ for all t.

Problem: Show this.

1.4 Allocations and Feasibility, More Definitions

An *allocation* is a list of the real parts of the above. That is, it is:

(A) Household quantities–$(c^i_t, x^i_t, k^i_t, n^i_t, \ell^i_t)$ $t = 0, 1, ...$ $i = 1, 2, ..., I$

(B) (i) Investment firms quantities– $\left(k^j_{xt}, n^j_{xt}, x^j_t\right)$ $t = 0, 1, ...j = 1, 2, ..., J_x$

(ii) Consumption firms quantities– $\left(k^j_{ct}, n^j_{ct}, c^j_t\right)$ $t = 0, 1, ...j = 1, 2, ..., J_c$

For simplicity in what follows, denote by z an allocation–

$$z = \{\{(c^i_t, x^i_t, k^i_t, n^i_t, \ell^i_t)\}^I_{i=1}, \{\left(k^j_{xt}, n^j_{xt}, x^j_t\right)\}^{J_x}_{j=1}, \{\left(k^j_{ct}, n^j_{ct}, c^j_t\right)\}^{J_c}_{j=1}\}$$

The allocation z is called *feasible* if:

FF) $c^j_t \leq F^j_{ct}(k^j_{ct}, n^j_{ct})$ for all j, t

$x^j_t \leq F^j_{xt}(k^j_{xt}, n^j_{xt})$ for all j, t

non-negativity of all variables.

FHH) $n^i_t + \ell^i_t \leq \bar{n}^i_t$ for all i, t

$k^i_{t+1} \leq (1 - \delta)k^i_t + x^i_t$ for all i, t

$k^i_0 = k^i_0$ for all i

non-negativity of all variables.

FM) $\sum_{j=1}^{J_c} n_{ct}^j + \sum_{j=1}^{J_x} n_{xt}^j \leq \sum_{i=1}^{I} n_t^i$ for all t

$\sum_{j=1}^{J_c} k_{ct}^j + \sum_{j=1}^{J_x} k_{xt}^j \leq \sum_{i=1}^{I} k_t^i$ for all t

$\sum_{i=1}^{I} c_t^i \leq \sum_{j=1}^{J_c} c_t^j$ for all t

$\sum_{i=1}^{I} x_t^i \leq \sum_{j=1}^{J_x} x_t^j$ for all t

Question: Should these be = ? Would it matter? Under what conditions would it make no difference?

Define the set of feasible allocations, $\mathcal{Z}$, by:

$$\mathcal{Z} = \{z | FF, FH, \text{ and } FM \text{ are satisfied}\}$$

Claim: $\mathcal{Z}$ is non-empty.

Problem: Show this.

Claim: Under F1) $\mathcal{Z}$ is a convex set.

Problem: Show this.

Claim: Some sort of monotonicity result.... I had something in the notes last year about this... something like if z is feasible and $z' \leq z$, then z' is also feasible. Pricila said I was either crazy, or just plain wrong (or maybe both...). Can anyone figure out some statement like this that is true, and prove it?

Claim: Under F1) $\mathcal{Z}$ is a compact set in the product topology.

Problem: Show this.

Claim: Under F1) and HH2) $\mathcal{Z}$ has a non-empty interior in the product topology.

Problem: Show this.

Define the function $U : \mathcal{Z} \to \mathcal{R}^I$, by:

$$U(z) = (U^1((c_t^1, \ell_t^1)_{t=0}^{\infty}), U^2((c_t^2, \ell_t^2)_{t=0}^{\infty}), ..., U^I((c_t^I, \ell_t^I)_{t=0}^{\infty}))$$

That is, U assigns to each feasible allocation the vector of utilities that would be received under that allocation.

Define $\mathcal{U} = U(\mathcal{Z})$, i.e., the range of U. This is the set of all feasible utility vectors that can be attained. It is called the Utility Possibilty Set.

Claim: $\mathcal{U}$ is non-empty.

Problem: Show this.

Claim: Under F1 and HH1, $\mathcal{U}$ is compact.

Problem: Show this.

Claim: Under F1 and HH1, $\mathcal{U}$ is convex.

Problem: Show this.

Define the Pareto Frontier as the upper contour of the Utility Possibility Set:

$$\mathcal{PF} = \{u \in \mathcal{U} | \nexists \hat{u} \in \mathcal{U} \text{ st } \hat{u}^i \geq u^i, \forall i \text{ and } \hat{u}^i > u^i \text{ some } i\}$$

Analogously, Define the set of Pareto Optimal Allocations, $\mathcal{PO}$, by:

$$\mathcal{PO} = \{z \in \mathcal{Z} | U(z) \in \mathcal{PF}\}$$

Claim: Under F1 and HH1, if $z \in \mathcal{PF}$,

$$k^i_{t+1} = (1-\delta)k^i_t + x^i_t, \text{ for all } i, t$$

$$n^i_t + \ell^i_t = \bar{n}^i_t, \text{ for all } i, t$$

$$c^j_t = F^j_{ct}(k^j_{ct}, n^j_{ct}), \text{ for all } j, t$$

$$x^j_t = F^j_{xt}(k^j_{xt}, n^j_{xt}), \text{ for all } j, t$$

$$\textstyle\sum_{j=1}^{J_c} n^j_{ct} + \sum_{j=1}^{J_x} n^j_{xt} = \sum_{i=1}^{I} n^i_t \text{ for all } t$$

$$\textstyle\sum_{j=1}^{J_c} k^j_{ct} + \sum_{j=1}^{J_x} k^j_{xt} = \sum_{i=1}^{I} k^i_t \text{ for all } t$$

$$\sum_{i=1}^{I} c_t^i = \sum_{j=1}^{J_c} c_t^j \text{ for all } t$$

$$\sum_{i=1}^{I} x_t^i = \sum_{j=1}^{J_x} x_t^j \text{ for all } t$$

I.e., PO allocations are 'non-wasteful.'

Problem: Show this. Show that it is also true for CE allocations.

Claim: Suppose the solution to the problem:

$$\max_{(u^1,u^2,\ldots,u^I)\in\mathcal{U}} U^1 \qquad s.t. \qquad u^i \geq \bar{u}^i, i = 2, ..., I$$

exists for some $\bar{u}^i, i = 2, ..., I$, and denote the solution by $(u^{1*}, ..., u^{I*})$. Then, $(u^{1*}, ..., u^{I*}) \in \mathcal{PF}$.

Problem: Show this.

Problem: Is the converse true? Under what conditions? Show this.

Claim: If $U(z) = (u^{1*}, ..., u^{I*})$ for some $z \in \mathcal{Z}$, then, $z \in \mathcal{PO}$.

Problem: Show this.

Claim: Under F1 and HH1, for any $(u^{1*}, ..., u^{I*}) \in \mathcal{PF}$ there is one and only one $z \in \mathcal{Z}$ satisfying this on the household side.

Problem: Show this.

Outline: Suppose there are two.... z_1 and z_2. Then $\frac{1}{2}z_1 + \frac{1}{2}z_2$ is also in $\mathcal{Z}$ and if it doesn't give the same allocations to all households, it's strictly better to at least one by strict concavity.

Claim: Under F1 and HH1, for any $(\lambda_1, ..., \lambda_I)$ there is one and only one solution to the problem:

$$\max_{z \in \mathcal{Z}} \qquad \sum_{i=1}^{I} \lambda_i U^i(z)$$

This solution is a PO allocation.

Problem: Show this.

Claim: Under F1 and HH1, the converse is also true.

Problem: Show this.

Claim: CE allocations are *productively efficient.* That is, there is no way of reorganizing the production plans of the firms, across firms, etc., to get the same net outputs, at a lower total discounted present value of cost.

Problem: Formalize and show this.

This is a first kind of 'efficiency' of decentralized decision-making result. It says that a centralized institution cannot 'improve' on the production side of the economy without lowering the output of that side.

2 The Welfare Theorems

One of the deepest and most fundamental results in Economics lies in the relationship between CE and PO allocations. This is that, under some circumstances, they 'coincide.' These two results, that CE allocations are PO, and the converse, that PO allocations are (sort of) CE allocations, are known as the First and Second Welfare Theorems. They are at the heart of much of economic thinking about policy questions: As a first cut, let the market work is always good policy advice. As a second cut, if you're concerned about 'fairness' or 'inequity' lump-sum tax, lump-sum redistribute and then let the market work. As a third cut, if there is a problem in X, fix X and let the market take care of the rest.

Where does this advice come from? This can be seen in the First and Second Welfare Theorems.

2.1 Theorem 1: CE allocations are PO

Theorem 1: Under HH1 and HH2, if z^* is a CE allocation with supporting prices $(p^*_{ct}, p^*_{xt}, r^*_t, w^*_t)$, then z^* is PO.

Comment: Most people are used to the informal type of proof for this kind of result, something about MRS's equalling MRT's... The formal proof has a very different strategy, however. The outline is:

1) Suppose not. Then we can make someone better off, without harming anyone.

2) The only way to make someone better off (or keep from harming them) is to increase (not decrease) their wealth.

3) Thus, 1) and 2) says you have to INCREASE aggregate wealth to make an improvement.

4) But this is not possible by feasibility. (Wealth is measured at the equilibrium prices, $(p_{ct}^*, p_{xt}^*, r_t^*, w_t^*)$.

Proof: Suppose not. Then there is an improving allocation, $\hat{z}$ that is also feasible. Without loss of generality, assume that household 1 is made strictly better off under $\hat{z}$ than under z. Since $(\hat{c}_t^1, \hat{x}_t^1, \hat{k}_t^1, \hat{n}_t^1, \hat{\ell}_t^1)$ improves 1's utility, it must not have been a feasible choice for 1, at the prices $(p_{ct}^*, p_{xt}^*, r_t^*, w_t^*)$. But, since $\hat{z}$ is feasible, it must be true that:

$\hat{n}_t^1 + \hat{\ell}_t^1 \leq \bar{n}_t^1, \qquad \forall t$

$\hat{k}_{t+1}^1 \leq (1-\delta)\hat{k}_t^1 + \hat{x}_t^1, \qquad \forall t$

$\hat{k}_0^1 = k_0^1$, it agrees with 1's initial endowment of k.

Thus, the fact that it improves 1's utility, and satisifies these constraints leaves only one option, that it does not satisfy 1's budget constaint. That is:

$$\sum_{t=0}^{\infty} (p_{ct}^{*}\hat{c}_t^1 + p_{xt}^{*}\hat{x}_t^1) > \sum_{t=0}^{\infty} \left(w_t^{*}\hat{n}_t^1 + r_t^{*}\hat{k}_t^1\right) + \pi^1$$

It is important that this inequality is strict.

Similar reasoning implies that, since household $i = 2, ..., I$ are not worse off under $\hat{z}$, gives us:

$$\sum_{t=0}^{\infty} (p_{ct}^{*}\hat{c}_t^i + p_{xt}^{*}\hat{x}_t^i) \geq \sum_{t=0}^{\infty} \left(w_t^{*}\hat{n}_t^i + r_t^{*}\hat{k}_t^i\right) + \pi^i$$

Summing over i, we conclude:

(*) $$\sum_{i=1}^{I}\sum_{t=0}^{\infty} (p_{ct}^{*}\hat{c}_t^i + p_{xt}^{*}\hat{x}_t^i) > \sum_{i=1}^{I}\sum_{t=0}^{\infty} \left(w_t^{*}\hat{n}_t^i + r_t^{*}\hat{k}_t^i\right) + \sum_{i=1}^{I}\pi^i$$

Now,

(**) $$\sum_{i=1}^{I}\pi^i = \sum_{i=1}^{I}\sum_{j=1}^{J_c}\theta_{ij}^c\left[\sum_{t=0}^{\infty}\left[p_{ct}^{*}c_t^{j*} - r_t^{*}k_{ct}^{j*} - w_t^{*}n_{ct}^{j*}\right]\right]$$
$$+\sum_{i=1}^{I}\sum_{j=1}^{J_x}\theta_{ij}^x\left[\sum_{t=0}^{\infty}\left[p_{xt}^{*}x_t^{j*} - r_t^{*}k_{xt}^{j*} - w_t^{*}n_{xt}^{j*}\right]\right]$$
$$\geq \sum_{i=1}^{I}\sum_{j=1}^{J_c}\theta_{ij}^c\left[\sum_{t=0}^{\infty}\left[p_{ct}^{*}\hat{c}_t^{j} - r_t^{*}\hat{k}_{ct}^{j} - w_t^{*}\hat{n}_{ct}^{j}\right]\right]$$
$$+\sum_{i=1}^{I}\sum_{j=1}^{J_x}\theta_{ij}^x\left[\sum_{t=0}^{\infty}\left[p_{xt}^{*}\hat{x}_t^{j} - r_t^{*}\hat{k}_{xt}^{j} - w_t^{*}\hat{n}_{xt}^{j}\right]\right]$$

Where the last inequality follows from the facts that

1) $\hat{z}$ is feasible and hence

$\hat{c}_t^j \leq F_{ct}^j(\hat{k}_{ct}^j, \hat{n}_{ct}^j), j = 1, ..., J_c, t = 0, ...$ and

$\hat{x}_t^j \leq F_{xt}^j(\hat{k}_{xt}^j, \hat{n}_{xt}^j), j = 1, ..., J_x, t = 0, ...$

2) $\left(k_{xt}^{j*}, n_{xt}^{j*}, x_t^{j*}\right)_{t=0}^{\infty}$ and $\left(k_{ct}^{j*}, n_{ct}^{j*}, c_t^{j*}\right)_{t=0}^{\infty}$ are profit maximizing at prices $(p_{ct}^*, p_{xt}^*, r_t^*, w_t^*)$.

Using (**) in (*), we get:

$$\sum_{i=1}^{I} \sum_{t=0}^{\infty} (p_{ct}^* \hat{c}_t^i + p_{xt}^* \hat{x}_t^i) >$$
$$\sum_{i=1}^{I} \sum_{t=0}^{\infty} \left(w_t^* \hat{n}_t^i + r_t^* \hat{k}_t^i\right) + \sum_{i=1}^{I} \sum_{j=1}^{J_c} \theta_{ij}^c \left[\sum_{t=0}^{\infty} \left[p_{ct}^* \hat{c}_t^j - r_t^* \hat{k}_{ct}^j - w_t^* \hat{n}_{ct}^j\right]\right]$$
$$+ \sum_{i=1}^{I} \sum_{j=1}^{J_x} \theta_{ij}^x \left[\sum_{t=0}^{\infty} \left[p_{xt}^* \hat{x}_t^j - r_t^* \hat{k}_{xt}^j - w_t^* \hat{n}_{xt}^j\right]\right]$$

Rearranging the order of summation and cancelling out the $\theta' s$ gives:

$$\sum_{i=1}^{I} \sum_{t=0}^{\infty} (p_{ct}^* \hat{c}_t^i + p_{xt}^* \hat{x}_t^i) >$$
$$\sum_{i=1}^{I} \sum_{t=0}^{\infty} \left(w_t^* \hat{n}_t^i + r_t^* \hat{k}_t^i\right) + \sum_{j=1}^{J_c} \sum_{t=0}^{\infty} \left[p_{ct}^* \hat{c}_t^j - r_t^* \hat{k}_{ct}^j - w_t^* \hat{n}_{ct}^j\right]$$
$$+ \sum_{j=1}^{J_x} \sum_{t=0}^{\infty} \left[p_{xt}^* \hat{x}_t^j - r_t^* \hat{k}_{xt}^j - w_t^* \hat{n}_{xt}^j\right]$$

Or,

(***) $$\sum_{i=1}^{I}\sum_{t=0}^{\infty}(p_{ct}^{*}\hat{c}_t^i + p_{xt}^{*}\hat{x}_t^i) + \sum_{j=1}^{J_c}\sum_{t=0}^{\infty}\left[r_t^*\hat{k}_{ct}^j + w_t^*\hat{n}_{ct}^j\right] + \sum_{j=1}^{J_x}\sum_{t=0}^{\infty}\left[r_t^*\hat{k}_{xt}^j + w_t^*\hat{n}_{xt}^j\right] > \sum_{i=1}^{I}\sum_{t=0}^{\infty}\left(w_t^*\hat{n}_t^i + r_t^*\hat{k}_t^i\right) + \sum_{j=1}^{J_c}\sum_{t=0}^{\infty}\left[p_{ct}^*\hat{c}_t^j\right] + \sum_{j=1}^{J_x}\sum_{t=0}^{\infty}\left[p_{xt}^*\hat{x}_t^j\right]$$

But, since $\hat{z}$ is feasible, it follows that:

$$\sum_{i=1}^{I}\hat{c}_t^i = \sum_{j=1}^{J_c}\hat{c}_t^j, \forall t,$$

$$\sum_{i=1}^{I}\hat{x}_t^i = \sum_{j=1}^{J_x}\hat{x}_t^j, \forall t,$$

$$\sum_{i=1}^{I}\hat{k}_t^i = \sum_{j=1}^{J_c}\hat{k}_{ct}^j + \sum_{j=1}^{J_x}\hat{k}_{xt}^j, \forall t,$$

$$\sum_{i=1}^{I}\hat{n}_t^i = \sum_{j=1}^{J_c}\hat{n}_{ct}^j + \sum_{j=1}^{J_x}\hat{n}_{xt}^j, \forall t,$$

Multiplying by the appropriate prices, and summing across t we get:

$$p_{ct}^*\sum_{i=1}^{I}\hat{c}_t^i = p_{ct}^*\sum_{j=1}^{J_c}\hat{c}_t^j, \forall t,$$

$$\sum_{t=0}^{\infty}p_{ct}^*\sum_{i=1}^{I}\hat{c}_t^i = \sum_{t=0}^{\infty}p_{ct}^*\sum_{j=1}^{J_c}\hat{c}_t^j, \qquad etc.$$

Summing across markets we obtain:

(@) $$\sum_{t=0}^{\infty}\sum_{i=1}^{I}p_{ct}^*\hat{c}_t^i + \sum_{t=0}^{\infty}\sum_{i=1}^{I}p_{xt}^*\hat{x}_t^i + \sum_{t=0}^{\infty}\sum_{j=1}^{J_c}r_t^*\hat{k}_{ct}^j + \sum_{t=0}^{\infty}\sum_{j=1}^{J_x}r_t^*\hat{k}_{xt}^j + \sum_{t=0}^{\infty}\sum_{j=1}^{J_c}w_t^*\hat{n}_t^j + \sum_{t=0}^{\infty}\sum_{j=1}^{J_x}w_t^*\hat{n}_t^j$$

$$= \sum_{t=0}^{\infty} \sum_{j=1}^{J_c} p_{ct}^* \hat{c}_t^j + \sum_{t=0}^{\infty} \sum_{j=1}^{J_x} p_{ct}^* \hat{x}_t^j$$
$$+ \sum_{t=0}^{\infty} \sum_{j=1}^{I} r_t^* \hat{k}_t^i + \sum_{t=0}^{\infty} \sum_{j=1}^{I} w_t^* \hat{n}_t^i$$

But, this contradicts (***) ending the proof.

NOTE: There are no assumptions on firms here. Why?

NOTE: How would this result fail if there were external effects? That is, which step of the proof would be false?

2.2 Theorem 2: PO allocations are 'CE'

A converse of Theorem 1 is also true. It requires some more assumptions and is quite a bit more complicated to prove however. Also, the statement is a bit less clear, and it comes in several different versions. The simplest version says that if z is a PO allocation, there is a redistribution of the endowments $(k_0^i, \bar{n}_t^i, \theta_{ij}^c, \theta_{ij}^x)$ among the agents and a set of prices $(p_{ct}, p_{xt}, r_t, w_t)$, such that z is a CE with those endowments and at those prices. Alternatively, z can be 'implemented' as a CE with a government using only lump-sum taxes and transfers. I will give, somewhat informal, statements of both versions in this section.

The proof of these results involves separation of an aggregate version of a 'better than' set from the set of feasible allocations to construct the new equilibrium prices. This requires convexity assumptions on both production functions and utility functions. Since I don't want us to get bogged down in this construction, I'll instead give a more informal argument constructing prices from MRS's etc. See Debreu, or etc. for the more complete and formal version of the proof.

Theorem 2: Suppose that z is a PO allocation, and assume that it is interior. Assume F1, F2, HH1 and HH2. Assume further that U^i and the F^j_{vt} are continuously differentiable. Then, there is an alternative assignment of endowments, $(\hat{k}^i_0, \hat{n}^i_t, \hat{\theta}^c_{ij}, \hat{\theta}^x_{ij})$, satisfying:

$$\sum_i \hat{k}^i_0 = \sum_i k^i_0,\ \sum_i \hat{n}^i_t = \sum_i \bar{n}^i_t,\ \sum_i \hat{\theta}^c_{ij} = \sum_i \theta^c_{ij},\text{ and } \sum_i \hat{\theta}^x_{ij} = \sum_i \theta^x_{ij}$$

and prices, $(p_{ct}, p_{xt}, r_t, w_t)$, such that z is a CE allocation at the prices $(p_{ct}, p_{xt}, r_t, w_t)$ given the initial endowments $(\hat{k}^i_0, \hat{n}^i_t, \hat{\theta}^c_{ij}, \hat{\theta}^x_{ij})$.

Sketch of Proof:

Since z is interior, $\sum_i x^i_t > 0$, and so $x^j_t > 0$ for some j. For simplicity, assume that $x^1_t > 0$ for all t. Similarly, assume that $c^1_t > 0$ for all t, and that $k^1_{ct} > 0$ and $n^1_{ct} > 0$. (You can fill in the details for the other combinations/cases.) Also note that $c_t^i > 0$ for all t, and $\ell_t^i > 0$ for all t. Using this, construct prices as:

$$p_{c0} = 1 \text{ (this is just a normalization)},$$

$$p_{ct} = \frac{\frac{\partial U^i}{\partial c_t}}{\frac{\partial U^i}{\partial c_0}}, \qquad w_t = \frac{\frac{\partial U^i}{\partial \ell_t}}{\frac{\partial U^i}{\partial c_0}}, \qquad \frac{r_t}{p_{ct}} = \frac{\partial F_{ct}^1}{\partial k_t}, \text{ and}$$

$$p_{xt} x_t^1 = r_t k_{xt}^1 + w_t n_{xt}^1.$$

Note that under our convexity assumptions, the fact that z is PO implies that it solves:

$$\max_{z \in \mathcal{Z}} \qquad \sum_{i=1}^{I} \lambda_i U^i(z^i)$$

For some choice of $\lambda_i' s$. Since z is interior, it must be that $\lambda_i > 0$ for all i. (Or else we could increase $\sum_{i=1}^{I} \lambda_i U^i(z)$ by moving consumption and leisure to agents with $\lambda_i > 0$.) This implies:

$$\frac{\frac{\partial U^i}{\partial c_t}}{\frac{\partial U^i}{\partial c_\tau}} = \frac{\frac{\partial U^{i'}}{\partial c_t}}{\frac{\partial U^{i'}}{\partial c_\tau}}, \text{ for all } i, i', t, \tau, \text{ and,}$$

$$\frac{\frac{\partial U^i}{\partial \ell_t}}{\frac{\partial U^i}{\partial c_0}} = \frac{\frac{\partial U^{i'}}{\partial \ell_t}}{\frac{\partial U^{i'}}{\partial c_0}}, \text{ for all } i, i', t.$$

This means that constructing prices from using any other agents allocation would give the same set of prices.

Define W by:

$$W = \sum_t w_t \sum_i \bar{n}_t^i + \sum_t r_t \sum_i (1-\delta)^t k_0^i.$$

W is the present discounted value of all initial resources in the economy evaluated at the constructed prices.

Let $W_i = \sum_t [p_{ct} c_t^i + w_t \ell_t^i]$, where c_t^i and ℓ_t^i are given from the allocation z.

This is the value of total spending, at the constructed prices, of all final consumption done by i in the allocation z.

Let $\eta_i = \frac{W_i}{W}$, this is the fraction of total wealth in the economy that is spent by i. Construct the endowments for i through using this fraction. For example:

Let $\hat{n}_t^i = \eta_i \sum_i \bar{n}_t^i$, and $\hat{k}_0^i = \eta_i \sum_i k_0^i$.

Since we have assume that the $F's$ are CRS, it doesn't matter how we assign the $\theta's$, (since profits will end up being 0), but for concreteness, let $\hat{\theta}_{ij}^c = \hat{\theta}_{ij}^x = \eta_i$ for all i, j.

Since the allocation is interior, it follows that investing in x is a zero profit activity at the prices as constructed. Thus, that part of the budget constraint is irrelevant.

Thus, we have constructed prices and endowments for this economy. What is left to show is that z is an equilibrium allocation. For this, we need to show that the assigned allocation (i.e., that part of z associated with each agent) is maximizing for each agent.

As an example, consider household 1. That the FOC's and BC hold at the proposed allocation follow by construction of the prices and endowments. What is left to show is that the FOC's and BC are sufficient. This is left to the reader. The argument for firms is similar.

This completes the construction.

The implication of this result is that, under convexity assumptions at least, the only kind of policy that ever need to be considered is to redistribute endowments of the resources in the economy.

An alternative way to do this is through the use of 'lump-sum' taxation. Formally:

An ADT equilibrium (Arrow-Debreu equilibrium with Transfers) is given by prices, $(p_{ct}, p_{xt}, r_t, w_t)$ a feasible allocation, z, profits π^i and a set of transfers, $(T^1, ..., T^I)$, such that:

(A) For all i $(c_t^i, x_t^i, k_t^i, n_t^i, \ell_t^i)_{t=0}^{\infty}$ solves

$Max \quad U^i(\cdot)$

s.t. $\sum_{t=0}^{\infty}(p_{ct}c_t^i + p_{xt}x_t^i) \leq \sum_{t=0}^{\infty}(w_t n_t^i + r_t k_t^i) + \pi^i + T^i$

$k_{t+1}^i \leq (1-\delta)k_t^i + x_t^i$

$\ell_t^i + n_t^i \leq \bar{n}_t^i$ for all t

k_0^i given (constraint on capital formulation by household)

……. non-negativity of all variables in all time periods.

(B1) For all $j,\ j = 1, 2, ..., J_x,$ for all t $\left(k_{xt}^j, n_{xt}^j, x_t^j\right)_{t=0}^{\infty}$ solves

$Max \quad \sum_{t=0}^{\infty}(p_{xt}x_t^j - w_t n_{xt}^j - r_t k_{xt}^j)$

s.t. $x_t^j \leq F_{xt}^j\left(k_{xt}^j, n_{xt}^j\right) \quad \forall\ t$

…….non-negativity of all variables in all time periods.

(B2) For all $j,\ j = 1, 2, ..., J_c,$ $\left(k_{ct}^j, n_{ct}^j, c_t^j\right)_{t=0}^{\infty}$ solves

$Max \quad \sum_{t=0}^{\infty}(p_{ct}c_t^j - w_t n_{ct}^j - r_t k_{ct}^j)$

s.t. $c_t^j \leq F_{ct}^j\left(k_{ct}^j, n_{ct}^j\right) \quad \forall\ t$

…… non-negativity of all variables in all time periods.

<Accounting> (Supply = Demand)

(i) $\sum_{i=1}^{I} c_t^i = \sum_{j=1}^{J_c} c_t^j \quad \forall\ t$

(ii) $\sum_{i=1}^{I} x_t^i = \sum_{j=1}^{J_x} x_t^j \quad \forall\ t$

(iii) $\sum_{i=1}^{I} n_t^i = \sum_{j=1}^{J_x} n_{xt}^j + \sum_{j=1}^{J_c} n_{ct}^j \quad \forall\ t$

(iv) $\sum_{i=1}^{I} k_t^i = \sum_{j=1}^{J_x} k_{xt}^j + \sum_{j=1}^{J_c} k_{ct}^j \quad \forall\ t$

<Profits> (are correct)

$$\sum_{i=1}^{I} \pi^i = \sum_{j=1}^{J_x} \sum_{t=0}^{\infty} \left(p_{xt} x_t^j - w_t n_{xt}^j - r_t k_{xt}^j\right) + \sum_{j=1}^{J_c} \sum_{t=0}^{\infty} \left(p_{ct} c_t^j - w_t n_{ct}^j - r_t k_{ct}^j\right)$$

<Government Budget Balance>

$$\sum_i T^i = 0.$$

Theorem 2': Suppose that z is a PO allocation, and assume that it is interior. Assume F1, F2, HH1 and HH2. Assume further that U^i and the F_{vt}^j are continuously differentiable. Then, there is a choice of transfers, $(T^1, ..., T^I)$ and prices $(p_{ct}, p_{xt}, r_t, w_t)$, such that z is an ADT equilibrium allocation.

2.3 Some Extra Related Problems

Problem: Show that if any of the production functions exhibits Increasing Returns to Scale, then, no CE can exist.

Problem: There are implicitly constraints on the behavior of prices so that the budget constraint of the consumer will make sense. Typically, it is necessary that the time path of prices $(p_{ct}, p_{xt}, r_t, w_t)$ has to decrease over time. In fact, . $\sum p_{ct} \leq \infty$, $\sum w_t \leq \infty$...)

Problem: Assume that labor supply is inelastic (i.e., ℓ_t^i does not enter U^i). State and prove a result to show that if a CE exists, it must be true that $\sum w_t \leq \infty$. What extra assumptions did you need for this result to hold? Can you do the same if labor is NOT inelastically supplied?

Problem: Show that CE is 'homogeneous of degree zero' in population size. That is, given a model economy such as the above, consider a new one with $2I$ people, 2 exactly like each of the agents in the original economy. Assuming that all $F's$ are CRS, show that the original equilibrium (prices and quantitities) are still an equilibrium.

Problem: Show that there is one degree of price indeterminacy in equilibrium. That is, if z and $(p_{ct}, p_{xt}, r_t, w_t)$ is an equilibrium, so is z with $(\lambda p_{ct}, \lambda p_{xt}, \lambda r_t, \lambda w_t)$ for any $\lambda > 0$.

2.4 Notes on Literature

This notion of equilibrium was first defined by Walras. The modern formal treatment of the existence of equilibrium and the welfare theorems for models like this with finitely many goods was developed in a series of papers by Arrow, Debreu and McKenzie in the 50's and 60's. The most well known source is Debreu, *Theory of Value*, 1959, Yale University Press.

An excellent treatment also appears in Mas-Colell, Whinston and Green, *Microeconomic Theory.*

The version analyzed here is not covered by those references since there are infinitely many goods. This is a special case of the models considered by Truman Bewley in his PhD dissertation. It appeared in JET in 1972 for those of you who are interested in looking into the technical details involved. (The math is hard.) Some of this also appears in Stokey, Lucas and Prescott, *Recursive Methods in Economic Dynamics*, 1989. See chapters 15-18.

3 Alternative Implementations of CE

Economists use the term 'implementation' to describe the maximizing model of markets which are open, how prices are formed, etc. which gives rise to a particular equilibrium allocation. For example, in the previous chapter, we 'implemented' the CE allocation through a complete set of time zero markets. That is, one can think of the mechanics of the equilibrium allocation as occurring one grand meeting of all agents in which an auctioneer calls out prices for all markets, both for those at time zero and for those at future dates. Thus, the trade is in 'futures contracts' on investment goods, consumption, etc. These contracts are all entered into before any actual consumption or production takes place. In this interpretation of the model, it is essential that the agents all believe that these contracts will actually be brought to fruition. That is, that they are enforced.

This 'time zero' trading is a bit tough to swallow as a positive theory of allocation. Although we do see some trading of futures contracts in the real world, activity in those markets is not large.

Because of this, alternative market structures to implement this same set of allocations have been developed. This lend themselves more easily to interpretation as trade occurring in 'spot markets' occurring each date. There are many different ways of doing this. Think of some of your own favorites! We'll talk about one in some detail here to give you a bit of a flavor for how they go.

3.1 AD Equilibrium Allocations and Borrowing

One of the aspects that often bothers newcomers is the lack of borrowing and lending in the formulation given above. Indeed, many authors treat the development differently exactly because of this. One of the things that looks most 'odd' about the presentation above is that there is only one budget constraint for all time periods for the household. A more natural, or reasonable, or easily interpretable version would have budget constraints for each period, with links between them given by borrowing and lending.

It's important to know that there is a kind of borrowing and lending going on in the AD equilibrium. This can be seen because there is nothing that forces spending in any period to equal income in that period. Thus, it is POSSIBLE for an agent to consume all of his income in period 1 if that is what he wants to do. But if he does that, he must be doing some sort of borrowing against future labor income, at least implicitly.

To see this, let's make it more explicit.

Define L_τ^i by:

$$L_\tau^i = \sum_{t=0}^{\tau} [p_{ct}c_t^i + p_{xt}x_t^i - w_t n_t^i - r_t k_t^i].$$

Thus, L_τ^i is the excess in spending over income by agent i in all periods up to and including period τ. This is the amount that the agent is implicitly 'financing through borrowing'. That is, it is the size, in period τ of the implicit loan to agent i. Note that this could be either positive or negative. However:

Problem: For all τ, $\sum_i L_\tau^i = 0$ in any CE allocation.

Also note:

Problem: $L_\tau^i \to 0$ in any CE allocation.

Thus, there is borrowing and lending allowed in the CE, it's just not explicit. But, this leaves open some questions, what is the interest rate, are there any limits on the amount of borrowing and lending done, etc.?

Is there a way of making this more explicit? Yes.

This, and many similar issues were first explored in a series of papers by Arrow in the 1950's.

To simplify notation, we drop production and leisure, so that we only consider a dynamic exchange economy in the one consumption good. We also assume that the horizon is finite and let the last period be denoted by T. Let e_t^i denote $i's$ endowment of the consumption good in period t.

Here's one version that will work:

A *Sequential Markets Equilibrium* (SME) is a sequence of prices p_t and interest rates R_t and an allocation for each agent, $(c_0^i, ..., c_T^i)$ and Loan balances, $(L_0^i, ..., L_T^i)$ such that:

For each i, $(c_0^i, ..., c_T^i)$ and $(L_0^i, ..., L_T^i)$ solve:

$$\max_{\{(c_0^i,...,c_T^i),(L_0^i,....,L_T^i)\}} U^i(c)$$

$$\text{subject to: } c_t^i \leq e_t^i + L_t^i - R_{t-1}L_{t-1}^i$$

$$L_T^i = 0$$

and, for all t, $\sum_i c_t^i = \sum_i e_t^i$.

Proposition:

A) If $(p_{c0}, ..., p_{cT})$ and $(c_0^i, ..., c_T^i)$ are an AD equilibrium, then $(c_0^i, ..., c_T^i)$, $(R_0, ..., R_T)$, $(L_0^i, ..., L_T^i)$ is an SME where:

$R_t = \frac{p_{t-1}}{p_t}$ and

$L_\tau^i = \frac{\sum_{t=0}^{\tau} p_{ct}(e_t^i - c_t^i)}{p_{c\tau}}$.

B) Conversely if $(c_0^i, ..., c_T^i)$, $(R_0, ..., R_T)$, $(L_0^i, ..., L_T^i)$ is an SME, then $(p_{c0}, ..., p_{cT})$ and $(c_0^i, ..., c_T^i)$ is an ADE, where:

$p_{ct} = \frac{p_{ct-1}}{R_{t-1}}$, ($p_{c0} = 1$, without loss of generality).

This result can be generalized to cover infinite horizon cases, but it is clear that the condition $L^i_T = 0$ is a problem. Moreover, it can't just be dropped. You should try this version of the definition to see that no equilibrium could ever exist (i.e., it's not an internally consistent definition) because of the possibility of Ponzi schemes. So, something else clearly needs to be added.

The thing that does the trick is a uniform bound on the amount of borrowing that a person can do. I.e., an extra constraint of the form:

$$L^i_\tau \leq \bar{L} \text{ for all } i, \tau.$$

Tim Kehoe has a very nice paper on this topic. See his webpage.

Problem: Do this for the version of the model with k, x, and n.

Problem: Can you find an 'arbitrage' relationship between interest rates on loans and the rental rate on capital from this?

4 Simplifying the Model: Aggregation

What we want to do next is see when we can reduce this CE problem to the single sector neoclassical growth model.

People use the standard single sector growth model to study a variety of issues when they need a model that generates explicit time series of output, savings, investment, labor supply etc. It looks like a very stark simplification relative to the real world. Typically, when people use this model, they have something a bit more complicated in mind, a rich model with many firms, multiple sectors, many households etc. The first part of these notes deals with "Aggregation." This means asking the question: Under what conditions is it true that a complex model with multiple firms, households and sectors can be reduced to the single sector model. It's useful to have some idea of the answer to this question. This is not meant to be an exhaustive study of the problem, rather an introduction to give you some idea of how these results look.

4.1 Simplifying the Firm Side

(Step1) Assume F_{xt}^{j} and F_{ct}^{j} are CRS(Constant Returns to Scale). Then profits are zero in every period.

(Step2) Suppose $F_{xt}^{j} = F_{xt}^{j'}$ $\forall$ t and $j\&j\prime$. (same technology for all firms)

(Step3) Same with consumption firms (i.e. $F_{ct}^{j} = F_{ct}^{j'}$ $\forall$ t and $j\&j\prime$)

Remark: These assumptions turn it into a 2 industry problem with no profits.

Remark: is investment side necessary for HH?

(Step4) Suppose $F_{xt} = F_{ct} \, \forall \, t$ (tech. for inv. firm = tech. for cons. firm)

Then the CE allocation is the same as the one in which there is ONE firm with technology given by F_{ct}.

i.e. the firm side of the model is one firm that chooses $\left(c_t^f, x_t^f, k_t^f, n_t^f\right)_{t=0}^{\infty}$ to maximize $\sum_{t=0}^{\infty}\left(p_{ct}c_t^f + p_{xt}x_t^f - r_t k_t^f - w_t n_t^f\right)$

s.t. $c_t^f + x_t^f \leq F_{ct}\left(k_t^f, n_t^f\right)$ for all t.

Remark: This is similar to Neoclassical single sector growth model version of firms.

Remark: In equilibrium you can drop either one of p_{ct} or p_{xt}. That is, if both c_t^f and x_t^f are positive in a given period, then $p_{ct} = p_{xt}$ in that period, with the obvious inequality if one is zero.

Remark: You could also do this with one firm for each period.

Problem: Give a formal statement and proof that the equilibrium with many firms is 'equivalent' to the equilibrium with one firm.

Problem: What if instead we had assumed that $F_{xt} = b_t F_{ct}$ for all t? (Assuming still that all firms within an industry have the same production function as each other in every period? Show that there is an aggregation result that holds here. Formally state and prove the result.

4.2 Simplifying the Household Side of the Model

→ Reducing it to a 'representative consumer' problem

<Method 1>

Everyone is the same.

<Method 2>

Heterogeneity exists but preferences are identical and homothetic.

4.2.1 Method 1: Identical Consumers

Household behavior is summarized by their utility function, initial capital stock(k_{oi}), and labor endowments(which is normalized to 1 $\forall$ i, t.).

If (1) all k_{oi} are all equal, (2) U_i are all the same and (3) labor endowments are the same, $\bar{n}_t^i = \bar{n}_t \; \forall \; i, \; t$), then all consumers make the same decision in any equilibrium.

NO!

(3) We also need: All U_i are strictly concave.

Cases where problems arise : non-convexities - not making the same decision under equilibrium

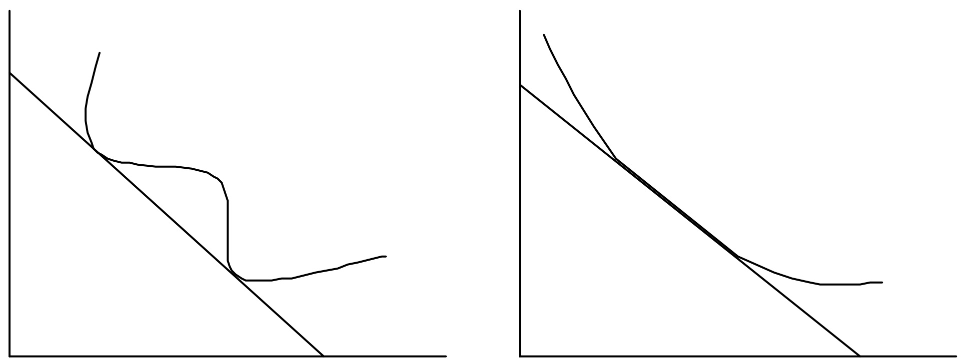

Figure 1:

(4) $k_{t+1} = (1-\delta)k_t + x_t$ implies constant returns to scale for investment(k_{t+1} as output, k_t and x_t as input). Thus, investment is irrelevant for consumers' side.

Remark: In equilibrium, it doesn't matter if consumer 1 makes all investment or consumer 2 makes all investment. Or any kind of combination of investment (ex. consumer 1 does all investment in period 1 and consumer 2 does all investment in period 2...). More completely, any reallocation of investment across households that leaves total investment unchanged will still be an equilibrium allocation. (Of course, the capital stocks of the individual agents must also be modified accordingly.)

Therefore, the results should be modified to "about c's and l's" (excluding investment).

Remark: The HH B/C $\sum(p_{ct}c_t + p_{xt}x_t) \leq \sum(r_t k_t + w_t n_t)$ becomes the same as

$\sum p_{ct}c_t \leq \sum(w_t n_t + (1-\delta)^t k_0 r_t)$ where investment doesn't enter into the B/C. If $x_t = 0$ then $k_{t+1} = (1-\delta)k_t$ and $k_t = (1-\delta)^t k_o$

What's missing from this? $(1-\delta+r_t)k_t p_t - k_t p_{t+1}$

Implication? If this is right, it must be true that $(1-\delta+r_t/p_t)k_t p_t - k_t p_{t+1} = 0$, that is, investment doesn't enter into the constraint for maximization at equilibrium prices. Only consumption, labor and initial holdings of capital matter.

(Modified) All consumers make the same decision in any equilibrium about c's and ℓ's (but not necessarily the same x's and k's) and one equilibrium has all the x_t^i and k_t^i equal. Thus the maximization part of equilibrium can be reduced to maximization by one of the households and equality by the others.

Remark: If all of the above conditions are satisfied, then HH problem is reduced into that of one representative agent.)

Remark: Then what needs to be done? Reducing accounting into one person problem.)

- $S = D$ can be done in per capita terms. Firms are indifferent to scale.

  For example, in period t, $S = D$ in output is

  $\sum_{i=1}^{I} c_t^i = \sum_{j=1}^{J_c} F_{ct}^j \left(k_{ct}^j, n_{ct}^j\right)$

  Since $\sum_{i=1}^{I} c_t^i = I \times c_t^1$,

  $c_t^1 = \frac{1}{I} \times \sum_{j=1}^{J_c} F_{ct}^j \left(k_{ct}^j, n_{ct}^j\right)$

  $= \sum_{j=1}^{J_c} F_{ct}^j \left(\frac{k_{ct}^j}{I}, \frac{n_{ct}^j}{I}\right)$ (since the production function is CRS)

  $= F_{ct}^1 \left(\frac{\sum_j k_{ct}^1}{I}, \frac{\sum_j n_{ct}^1}{I}\right)$

< Summary >

If (A) all firms are identical within and across sectors, and the technology are CRS

(B) all HH have the same k_{oi} and U_i , and

(C) U_i is strictly concave,

then, CE of original economy is the same as one with one firm and one household.

(i) (HH) $Max \quad U_1 \left(\underset{\sim}{c}, \underset{\sim}{\ell}\right)$

s.t. $\sum_{t=0}^{\infty} p_t \left(c_t + x_t\right) \leq \sum_{t=0}^{\infty} \left(w_t n_t + r_t k_t\right)$

$k_{t+1} \leq (1-\delta) k_t + x_t$

$\ell_t + n_t \leq \bar{n}_t$

.... non-negativity.

(ii) (Firm) $Max \quad p_t F\left(k_t^f, n_t^f\right) - w_t n_t^f - r_t k_t^f$

(iii) $k_t = k_t^f$

$n_t = n_t^f$

$c_t + x_t = F\left(k_t^f, n_t^f\right)$

Theorem 3: The CE of the above economy solves

$$Max \quad U\left(\underset{\sim}{c}, \underset{\sim}{\ell}\right)$$

s.t. $c_t + x_t \leq F_t(k_t, n_t)$

$k_{t+1} \leq (1-\delta)k_t + x_t$

k_o *given*

$\ell_t + n_t \leq \bar{n}_t$

Proof: Use First Welfare Theorem to show that CE is PO. There is exactly one PO allocation for the economy and it is given by $(*)$.

Problem: Make the statement precise and prove it.

Remark: Thus, the CE is the same as the one sector growth model.

Remark: This uses the fact that with only one agent CE maximized the utility of the representative agent subject to feasibility. (This is PO in this case.)

Remark: So far, all we need about the utility function is that it is strictly increasing and strictly concave. (We also used that the production function is concave, and CRS.)

4.2.2 Method 2: Homothetic Aggregation

Similar result holds under "heterogeneous endowments" but you need stronger assumptions in the utility function.

Result: If k_o^i are different but $U_i = U$ and U is "*homothetic*" then planning problem representation of CE still holds.

U is "*homothetic*" if $U(x_1) = U(x_2) \implies U(\lambda x_1) = U(\lambda x_2)\ \forall\ \lambda \geq 0$

Remark: This means that we have the same shape for indifference curves but they are shifted out parallel.

Examples:

1. Suppose U is homogeneous of degree η s.t. $U(\lambda x) = \lambda^{\eta} U(x)\ \forall\ \lambda \geq 0$

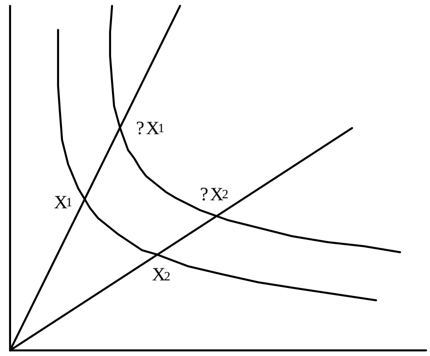

Figure 2:

Then if $U(x_1) = U(x_2) \iff U(\lambda x_1) = \lambda^{\eta} U(x_1) = \lambda^{\eta} U(x_2) = U(\lambda x_2)$

$\forall \lambda \geq 0$

Thus homogeneous of degree $\eta \implies$ "*homothetic*"

2. $U(x, y) = x^{\alpha} + by^{\alpha}$

Homogeneous of degree $\alpha \implies \therefore$ *homothetic*

3. $U(c_1, c_2, c_{3, \ldots..}) = \frac{1}{1-\gamma}\left(b_1 c_1^{1-\gamma} + b_2 c_2^{1-\gamma} + b_3 c_3^{1-\gamma} + \ldots\right)$

Homogeneous of degree $1 - \gamma$

4. $U(x, y) = x^{\alpha 1} + by^{\alpha 2}$

Not homogenous unless $\alpha 1 = \alpha 2$

ex. $x + by^2$

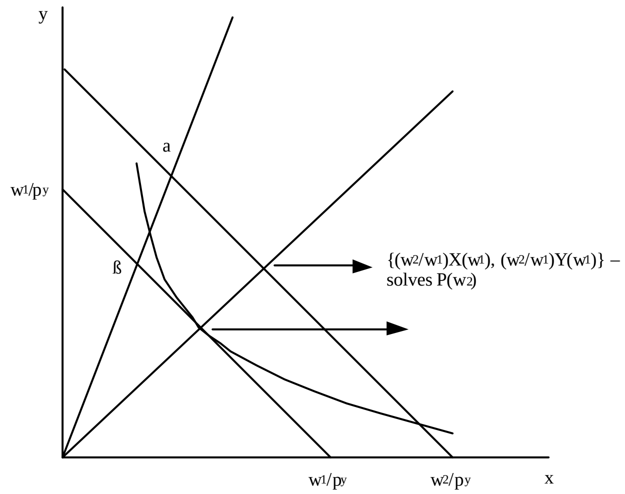

Figure 3:

5. $U(x,y) = \exp\left[(x^{\alpha} + by^{\alpha})^{3} + r\right]$

Homothetic but not homogeneous

6. $U(x,y) = a \log x + b \log y$

Homothetic but not homogeneous

Consider the maximization problem:

$P(W) \qquad Max\ U(x,y)$

$\qquad\qquad \text{s.t. } p_x x + p_y y \leq W$

Denote the solution to this problem by: $(x(W), y(W))$

If $W \uparrow (W_1 \to W_2)$, then budget constraint shifts upwards.

If we assume homotheticity, then one would choose $\left[\frac{W_2}{W_1} x(W_1), \frac{W_2}{W_1} y(W_1)\right]$

Remark: If not, say (α_x, α_y) is strictly better than $\left[\frac{W_2}{W_1} x(W_1), \frac{W_2}{W_1} y(W_1)\right]$ then, by going back, $(\frac{W_1}{W_2}\alpha_x, \frac{W_1}{W_2}\alpha_y)$ would be better than $(x(W_1), y(W_1))$ which is a contradiction.

Problem: State and prove this formally.

Proposition: If U is homothetic, then $(x(W_2), y(W_2)) = W_2(x(1), y(1))$ $\forall W_2 \geq 0$

(i.e. (x, y) is homogeneous of degree 1 in W)

Consider a 2 good CE model with I consumers each with $U_i = U$ which is homothetic and initial endowment being W_i

Proposition: Let $D(p; W_1, W_2, ..., W_I)$ = aggregate demand if prices are $p = (p_x, p_y)$ and consumer 1 has all the wealth $W_1 + W_2 + ... + W_I$.

Then $D(p; W_1, W_2, ..., W_I) = \sum_{i=1}^{I} D_i(p; W_i)$

$$= D_1(p; \sum W_i)$$

Remark: This is what you would get if person 1 has all the wealth. The conclusion is: For aggregate demand, the wealth distribution doesn't matter.

Proof:

For all p, $D_i(p; W_i) = (x(W_i), y(W_i)) = W_i(x(1), y(1))$

So, $\sum_{i=1}^{I} D_i(p; W_i) = \sum_{i=1}^{I}(x(W_i), y(W_i)) = \sum_{i=1}^{I} W_i(x(1), y(1))$

On the other hand, $D_1(p; \sum W_i) = (x(\sum W_i), y(\sum W_i)) = \sum W_i(x(1), y(1))$.

Proposition: Assume $U^i = U \;\; \forall\, i$ and that U is homothetic and let $\left[(p_t, r_t, w_t)_{t=0}^{\infty}, (c_t^i, x_t^i, k_t^i, n_t^i, \ell_t^i)_{i=1}^{I}\ {}_{t=0}^{\infty}, \left(c_t^f, x_t^f, k_t^f, n_t^f\right)_{t=0}^{\infty}\right]$ be a CE for the economy E with $\left(k_0^1, k_0^2, ..., k_0^I\right)$.

Then

$$\left[(p_t, r_t, w_t)_{t=0}^{\infty}, \left(\sum_{i=1}^{I} c_t^i, \sum_{i=1}^{I} x_t^i, \sum_{i=1}^{I} k_t^i, \sum_{i=1}^{I} n_t^i, \sum_{i=1}^{I} \ell_t^i\right)_{t=0}^{\infty}, \left(c_t^f, x_t^f, k_t^f, n_t^f\right)_{t=0}^{\infty}\right]$$

is a CE for the economy $\widehat{E}$ which has one consumer with initial capital stock $\overline{k_0} = \sum_{i=1}^{I} k_0^i$ and $\bar{n}_t = \sum_{i=1}^{I} \bar{n}_t^i$ units of leisure.

Proof: Obvious

Remark: What determines initial wealth? Initial capital stock and labor supply.

Note that the consumer's budget cconstraint is usually written as:

$\sum p_t(c_t + x_t) \leq \sum(r_t k_t + w_t n_t)$,

but this is equivalent to:

$$\sum \left[p_t \left(c_t + x_t\right) + w_t \ell_t\right] \leq \sum \left(r_t k_t + w_t \bar{n}_t\right) \qquad (i.e.,\ n_t = \bar{n}_t - \ell_t)$$

then it is easy to show that aggregate consumption is the same as one consumer problem.

Summarizing:

Theorem 4:

The CE solves the following maximization problem model

$$Max \quad U\ ()$$

s.t. $c_t + x_t \leq F_t\left(k_t, n_t\right)$

$$k_{t+1} \leq (1-\delta)k_t + x_t$$

$$k_0 = \sum_i k_0^i$$

$$\ell_t + n_t \leq \sum_i \bar{n}_t^i$$

Each individual consumer consumes c and ℓ in proportion to their price weighted, present discounted value of the aggregate endowment. That is, the solution to $(*)$ gives the aggregate consumption of c and ℓ. Then, household i consumes the fraction, η_i of this aggregate quantity where η_i is constructed as in Theorem 2 above.

Problem: State this formally and prove it.

4.3 Summary:

If (1) $U_i = U$ and $k_0^i = k_0 \ \ \forall \ i$

or (2) $U_i = U$ and U is homothetic,

then CE is the solution to a Neoclassical growth model planner's problem.

In this case, the problem of finding the CE can be reduced to solving a deterministic Dynamic Programming Problem. This is the topic that we will turn to next.

One should not get the impression that things are hopeless without this, however. If not, there are still things

that can be done to make the problem manageable. Examples:

Approach 1) System of equations defining CE - Solve them!

Approach 2) CE is still PO.

Since CE is PO $\Longrightarrow$ for some set of λ's it solves:

$(*) \ \ Max \quad \lambda_i U_i \left(\underset{\sim}{c}, \underset{\sim}{\ell} \right)$

s.t. feasibility

For example, If $U_i \left(\underset{\sim}{c}, \underset{\sim}{\ell} \right) = \sum_{t=0}^{\infty} \beta^t u_i \left(c_t^i, \ell_t^i \right)$

then $\sum \lambda_i U_i \left(\underset{\sim}{c}, \underset{\sim}{\ell} \right) = \sum_{t=0}^{\infty} \beta^t \left[\sum_{i=1}^{I} \lambda_i u_i \left(c_t^i, \ell_t^i \right) \right]$

Pick λ's $\longrightarrow$ Solve $Max \quad \sum \lambda_i U_i \left(\underset{\sim}{c}, \underset{\sim}{\ell} \right)$

$\longrightarrow$ Use U_i to calculate "supporting prices" using dynamic programming

Remark: We get the same prices whichever U_i we use.

$\longrightarrow$ Calculate value of consumption at those prices and compare to value of k_o^i, labor supply at those prices

$\longrightarrow$ If these are unequal, adjust λ's and repeat the process

Remark:. This is a way of calculating CE without using strong assumptions.

Remark:. Related to 2nd Welfare Theorem. For any weights, solving the maximization problem above will give the CE for SOME initial endowments!

5 Properties of the Model's Time Series

The previous discussion leads us to study the properties of the solutions to maximization problems of the form:

$$P(k_0): \qquad Max_{\{(c_t,x_t,k_t,n_t,\ell_t)\}_{t=0}^{\infty}} \quad U\,()$$

s.t. $c_t + x_t \leq F_t\,(k_t, n_t)\,;$

$k_{t+1} \leq (1-\delta)k_t + x_t;$

$\ell_t + n_t \leq \bar{n}_t;$

k_0 fixed.

We have noted, in our notation that this problem depends on the initial capital stock, k_0. It also depends on a variety of other things too, the $\bar{n}_t$, etc., but k_0 will play a special role in what comes below.

Note carefully what we have gotten rid of in this reformulation of our equilibrium problem:

1. We only have one firm now– this depends on the assumptions of CRS and that all firms have the same production function;

2. There is only one consumer– this depends on the assumptions on the consumer side, either they are all identical or, the differ in endowments and their common utility function is homothetic;

3. We have replaced the Arrow-Debreu budget constraint in the problem with the physical feasibility restriction for the economy as a whole.

Thus, we no longer the consumer the option of trading across time, etc. In essence, the argument is: Since everyone is identical, there is no need to allow for trade across agents since there will be none in equilibrium (a similar, but more subtle version holds if agents are not identical, but preferences are homothetic). Because of this, in both cases, the economy acts as if there is only one agent. Therefore, due to the First Welfare Theorem, we can replace individual constraints with aggregate ones.

Even after these simplifications, at this level of generality it is hard to say much about the solution to this problem. However, under a few additional assumptions, some progress can be made. Basically, these amount to removing the 'time dependence' from the problem. Viz.,

1. Assume that $F_t(k_t, n_t) = F(k_t, n_t)$ for all t;

2. Assume that $\bar{n}_t = n_t$ for all t;

3. Assume that $U\ () = \sum_t \beta^t u(c_t, \ell_t)$.

Under these assumptions the parametric family of maximization problems, $P(k_0)$, can be written as:

$$P(k_0): \qquad Max_{\{(c_t, x_t, k_t, n_t, \ell_t)\}_{t=0}^{\infty}} \quad \sum_t \beta^t u(c_t, \ell_t)$$

s.t. $c_t + x_t \leq F(k_t, n_t)$;

$k_{t+1} \leq (1-\delta)k_t + x_t$;

$\ell_t + n_t \leq \bar{n}$;

k_0 fixed.

The thing that is special about this sort of maximization problem is that, although time is explicitly included, the way in which time enters the problem is very limited. That is, from any period forward, the 'continuation' of the problem looks a lot like the problem we started with, except perhaps that we have a different initial condition.

Problems that have this form are called Stationary Dynamic Programming Problems. And there is a lot known about the properties of the solutions to problems of this form. There is also a lot known about how to solve them on the computer. This, it turns out, is indespensible, since there are very few cases for which analytic solutions are available.

What follows is a quick summary of some of the properties of these problems. This is basically taken from Stokey, Lucas and Prescott (otherwise known as the Bible).

5.1 Summary of Solution Characteristics

For any given initial condition, k_0, define $V^*(k_0)$ by:

$$V^*(k_0) = Sup_{\{(c_t, x_t, k_t, n_t, \ell_t)\}_{t=0}^{\infty}} \quad \sum_t \beta^t u(c_t, \ell_t)$$

$$\text{s.t. } \; c_t + x_t \leq F(k_t, n_t);$$

$$k_{t+1} \leq (1-\delta)k_t + x_t;$$

$$\ell_t + n_t \leq \bar{n};$$

k_0 fixed.

That is, $V^*(k_0)$ is the utility that the agent gets when he solves the problem $P(k_0)$ – the indirect utility of starting with initial condition k_0. Note that we have replaced Max inside the problem with Sup. This is to allow for the possibility that there may not be a finite valued solution to the maximization problem as stated. IF you knew that there was indeed a solution, you could replace the Sup with Max. But, even in situations where there need not be a solution, the Sup is always well-defined. This is the advantage of this formulation of the problem.

The basic results of Dynamic Programming when applied to this problem give us the following results:

1. The function V^* satisfies what is known as Bellman's Equation. This is an operator that takes in functions and gives out new functions. Formally, define T by:

   $T(v(k)) = \sup_{\{(c,x,k',n,\ell)\}} u(c, \ell) + \beta v(k')$

   s.t., $c + x \leq F(k, n)$;

   $k' \leq (1 - \delta)k + x$;

   $\ell + n \leq \bar{n}$;

   k fixed;

   non-negativity of all variables.

   Under this definition, you can see, if you put a function $v(k)$ in, performing the right hand side optimization for each k gives a new function of k, known as $T(v)$.

   What Bellman's Equation says then is that:

   $T(V^*(k)) \equiv V^*(k)$ – this is an identity in k – it holds for all k.

   That is, the indirect utility of the original problem is a fixed point of the mapping T.

   Conversely, under some restrictions, the only finite valued fixed point to T is V^*.

These results are known as 'The Principle of Optimality.'

2. Under certain conditions on the technology and the utility (boundedness sorts of things) it can be shown that there is a finite valued solution to the problem for all initial conditions and hence V^* is a real valued function.

3. Under more or less these same conditions, and assuming that $0 < \beta < 1$, the proof that a solution exists also gives rise to a method for calculating V^*. Under standard conditions, known as Blackwell's sufficient conditions, the operator defined in part 1 is what is known as a contraction mapping. That is, for any two functions $v_1(k)$ and $v_2(k)$:

   $$\sup_k |T(v_1(k)) - T(v_2(k))| < \beta \sup_k |v_1(k) - v_2(k)|.$$

   That is, $T(v_1)$ is closer to $T(v_2)$ than v_1 is to v_2.

4. From 3, it follows that from any initial function, $v(k)$, the sequence of functions defined by:

   $$v_n(k) = T^n(v(k))$$

   has the property that:

$\sup_k |v_n(k) - V^*(k)| \to 0.$

5. In addition to the conditions in 2., assume that $0 < \beta < 1$ that u is continuous, strictly increasing and strictly concave and that F is continuous, strictly increasing and strictly quasi-concave. Then, V^* is continuous, strictly increasing and strictly concave. Because of this it follows that the solution to the maximization problem on the RHS of BE:

   $\max_{(c,x,k',n,\ell)} u(c,\ell) + \beta v(k')$

   s.t., $c + x \leq F(k,n)$;

   $k' \leq (1-\delta)k + x$;

   $\ell + n \leq \bar{n}$;

   k fixed;

   exists and is unique for all k. Let this solution be denoted by:

   $(g_c(k), g_x(k), g_k(k), g_n(k), g_\ell(k))$.

   These are known as the 'policy functions' for the problem.

6. Let $\{(c_0^*(k_0), x_0^*(k_0), ...); (c_1^*(k_0), ...); ..\}$ denote the solution to the original, infinite horizon problem when the initial condition is k_0. Under

the assumptions in 2 and 5 above, this solution is given by:

$$(c_0^*(k_0), x_0^*(k_0), k_0^*(k_0), n_0^*(k_0), \ell_0^*(k_0)) = (g_c(k_0), g_x(k_0), k_0, g_n(k_0), g_\ell(k_0));$$

$$(c_1^*(k_0), x_1^*(k_0), k_1^*(k_0), n_1^*(k_0), \ell_1^*(k_0)) = (g_c(g_k(k_0)), g_x(g_k(k_0)), g_k(k_0), g_n(g_k(k_0)), g_\ell$$

etc.

Note something here. In the first period, $k_0^*(k_0) = k_0$. This comes from feasibility. In the second period, $k_1^*(k_0) = g_k(k_0)$ and all other variables are defined in terms of this.

Thus, this shows that the system is a First Order Difference Equation in k.

7. If it is also true that u and F are continuously differentiable, then V^* is also differentiable at every k such that the policy functions are 'interior.'

8. Under the additional assumption that $u_\ell = 0$ (i.e., inelastic labor supply) so that $g_n(k) = \bar{n}$ for all k, it can be shown that $g_c(k)$, $g_x(k)$, and $g_k(k)$ are all strictly increasing functions of k.

9. Under the additional assumption that $u_\ell = 0$, and some extra assumptions on u and F – in particular that $F(0, \bar{n}) = 0$ – it can be shown

that there are exactly two steady state values for the system, i.e., values of k such at $g_k(k) = k$. Note that at these values of k it follows that $k_t^*(k) = k$ for all t and the system never moves. One of these two values is $k = 0$. The second is strictly positive and under differentiability of u and F is the unique solution to:

$$1 = \beta [1 - \delta + F_k(k, \bar{n})].$$

10. Let k^* denote the positive steady state identified in part 9. Under these same assumptions, it follows that the system is 'globally asymptotically stable.' That is,

    a) If $k_0 > k^*$, then $k_t^* > k_{t+1}^* > ... > k^*$ and $\lim_t k_t^* = k^*$;

    b) If $0 < k_0 < k^*$, then $k_t^* < k_{t+1}^* < ... < k^*$ and $\lim_t k_t^* = k^*$;

    c) If $k_0 = 0$ then $k_t^* = 0$ for all t.

    This gives a complete characterization of the general properties of the time paths of all of the endogenous variables in the model for arbitrary initial conditions, at least qualitatively.

11. Under the assumptions above plus the additional assumptions that:

    a) $u(c, \ell) = \log(c)$;

b) $\quad F(k,n) = Ak^{\alpha}n^{1-\alpha};$

c) $\quad \delta = 1;$

The solution to the problem is:

$$c_t^* = g_c(k_t^*) = \phi_c A k_t^{*\alpha};$$

$$k_{t+1}^* = g_k(k_t^*) = \phi_k A k_t^{*\alpha};$$

where $\phi_c + \phi_k = 1$.

12. Even when $u_\ell \neq 0$ most of the properties given above tend to hold and even moreso when the system is started out 'near' a steady state. The exception is, as a rule, in certain cases when β is very low.

Graph of the Time Series goes Here.

As you can see, this gives us a problem when trying to use the model as a data generating device to compare with US time series. This is that although it is possible that the time series of k (and consumption and output and ...) MIGHT grow if k_0 is less than k^*, this growth will only be temporary – along the transition path to the steady state. In fact, it can be shown that this transition happens very quickly for the kinds of production functions and utility functions that people usually use.

The reason for this is very simple. The first order conditions from the problem given above include one known as the Euler Equation, which is the basic equation describing the dynamics of the system. For this model, it is given by assuming that labor supply is inelastic :

$$(EE) \qquad u'(c_t^*) = \beta u'(c_{t+1}^*)\left[1-\delta+F_k(k_{t+1}^*,\bar{n})\right].$$

Assuming that $F(k,n) = Ak^{\alpha}n^{1-\alpha}$ this becomes:

$$(EE) \qquad \frac{u'(c_t^*)}{u'(c_{t+1}^*)} = \beta\left[1-\delta+A^*\left[k_{t+1}^*\right]^{\alpha-1}\right].$$

The right hand side of this equation can be recognized as $\beta(1+R_t)$, where $(1+R_t)$ is the interest rate a consumer must pay to borrow a dollar in period t to be paid back in period $t+1$. Since α is typically estimated to be in the .3 to .35 range, any deviation from $k_{t+1}^* = k^*$ implies a very large movement in interest rates, and causes savings (or dis-savings) to be very large.

Thus, although the model does give rise to some growth in transition, it is not quantitatively important typically.

This is in marked contrast to what is seen in the US time series where growth is substantial, and perhaps more importantly, shows no sign of slowing down whatsoever.

The model also shows no 'wiggles' in the time series, but this is clearly of second order importance at this stage.

5.2 Adding Trend Growth to the Formulation

Because of the difficulty described in the last section typically the model is adjusted to allow for trend growth. This is done in an exogenous, and labor augmenting way. This is described loosely as technological progress that makes labor more productive. But note below that it will not be the product of any modelled R&D decisions, and moreover, it takes up no resources from the economy. We will turn back to these issues later in the course.

In sum then, the model is modified to the form:

$$P(k_0): \qquad Max_{\{(c_t, x_t, k_t, n_t, \ell_t)\}_{t=0}^{\infty}} \quad \sum_t \beta^t u(c_t, \ell_t)$$

$$\text{s.t.} \quad c_t + x_t \leq F\left(k_t, \gamma^t n_t\right);$$

$$k_{t+1} \leq (1-\delta)k_t + x_t;$$

$$\ell_t + n_t \leq \bar{n};$$

$$k_0 \text{ fixed.}$$

Where $\gamma = 1$ corresponds to the original version and $\gamma > 1$ means that the technology is improving over time.

The difficulty is, this problem is no longer a Stationary DP. But, with a little work and an extra assumption we can make it one:

Under the assumption that F is CRS, note that we can rewrite the feasibility restriction as:

$$c_t + x_t \leq \gamma^t F\left(\frac{k_t}{\gamma^t}, n_t\right);$$

$$\frac{c_t}{\gamma^t} + \frac{x_t}{\gamma^t} \leq \gamma^t F\left(\frac{k_t}{\gamma^t}, n_t\right).$$

This suggests introducing new variables to the problem by detrending everything by γ^t. Accordingly, let:

$\hat{c}_t = \frac{c_t}{\gamma^t}$;

$\hat{x}_t = \frac{x_t}{\gamma^t}$;

$\hat{k}_t = \frac{k_t}{\gamma^t}$;

Using this, we can rewrite our original problem as:

$$\hat{P}(k_0): \qquad Max_{\{(\hat{c}_t, \hat{x}_t, \hat{k}_t, n_t, \ell_t)\}_{t=0}^{\infty}} \quad \sum_t \beta^t u(\gamma^t \hat{c}_t, \ell_t)$$

s.t. $\hat{c}_t + \hat{x}_t \leq F\left(\hat{k}_t, n_t\right)$;

$\hat{k}_{t+1} \leq (1-\hat{\delta})\hat{k}_t + \phi\hat{x}_t$;

$\ell_t + n_t \leq \bar{n}$;

$\hat{k}_0 = k_0$ fixed.

Here, $(1-\hat{\delta}) = \frac{1-\delta}{\gamma}$, and $\phi = \frac{1}{\gamma}$.

This looks a lot like the problem that we stated above with the exception of the objective function. Note that if the utility is of the form $u(c,\ell) = \frac{c^{1-\sigma}}{1-\sigma}v(\ell)$ with $\sigma > 0$, we have that:

$$\beta^t u(\gamma^t \hat{c}_t, \ell_t) = \beta^t \frac{\gamma^{t(1-\sigma)}\hat{c}_t^{1-\sigma}}{1-\sigma} v(\ell_t) = \left(\beta\gamma^{1-\sigma}\right)^t \frac{\hat{c}_t^{1-\sigma}}{1-\sigma} v(\ell_t) = \hat{\beta}^t \frac{\hat{c}_t^{1-\sigma}}{1-\sigma} v(\ell_t),$$

where $\hat{\beta} = \beta\gamma^{1-\sigma}$.

Thus, under this assumption about utility, we can rewrite our maximization problem as:

$$\hat{P}(k_0): \qquad Max_{\{(\hat{c}_t,\hat{x}_t,\hat{k}_t,n_t,\ell_t)\}_{t=0}^{\infty}} \quad \sum_t \hat{\beta}^t u(\hat{c}_t, \ell_t)$$

s.t. $\hat{c}_t + \hat{x}_t \leq F\left(\hat{k}_t, n_t\right)$;

$\hat{k}_{t+1} \leq (1-\hat{\delta})\hat{k}_t + \phi\hat{x}_t$;

$\ell_t + n_t \leq \bar{n}$;

$\hat{k}_0 = k_0$ fixed.

This problem can be shown to have all of the same properties that the original bounded one has and hence we're off to the races.

In particular, it follows that (under inelastic labor supply, etc etc etc....)

$$\hat{k}_t^* = \frac{k_t^*}{\gamma^t} \to \hat{k}^*$$

where $\hat{k}^*$ is the unique solution to:

$$1 = \phi\hat{\beta}\left[\frac{1}{\phi}\left(1-\hat{\delta}\right) + F_k(\hat{k}^*, \bar{n})\right];$$

$$\gamma^\sigma = \beta\left[1-\delta + F_k(\hat{k}^*, \bar{n})\right];$$

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In cases like this, when initial conditions are such that this equation is satisfied, we are at the steady state capital stock in period zero and hence, in the detrended version of the model, all variables are constant over time. This implies that the variables (c_t, x_t, k_t, y_t) all grow at the constant rate, γ from period to period. Morover, $n_t = n$ and hence it grows at a constant rate too, $\gamma_n = 1$. Because of this constancy of growth rates along the solution path, it is called a Balanced Growth Path. This is a situation in which every variable grows at a constant, not necessarily equal, rate in every period. This is the analog of a steady state when there is trend growth.

With this 'fix' of the model, even this very simple version can generate time series that are very much like those seen in the U.S. economy.