Econ 8106 MACROECONOMIC THEORY Part II

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Part One: Aggregation and the Growth Model

People use the standard single sector growth model to study a variety of issues when they need a model that generates explicit time series of output, savings, investment, labor supply etc. It looks like a very stark simplification relative to the real world. Typically, when people use this model, they have something a bit more complicated in mind, a rich model with many firms, multiple sectors, many households etc. The first part of these notes deals with "Aggregation." This means asking the question: Under what conditions is it true that a complex model with multiple firms, households and sectors can be reduced to the single sector model. It's useful to have some idea of the answer to this question. This is not meant to be an exhaustive study of the problem, rather an introduction to give you some idea of how these results look.

10/26/2000

Consider an infinite horizon competitive general equilibrium model.

(A) Utility Maximizing Households (HH)

- Indexed by $i = 1, 2, \ldots, I$
- Infinitely lived agents
- Decide about their own consumption, labor supply, leisure, investment, etc.
- Have endowments of firm ownership (see below) and leisure, $\hat{h}_t^i$

(B) Two types of firms

- Investment firms (from capital and labor, goes to HH for investment use) : $j_x = 1, 2, \ldots, J_x$
- Consumption firms (from capital and labor, goes to HH for consumption use) : $j_c = 1, 2, \ldots, J_c$
- These are owned by the households: μ_{ij}^c, μ_{ij}^x : consumer i's ownership of firm j $(0 \leq \mu_{ij}^c, \mu_{ij}^x \leq 1)$

  $\sum_{i=1}^{I} \mu_{ij}^c = 1$

$\sum_{i=1}^{I} \theta_{ij}^{x} = 1$

A Competitive Equilibrium is a sequence of prices for consumption(c), investment(x), capital rental(k), and labor wage(n)

$(p_{ct}, p_{xt}, r_t, w_t) \quad t = 0, 1, \ldots$

and quantities

(A) Household quantities $\{(c_t^i, x_t^i, k_t^i, n_t^i, \ell_t^i) \quad t = 0, 1, \ldots \quad i = 1, 2, \ldots, I$

(B) (i) Investment firms quantities $\{\left(k_{xt}^j, n_{xt}^j, x_t^j\right) \quad t = 0, 1, \ldots\ j = 1, 2, \ldots, J_x$

(ii) Consumption firms quantities $\{\left(k_{ct}^j, n_{ct}^j, c_t^j\right) \quad t = 0, 1, \ldots\ j = 1, 2, \ldots, J_c$

(C) Profits for each household $\pi^i; \quad i = 1, \ldots, I$

< Maximizing Behavior >

(A) For all i $(c_t^i, x_t^i, k_t^i, n_t^i, \ell_t^i)_{t=0}^{\infty}$ solves

Max $U_i\,()$

s.t. $\sum_{t=0}^{\infty} (p_{ct} c_t^i + p_{xt} x_t^i) \leq \sum_{t=0}^{\infty} (w_t n_t^i + r_t k_t^i) + \pi^i$

$k_{t+1}^i \leq (1 - \delta) k_t^i + x_t^i$

k_0^i given (constraint on capital formulation by household)

...... $\ell_t^i + n_t^i \leq \bar{n}_t^i$ for all t

.... non-negativity of all variables in all time periods.

where $U_i(\cdot)$ is a function of $(c_t^i, \ell_t^i)_{t=0}^{\infty}$ and is increasing in c and in ℓ.

(B1) For all j $\left(k_{xt}^j, n_{xt}^j, x_t^j\right)_{t=0}^{\infty}$ solves

$$\text{Max} \quad \sum_{t=0}^{\infty}(p_{xt}x_t^j - w_t n_{xt}^j - r_t k_{xt}^j)$$

$$\text{s.t.} \quad x_t^j \le F_{xt}^j\left(k_{xt}^j, n_{xt}^j\right) \quad \forall t$$

.......non-negativity of all variables in all time periods.

(B2) For all j $\left(k_{ct}^j, n_{ct}^j, c_t^j\right)_{t=0}^{\infty}$ solves

$$\text{Max} \quad \sum_{t=0}^{\infty}(p_{ct}c_t^j - w_t n_{ct}^j - r_t k_{ct}^j)$$

$$\text{s.t.} \quad c_t^j \le F_{ct}^j\left(k_{ct}^j, n_{ct}^j\right) \quad \forall t$$

...... non-negativity of all variables in all time periods.

<Accounting> (Supply = Demand)

(i) $\sum_{i=1}^{I} c_t^i = \sum_{j=1}^{J_c} c_t^j \quad \forall t$

(ii) $\sum_{i=1}^{I} x_t^i = \sum_{j=1}^{J_x} x_t^j \quad \forall t$

(iii) $\sum_{i=1}^{I} n_t^i = \sum_{j=1}^{J_x} n_{xt}^j + \sum_{j=1}^{J_c} n_{ct}^j \quad \forall t$

(iv) $\sum_{i=1}^{I} k_t^i = \sum_{j=1}^{J_x} k_{xt}^j + \sum_{j=1}^{J_c} k_{ct}^j \quad \forall t$

<Profits> (are correct)

$$\sum_{i=1}^{I} \pi^i = \sum_{j=1}^{J_x} \sum_{t=0}^{\infty} \left(p_{xt}x_t^j - w_t n_{xt}^j - r_t k_{xt}^j\right) + \sum_{j=1}^{J_c} \sum_{t=0}^{\infty} \left(p_{ct}c_t^j - w_t n_{ct}^j - r_t k_{ct}^j\right)$$

Note: This is sometimes also called a Walrasian Equilibrium. The essence of this is price taking behavior by all agents. This was first formally noted by Walras.

A CE implicitly generates a whole time-series of individual household's allocations and outputs of the firms as well as prices and interest rates as well as the appropriate aggregate quantities for the agents taken as a group.

Problem: Show that if any of the production functions exhibits Increasing Returns to Scale, then, no CE can exist.

Note: There implicitly constraints on the behavior of prices so that the budget constraint of the consumer will make sense. Typically, it is necessary that the time path of prices $(p_{ct}; p_{xt}; r_t; w_t)$ has to decrease over time. In fact, . $\sum p_{ct} < \infty$, $\sum w_t < \infty$ ...)

Problem: Assume that labor supply is inelastic (i.e., ℓ_t^i does not enter U^i). State and prove a result to show that if a CE exists, it must be true that $\sum w_t < \infty$. What extra assumptions did you need for this result to hold? Can you do the same if labor is NOT inelastically supplied?

This type of CE model in a dynamic setting is a special case of the models considered by Truman Bewley in his PhD dissertation. It appeared in JET in 1972 for those of you who are interested in looking into the technical details involved.

Simplifying the Model

What we want to do next is see when we can reduce this CE problem to the single sector neoclassical growth model.

(Step1) Assume F^j_{xt} and F^j_{ct} are CRS(Constant Returns to Scale). Then profits are zero in every period.

(Step2) Suppose $F^j_{xt} = F^{j'}_{xt}$ $\forall$ t and $j \neq j'$. (same technology for all firms)

(Step3) Same with consumption firms (i.e. $F^j_{ct} = F^{j'}_{ct}$ $\forall$ t and $j \neq j'$)

Remark: These assumptions turn it into a 2 industry problem with no profits.

Remark: is investment side necessary for HH? (HW1 Q2)

(Step4) Suppose $F_{xt} = F_{ct}$ $\forall$ t (tech. for inv. firm = tech. for cons. firm)

Then the CE allocation is the same as the one in which there is ONE firm with technology given by F_{ct}.

i.e. the firm side of the model is one firm that chooses $\left(c^f_t; x^f_t; k^f_t; n^f_t\right)_{t=0}^{\infty}$ to maximize $\sum_{t=0}^{\infty}\left(p_{ct}c^f_t + p_{xt}x^f_t - r_t k^f_t - w_t n^f_t\right)$

s.t. $c^f_t + x^f_t \leq F_{ct}\left(k^f_t; n^f_t\right)$ for all t.

Remark: This is similar to Neoclassical single sector growth model version of ¯rms.

Remark: In equilibrium you can drop either one of p_{ct} or p_{xt}. That is, if both c_t^f and x_t^f are positive in a given period, then $p_{ct} = p_{xt}$ in that period, with the obvious inequality if one is zero.

Remark: You could also do this with one ¯rm for each period.

Problem: Give a formal statement and proof that the equilibrium with many ¯rms is 'equivalent' to the equilibrium with one ¯rm.

Problem: What if instead we had assumed that $F_{xt} = b_t F_{ct}$ for all t? (Assuming still that all ¯rms within an industry have the same production function as each other in every period? Show that there is an aggregation result that holds here. Formally state and prove the result.

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A Competitive Equilibrium is a

i) system of prices $(p_{ct}; p_{xt}; r_t; w_t)_{t=0}^{1}$

ii) quantities for households

iii) quantities for ¯rms

such that :

a) HH are maximizing taking prices as given

b) Firms are maximizing taking prices as given

c) Accounting

1) Quantities supplied = Quantities demanded

2) Pro¯ts in HH = Pro¯ts in Firms

<Simplify>

i) If ¯rms' technology are

a) CRS

b) The same within each sector

c) The same across the two sectors

(rmk) enables you to aggregate into one ¯rm (one representative ¯rm with one representative technology)

then the equilibria of the model are the same as a model in which there is only one industry and one in that industry - it has the 'representative' technology

Simplifying the Household Side of the Model:

! Reducing it to a `representative consumer' problem

<u><Method 1></u>

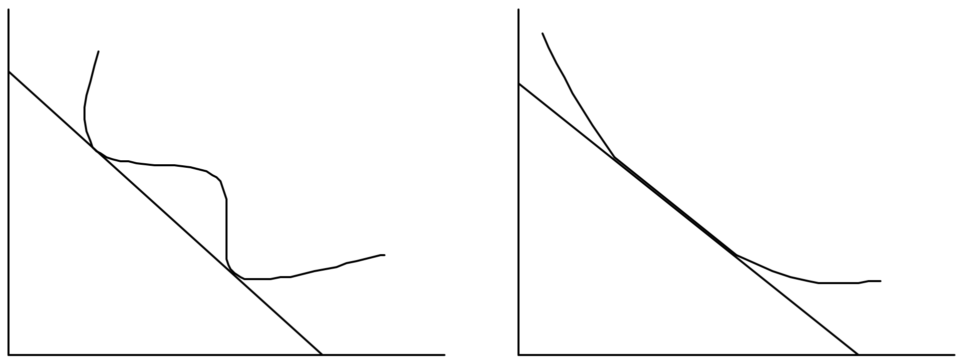

Figure 1:

Everyone is the same.

<Method 2>

Heterogeneity exists but preferences are identical and homothetic.

<Method 1>

Household behavior is summarized by their utility function, initial capital stock(k_{oi}), and labor endowments(which is normalized to 1 8 i, t.).

If (1) all k_{oi} are all equal, (2) U_i are all the same and (3) labor endowments are the same, $\hbar^i_t = \hbar_t$ 8 i, t), then all consumers make the same decision in any equilibrium.

NO!

(3) We also need: All U_i are strictly concave.

Cases where problems arise : non-convexities - not making the same decision under equilibrium

(4) $k_{t+1} = (1-\delta)k_t + x_t$ implies constant returns to scale for investment(k_{t+1} as output, k_t and x_t as input). Thus, investment is irrelevant for consumers' side.

Remark: In equilibrium, it doesn't matter if consumer 1 makes all investment or consumer 2 makes all investment. Or any kind of combination of investment (ex. consumer 1 does all investment in period 1 and consumer 2 does all investment in period 2...). More completely, any reallocation of investment across households that leaves total investment unchanged will still be an equilibrium allocation. (Of course, the capital stocks of the individual agents must also be modified accordingly.)

Therefore, the results should be modified to "about c's and l's" (excluding investment).

Remark: The HH B/C $\sum (p_{ct}c_t + p_{xt}x_t) \leq \sum (r_t k_t + w_t n_t)$ becomes the same as

$\sum p_{ct}c_t \leq \sum (w_t n_t + (1-\delta)^t k_0 r_t)$ where investment doesn't enter into the B/C. If $x_t = 0$ then $k_{t+1} = (1-\delta)k_t$ and $k_t = (1-\delta)^t k_o$

What's missing from this? $(1-\delta+r_t)k_t p_t - k_t p_{t+1}$

Implication? If this is right, it must be true that $(1 - \delta + r_t/p_t)k_t p_t - k_t p_{t+1} = 0$, that is, investment doesn't enter into the constraint for maximization at equilibrium prices. Only consumption, labor and initial holdings of capital matter.

(Modified) All consumers make the same decision in any equilibrium about c's and ℓ's (but not necessarily the same x's and k's) and one equilibrium has all the x_t^i and k_t^i equal. Thus the maximization part of equilibrium can be reduced to maximization by one of the households and equality by the others.

Remark: If all of the above conditions are satisfied, then HH problem is reduced into that of one representative agent.)

Remark: Then what needs to be done? Reducing accounting into one person problem.)

- S = D can be done in per capita terms. Firms are indifferent to scale.

  For example, in period t, S = D in output is

  $\sum_{i=1}^{I} c_t^i = \sum_{j=1}^{J_c} F_{ct}^j\left(k_{ct}^j, n_{ct}^j\right)$

  Since $\sum_{i=1}^{I} c_t^i = I \times c_t^1$,

$$
\begin{aligned}
c_t^1 &= \frac{1}{I}\left[\sum_{j=1}^{J_c} F_{ct}^j\left(k_{ct}^j, n_{ct}^j\right)\right] \\
&= \sum_{j=1}^{J_c} F_{ct}^j\left(\frac{k_{ct}^j}{I}, \frac{n_{ct}^j}{I}\right) \qquad \text{(since the production function is CRS)} \\
&= F_{ct}^1\left(\frac{\sum_j k_{ct}^1}{I}, \frac{\sum_j n_{ct}^1}{I}\right)
\end{aligned}
$$

< Summary >

If (A) all firms are identical within and across sectors, and the technology are CRS

(B) all HH have the same k_{oi} and U_i , and

(C) U_i is strictly concave,

then, CE of original economy is the same as one with one firm and one household.

(i) (HH) Max $\quad U_1\left(\underset{\sim}{c}, \underset{\sim}{\ell}\right)$

s.t. $\quad \sum_{t=0}^{\infty} p_t\left(c_t + x_t\right) \le \sum_{t=0}^{\infty}\left(w_t n_t + r_t k_t\right)$

$\qquad k_{t+1} \le (1-\delta)k_t + x_t$

$\qquad \ell_t + n_t \le \hat{n}_t$

.... non-negativity.

(ii) (Firm) Max $\quad p_t F\left(k_t^f, n_t^f\right) - w_t n_t^f - r_t k_t^f$

(iii) $k_t = k_t^f$

$$n_t = n_t^f$$

$$c_t + x_t = F\left(k_t^f ; n_t^f\right)$$

Theorem: The CE of the above economy solves $(*)$

$$(*) \quad \text{Max} \quad U\left(\underset{\sim}{c}; \underset{\sim}{\ell}\right)$$

$$\text{s.t.} \quad c_t + x_t \leq F_t(k_t; n_t)$$

$$k_{t+1} \leq (1 - \delta)k_t + x_t$$

$$k_o \ \text{given}$$

$$\ell_t + n_t \leq \hat{n}_t$$

pf) Use First Welfare Theorem to show that CE is PO. There is exactly one PO allocation for the economy and it is given by $(*)$.

Problem: Make the statement precise and prove it.

Remark: Thus, the CE is the same as the one sector growth model.

Remark: This uses the fact that with only one agent CE maximized the utility of the representative agent subject to feasibility. (This is PO in this case.)

If all preferences and endowments are identical, then CE allocation solves (*)

$$(*) \quad \max \; U\left(\underset{\sim}{c}, \underset{\sim}{\ell}\right)$$

$$\text{s.t.} \quad c_t + x_t \leq F_t(k_t, n_t)$$

$$k_{t+1} \leq (1-\delta)k_t + x_t$$

$$k_0 \text{ given}$$

$$\ell_t + n_t \leq \bar{h}_t$$

Remark: So far, all we need about the utility function is that it is strictly increasing and strictly concave.

Method 2:

Similar result holds under "heterogeneous endowments" but you need stronger assumptions in the utility function.

Result: If k_0^i are different but $U_i = U$ and U is "homothetic" then planning problem representation of CE still holds.

U is "homothetic" if $U(x_1) = U(x_2) \iff U(\lambda x_1) = U(\lambda x_2) \; \forall \lambda, \lambda \geq 0$

Remark: This means that we have the same shape for indifference curves but they are shifted out parallel.

Examples:

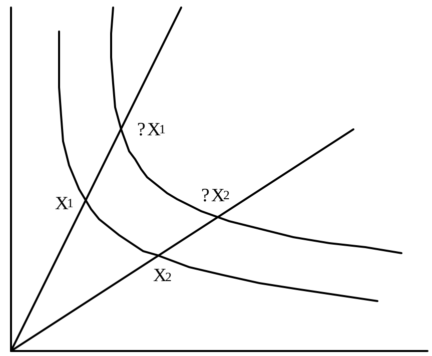

Figure 2:

1. Suppose U is homogeneous of degree η s.t. $U(\lambda x) = \lambda^{\eta} U(x) \ \forall \lambda \geq 0$

Then if $U(x_1) = U(x_2) \iff U(\lambda x_1) = \lambda^{\eta} U(x_1) = \lambda^{\eta} U(x_2) = U(\lambda x_2)$

$\forall \lambda \geq 0$

Thus homogeneous of degree $\eta \Rightarrow$ "homothetic"

2. $U(x; y) = x^{\alpha} + by^{\alpha}$

Homogeneous of degree $\alpha \Rightarrow$ homothetic

3. $U(c_1; c_2; c_{3;\dots}) = \frac{1}{1-\gamma}\left(b_1 c_1^{1-\gamma} + b_2 c_1^{1-\gamma} + b_1 c_1^{1-\gamma} + \dots\right)$

Homogeneous of degree $1-\gamma$

4. $U(x; y) = x^{\alpha_1} + by^{\alpha_2}$

Not homogenous unless $\alpha_1 = \alpha_2$

ex. $x + by^2$

5. $U(x;y) = \exp\left[(x^{\alpha} + by^{\alpha})^3 + r\right]$

Homothetic but not homogeneous

6. $U(x;y) = a \log x + b \log y$

Homothetic but not homogeneous

Consider the maximization problem:

$$P(W) \qquad \operatorname{Max} U(x;y)$$

$$\text{s.t. } p_x + p_y \leq W$$

Denote the solution to this problem by: $(x(W); y(W))$

If $W \uparrow (W_1 \to W_2)$, then budget constraint shifts upwards.

If we assume homotheticity, then one would choose $\left[\frac{W_2}{W_1} x(W_1); \frac{W_2}{W_1} y(W_1)\right]$

Remark: If not, say $(\alpha_x; \alpha_y)$ is strictly better than $\left[\frac{W_2}{W_1} x(W_1); \frac{W_2}{W_1} y(W_1)\right]$ then, by going back, $(\frac{W_1}{W_2}\alpha_x; \frac{W_1}{W_2}\alpha_y)$ would be better than $(x(W_1); y(W_1))$ which is a contradiction.

Problem: State and prove this formally.

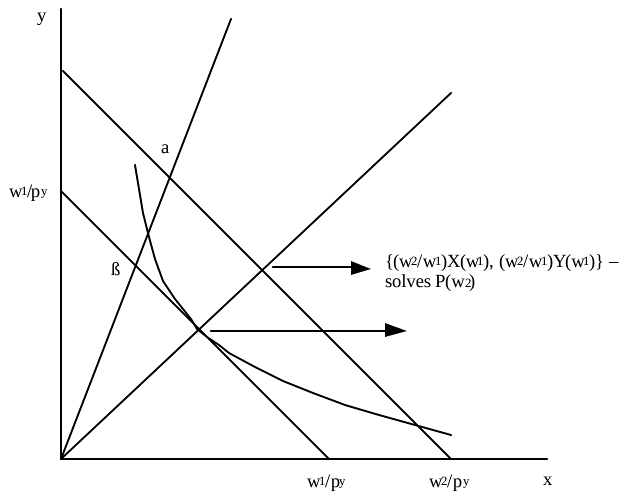

Figure 3:

(¤) If U is homothetic, then $(x(W_2); y(W_2)) = W_2(x(1); y(1)) \quad \forall W_2 \geq 0$

(i.e. $(x; y)$ is homogeneous of degree 1 in W)

Consider a 2 good CE model with I consumers each with $U_i = U$ which is homothetic and initial endowment being W_i

(rmk. all wealth to consumer 1... what happens?)

Proposition: Let $D(p; W_1; W_2; \ldots; W_I)$ = aggregate demand if prices are $p = (p_x; p_y)$ and consumer 1 has all the wealth $W_1 + W_2 + \ldots + W_I$.

Then $D(p; W_1; W_2; \ldots; W_I) = \sum_{i=1}^{I} D_i(p; W_i)$

$$= D_1(p; \sum W_i)$$

Remark: This is what you would get if person 1 has all the wealth. The conclusion is: For aggregate demand, the wealth distribution doesn't matter.

pf) $D_i(p, W_i) = (x(W_i), y(W_i))$

$$= W_i(x(1), y(1))$$

So, $\sum_{i=1}^{I} D_i(p, W_i) = \sum_{i=1}^{I}(x(W_i), y(W_i))$

$$= \sum_{i=1}^{I} W_i(x(1), y(1))$$

On the other hand, $D_1(p, \sum W_i) = (x(\sum W_i), y(\sum W_i)) = \sum W_i(x(1), y(1))$

<Homothetic Aggregation in the Growth Model>

Assume $U_i = U \ \forall i$ and that U_i's are homothetic and let $\left[(p_t, r_t, w_t)_{t=0}^{\infty}, (c_t^i, x_t^i, k_t^i, n_t^i, \ell_t^i)_{i=1}^{I} {}_{t=0}^{\infty}, \left(c_t^f, x_t^f, k_t^f, n_t^f\right)_{t=0}^{\infty}\right]$ be a CE for the economy E with $\left(k_0^1, k_0^2, \ldots, k_0^I\right)$.

Then $\left[(p_t, r_t, w_t)_{t=0}^{\infty}, \left(\sum_{i=1}^{I} c_t^i, \sum_{i=1}^{I} x_t^i, \sum_{i=1}^{I} k_t^i, \sum_{i=1}^{I} n_t^i, \sum_{i=1}^{I} \ell_t^i\right)_{t=0}^{\infty}, \left(c_t^f, x_t^f, k_t^f, n_t^f\right)_{t=0}^{\infty}\right]$ is a CE for the economy $\hat{E}$ which has one consumer with initial capital stock $\overline{k_0} = \sum_{i=1}^{I} k_0^i$ and $\sum_{i=1}^{I} \hat{n}_t^i$ units of leisure.

pf. Obvious

(rmk. making it into one consumer problem)

Remark:. What determines initial wealth? Initial capital stock and labor supply.

Note that the consumer's b/c is usually written as:

$$\sum p_t(c_t + x_t) \leq \sum (r_t k_t + w_t n_t),$$

but this is equivalent to:

$$\sum [p_t(c_t + x_t) + w_t \ell_t] \leq \sum (r_t k_t + w_t \bar{n}_t) \qquad (i.e.; \; n_t = \bar{n}_t - \ell_t)$$

then it is easy to show that aggregate consumption is the same as one consumer problem.

Summarizing:

Theorem:

The CE (for aggregate) solves the following maximiztion problem model

$(*)$ Max $U()$

s.t. $c_t + x_t \leq F_t(k_t; n_t)$

$k_{t+1} \leq (1 - \delta)k_t + x_t$

$k_o = \sum_i k_o^i$

$\ell_t + n_t \leq \sum_i \bar{n}_t^i$

Problem: State this formally and prove it.

<Summary>

If (1) $U_i = U$ and $k_0^i = k_0$ $\forall i$

or (2) $U_i = U$ and U is homothetic,

then CE is the solution to a Neoclassical growth model planner's problem.

If not, then what?

Approach 1) System of equations defining CE - Solve them!

Approach 2) CE is still PO.

Since CE is PO $\Rightarrow$ for some set of λ's it solves:

$(*)$ Max $\lambda_i U_i\left(\tilde{c}, \tilde{\ell}\right)$

s.t. feasibility

For example, If $U_i\left(\tilde{c}, \tilde{\ell}\right) = \sum_{t=0}^{\infty} \beta^t u_i(c_t^i, \ell_t^i)$

then $\sum \lambda_i U_i\left(\tilde{c}, \tilde{\ell}\right) = \sum_{t=0}^{\infty} \beta^t \left[\sum_{i=1}^{I} \lambda_i u_i(c_t^i, \ell_t^i)\right]$

Pick λ's $\rightarrow$ Solve Max $\sum \lambda_i U_i\left(\tilde{c}, \tilde{\ell}\right)$

$\rightarrow$ Use U_i to calculate "supporting prices" using dynamic programming

Remark: We get the same prices whichever U_i we use.

¡ ! Calculate value of consumption at those prices and compare to value of k_o^i, labor supply at those prices

¡ ! If these are unequal, adjust ¸'s and repeat the process

Remark:. This is a way of calculating CE without using strong assumptions.

Remark:. Related to 2nd Welfare Theorem. For any weights, solving the maximization problem above will give the CE for SOME initial endowments!