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Class Notes: Part II

1 Knowledge as a Public Good

This section is meant to capture, in some loose sense how one might try to model knowledge as a public good, what the difficulties are with different approaches, and, to some extent, what the difficulties are with ALL approaches.

The loose idea that people are trying to capture in models like this is the idea that one person using an 'idea' or a 'piece of knowledge' does not prevent others from using it to.

It is NOT clear how the $A(k, h)$ model does NOT fit into this category. That is, if we interpreted this as meaning that there is a once and for all give stock of "Potential Knowledge" (i.e., "Truth") out there, and interpret $h_{it} = h^*$ as $i's$ attainment of that by date t, does this prevent other agents from also attaining h^* as well? It's not clear that it does. In this sense, that knowledge is 'freely' available to all. (Well it's costly for any individual agent to attain it, but having one person attain it does NOT prevent others from attaining it.)

A second, related, idea is that while it is costly for any agent to attain it, it is more costly for it to be attained the 'first' time. The most extreme version of this is that it is costly for the first agent, but then free for any subsequent agent. Other possibilities can easily be imagined.

Let's Try:
An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_{it}, k_{it}, x_{ikt}, h_{it}, x_{iht}, \ell_{it}, z_{it})\}_{t=0}^{\infty} = z_i^{HH}$

Quantity decisions for the output firms: $\{(c_{jt}^f, x_{jkt}^f, x_{jht}^f, k_{jt}^f, z_{jt}^f)\}_{t=0}^{\infty} = z_j^f$,

SUCH THAT:

1) For each $i \in [0, 1]$, z_i^{HH} is the solution to:

$Max_{\{(c_{it},k_{it},x_{ikt},h_{it},x_{iht},\ell_{it},z_{it})\}_{t=0}^{\infty}} \sum_t \beta^t u(c_{it}, \ell_{it} h_{it})$

subject to:

$\sum_{t=0}^{\infty} p_t [c_{it} + x_{ikt} + x_{iht}] \leq \sum_{t=0}^{\infty} [r_t k_{it} + w_t z_{it}] + \Pi_i$

$k_{it+1} \leq (1 - \delta_k) k_{it} + x_{ikt}$

$h_{it+1} \leq (1 - \delta_h) h_{it} + G(x_{hit}, x_{hjt}^{-i})$

$z_{it} = n_{it} h_{it}$

$n_{it} + \ell_{it} \leq 1,$

(k_{i0}, h_{i0}) fixed.

2) For each $j \in [0,1]$, z_j^f is the solution to:

$Max_{\{(c_{jt}^f, x_{jkt}^f, x_{jht}^f, k_{jt}^f, z_{jt}^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t (c_{jt}^f + x_{jkt}^f + x_{jht}^f) - r_t k_{jt}^f - w_t z_{jt}^f \right]$

subject to: $c_{jt}^f + x_{jkt}^f + x_{jht}^f \leq F(k_{jt}^f, n_{jt}^f)$.

AND

$\int_0^1 c_{it} di = \int_0^1 c_{jt}^f dj$

$\int_0^1 x_{ikt} di = \int_0^1 x_{jkt}^f dj$

$\int_0^1 k_{it} di = \int_0^1 k_{jt}^f dj$

$\int_0^1 z_{it} di = \int_0^1 z_{jt}^f dj$

$\int_0^1 \Pi_i di = \int_0^1 \sum_{t=0}^{\infty} \left[p_t (c_{jt}^f + x_{jkt}^f + x_{jht}^f) - r_t k_{jt}^f - w_t z_{jt}^f \right] dj$

Notice that h_{it+1} depends not only on the investments of i (i.e., x_{hit}), but also on the investments of the other households too, x_{hjt}^{-i}. Thus, there is an external effect in the accumulation of h.

As in the discussion with the $A(k,h)$ model, there are lots of different ways to fill in the details about how the accumulation of knowledge is modeled and hence, the exact form of this external effect. For example:

$Max_{\{(c_{it},k_{it},x_{ikt},h_{it},x_{iht},\ell_{it},z_{it})\}_{t=0}^{\infty}} \sum_t \beta^t u(c_{it}, \ell_{it} h_{it})$

subject to:

$\sum_{t=0}^{\infty} p_t \left[c_{it} + x_{ikt} + x_{iht}\right] \leq \sum_{t=0}^{\infty} \left[r_t k_{it} + w_t z_{mit}\right] + \Pi_i$

$k_{it+1} \leq (1-\delta_k)k_{it} + x_{ikt}$

$h_{it+1} \leq (1-\delta_h)h_{it} + G(x_{hit}, z_{hit}; x_{hjt}^{-i}, z_{hjt}^{-i})$

$z_{mit} = n_{mit}h_{it}$, $z_{hit} = n_{hit}h_{it}$

$n_{hit} + n_{mit} + \ell_{it} \leq 1,$

(k_{i0}, h_{i0}) fixed.

ETC.

Both of these share the property that the notion of equilibrium is a mixture between Walrasian (or Arrow-Debreu) and Nash. The A-D part is obvious, all agents view themselves as having no impact on prices at the aggregate level. The Nash part is more subtle. h_{it+1} depends on the actions of the other agents in the economy through G, and hence, in the maximization problem, it can be seen that we have assumed that agent i takes as given the entire time series of x_{hjt}^{-i} when it chooses the entire time series x_{hit}. This is rather an odd way of modelling this (from the perspective of Game Theory it is anyway) and we will return to this (briefly) below.

Under this formulation, it is the properties of G that determine the 'publicness' of knowledge.

For example, if $G(x_{hit}, x_{hjt}^{-i}) = x_{hit}$, then we are back in the $A(k,h)$ world, where knowledge is purely private.

Other examples:

1. $G(x_{hit}, x_{hjt}^{-i}) = x_{hit}$, here we are in the $A(k,h)$ world.
2. $G(x_{hit}, x_{hjt}^{-i}) = G(x_{hjt}^{-i})$, that is literally $\partial G(x_{hit}, x_{hjt}^{-i})/\partial x_{hit} = 0$.
3. $G(x_{hit}, x_{hjt}^{-i}) = \frac{1}{I}\sum_j x_{hit}$.
4. $G(x_{hit}, x_{hjt}^{-i}) = \int_I x_{hjt}dj$ or, $G(x_{hit}, x_{hjt}^{-i}) = \frac{1}{I}\int_I x_{hjt}dj$.
5. $G(x_{hit}, x_{hjt}^{-i}) = \sup_j\{x_{hjt}\}$ $j = 1, ..., I$.
6. $G(x_{hit}, x_{hjt}^{-i}) = ess\sup_j\{x_{hjt}\}$ $j \in [0,1]$.

If the idea is supposed to be that once one person has attained a given level of knowledge, it is freely available to all other households, it seems to me that this is best captured by formulations 4 and 5.

There are problems here however. First and foremost is that in many of these formulations (all except #1 in fact) knowledge is VERY public. This is reflected in the following:

Claim 1 *In formulations 2, 4, and 6,* $x_{hit} = 0$ *in equilibrium.*

Claim 2 *In formulation 3,* $x_{hit} \approx 0$ *if* I *is large.*

Conjecture 3 *In formulation 5,* $x_{hit} \approx 0$ *if* I *is large.*

This creates a problem!

Note that these is another one, of a more technical nature, which is alluded to above. The way this model is constructed, at time $t = 0$, all agents simultaneously and independently choose the entire time paths for all of their choice variables. This is an odd way to construct a game in a dynamic setting. More typically, one would model the choices in peroid t as being made as functions of the entire history up to and including period t. Thus, we would have $x_{hit}(H^t)$ where H^t includes, in particular, all of the past decisions by ALL agents on $x_{hj\tau}$ $\tau < t$. This may seem like only a technical issue, but it raises the possibility of repeated game equilibria, other than those described in the Claim's and Conjecture above. For example, might there be an equilibrium path in which $x_{hit} \neq 0$ which is 'enforced' through some sort of trigger strategies? This is almost certainly true for the versions of the models with finitely many agents. Moreover, there are almost certainly many, many equilibria. Why? It seems likely (maybe this should go under the category of Conjecture again) that even with this alternative formulation, the $'0'$ equilibrium described above will still be a subgame perfect equilibrium of the new model constructed with this richer set of strategies. That is, if for all j $x_{hjt}(H^t) = 0$ for all H^t a best response by i is likely to be $x_{hit}(H^t) = 0$ for all H^t as well. But, the presence of this 'bad' equilibrium is exactly the kind of thing that gives the Folk Theorem the most bite. Using this equilibrium as a 'threat' (and a credible one since it represents a subgame perfect equilibrium set of strategies), it then become possible to implement almost any feasible outcome exactly because the outcome under the threat is SO bad.

Thus, Alternative modelling devices need to be explored. Some of the most obvious are:

1. Abandon Competitive Behavior. I.e., look at equilibrium notions where the government provides x_{hit} directly and/or requires some minimum level of x_{hit}. This clearly does away with (some of) the difficulties described in the claims.

2. Alter the form of G to something like:

   $G(x_{hit}, x_{hjt}^{-i}) = B(x_{hit})^{\eta}\bar{x}^{1-\eta}$, where,

   $\bar{x} = \frac{1}{I}\int_I x_{hjt}dj$.

   This will take care of the '0' problem since under this formulation, $\partial G/\partial x_{hit} = \infty$ at $(0, \bar{x})$ as long as $\bar{x} > 0$. Thus, although it may be true that '0' is still an equilibrium, it is likely that there is an interior one too.

Note that in these formulations, as in the $A(k,h)$ model with heterogeneity, we will still have:

$$TFP_t = A\frac{\left[n_{mt}\int_I h_{it}di\right]^{.67}}{\left[In_{mt}\right]^{.67}} = A\frac{\left[In_{mt}\bar{h}_t\right]^{.67}}{\left[In_{mt}\right]^{.67}} = A\left[\bar{h}_t\right]^{.67}$$

where $\bar{h}_t = \frac{1}{I}\int_I h_{it}di$.

2 Back to the Examples

I guess we didn't learn. Maybe we should try some simpler examples first?

Here we go....

Here, we begin to try and integrate the idea that 'knowledge' is a public good. We'll begin with the same sort of simple examples to try and anticipate pitfalls/problems we might encounter in more complicated settings.

To formalize the idea of the public good aspect it is useful to complicate the model somewhat. Since it is intrinsically a statement about the 'interactions' among multiple agents. With this in mind, we'll look at a series of examples which share several common features:

1. Continuum of agents of each type– type is consumer, R&D firm, Output firm.

2. Identical Agents within a type.

3. Learning takes place in the first period.

4. The Problem solved by each individual agent is an CRS/convex maximization problem.
5. Productivity in the second period depends not only own investment, but also on how much 'knowledge creation' other agents do.

2.1 Try #1

Households: $i \in [0, I]$
R&D Firms: $j_{RD} \in [0, J_{RD}]$
Output Firms: $j_y \in [0, J_y]$

An equilibrium is:

Prices: (p, r, w, r_0^A, r_1^A)

Quantity decisions for the households: $(c_i, k_i, \ell_i, h_i, n_i^h, n_i^y, e_i^{HH}, A_i^{HH}) = z_i^{HH}$;

Quantity decisions for the output firms: $(c_j^f, k_j^f, e_j^f, A_{j1}^f) = z_j^f$;

Quantity decisions for the R&D firms: $(A_{0j}^{RD}, A_{1j}^{RD}, k_j^{RD}, e_j^{RD}) = z_j^{RD}$;

Specifications of profits for HH, output firm, R&D firm: $(\Pi_i^{HH}, \Pi_j^f, \Pi_j^{RD})$;

SUCH THAT:

1) z_i^{HH} is the solution to:

$$Max_{(c_i,k_i,\ell_i,h_i,n_i^h,n_i^y,e_i^{HH},A_i^{HH})} U(c, \ell)$$

subject to:

$$pc_i \le rk_i + we_i^{HH} + r_0^A A_i^{HH} + \Pi_i^{HH};$$

$$n_i^h + n_i^y + \ell \le 1;$$

$$h_i \le g^1(n_i^h; \bar{n}^h);$$

$$e_i^{HH} \le g^2(n_i^y, h_i; \bar{n}^y, \bar{h}^y);$$

$k_i \le k_0$ $\quad$ k_0 fixed and identical across households;

$A^{HH} \leq A_0^{HH} \qquad A_0^{HH}$ fixed.

2) $\quad z_j^f$ is the solution to:

$$Max_{(c_j^f, k_j^f, e_j^f, A_{j1}^f)} \qquad pc_j^f - rk_j^f - we_j^f - r_1^A A_{j1}^f$$

subject to: $c_j^f \leq F(A_{1j}^f, k_j^f, e_j^f; \bar{A}_1^f; \bar{A}_1^{RD})$.

3) $\quad z^{RD}$ is the solution to:

$$Max_{(A_{0j}^{RD}, A_{1j}^{RD}, k_j^{RD}, e_j^{RD})} \qquad r_1^A A_{1j}^{RD} - rk_j^{RD} - we_j^{RD} - r_0^A A_{0j}^{RD}$$

subject to: $A_{1j}^{RD} \leq G(A_{0j}^{RD}, k_j^{RD}, e_j^{RD}; \bar{A}^{RD}, \bar{k}^{RD}, \bar{e}^{RD})$.

AND

$\int c_i di = \int c_j^f dj;$

$\int k_i di = \int k_j^f dj + \int k_j^{RD} dj;$

$\int e_i di = \int e_j^f dj + \int e_j^{RD} dj;$

$\int A_{1j}^f dj = \int A_{1j}^{RD} dj;$

$\int A_i^{HH} di = \int A_{0j}^{RD} dj;$

$\Pi_i^{HH} = \frac{1}{I}\left[\int_j \Pi_j^f dj + \int \Pi_j^{RD} dj\right];$

$\Pi_j^f = pc_j^f - rk_j^f - we_j^f - r_1^A A_{1j}^f;$

$\Pi_j^{RD} = r_1^A A_{1j}^{RD} - rk_j^{RD} - we_j^{RD} - r_0^A A_{0j}^{RD};$

$\bar{n}^h = \int n_i^h di \qquad or \qquad \bar{n}^h = \frac{1}{I}\int n_i^h di;$

$\bar{n}^y = \int n_i^y di \qquad or \qquad \bar{n}^y = \frac{1}{I}\int n_i^y di;$

$\bar{h}^y = \int h_i^y di \qquad or \qquad \bar{h}^y = \frac{1}{I}\int h_i^y di;$

$\bar{A}_1^f = \int A_{1j}^f dj \qquad or \qquad \bar{A}_1^f = \frac{1}{J_f}\int A_{1j}^f dj;$

$\bar{A}_1^{RD} = \int A_{1j}^{RD} dj \qquad or \qquad \bar{A}_1^{RD} = \frac{1}{J_{RD}}\int A_{1j}^{RD} dj \qquad or \qquad \bar{A}_1^{RD} = \frac{1}{J_f}\int A_{1j}^{RD} dj;$

$$\bar{A}^{RD} = \int A_{1j}^{RD} dj \qquad or \qquad \bar{A}^{RD} = \frac{1}{J_{RD}} \int A_{1j}^{RD} dj;$$

$$\bar{k}^{RD} = \int k_j^{RD} dj \qquad or \qquad \bar{k}^{RD} = \frac{1}{J_{RD}} \int k_j^{RD} dj;$$

$$\bar{e}^{RD} = \int e_j^{RD} dj \qquad or \qquad \bar{e}^{RD} = \frac{1}{J_{RD}} \int e_j^{RD} dj.$$

ASSUME: g^1, g^2, F, and G are all CRS in the private choices, but may be IRS in the public ones.

WHAT A MESS!!!!!!!!

What is the notion of Equilibrium?

Is everyone choosing 0 an equilibrium?

What if knowledge is truly public – I.e., YOUR n^h doesn't affect YOUR h only $\bar{n}^h$ does? Related questions otherwise.

What about teaching?

What is the 'right' technology?

2.2 Try #2– Much Simpler

Kill R&D, Only one type of External Effect:

Households: $i \in [0, I]$
R&D Firms: $j_{RD} \in [0, J_{RD}]$

An equilibrium is:

Prices: (p, r, w, r_0^A, r_1^A)

Quantity decisions for the households: $(c_i, k_i, \ell_i, h_i, n_i^h, n_i^y, e_i^{HH}) = z_i^{HH}$;

Quantity decisions for the output firms: $(c_j^f, k_j^f, e_j^f) = z_j^f$;

Specifications of profits for HH, output firm: (Π_i^{HH}, Π_j^f);

SUCH THAT:

1) z_i^{HH} is the solution to:

$$Max_{(c_i,k_i,\ell_i,h_i,n_i^h,n_i^y,e_i^{HH})}U(c_i,\ell_i,h_i)$$

subject to:

$$pc_i \leq rk_i + we_i^{HH} + \Pi_i^{HH};$$

$$n_i^h + n_i^y + \ell_i \leq 1;$$

$$h_i \leq g^1(n_i^h;\bar{n}^h) = b\left(n_i^h\right)^{\nu}\left(\bar{n}^h\right)^{\eta};$$

$$e_i^{HH} \leq g^2(n_i^y, h_i) = [n_i^y]^{\delta}[h_i]^{1-\delta};$$

$k_i \leq k_0 \qquad k_0$ fixed and identical across households.

2) z_j^f is the solution to:

$$Max_{(c_j^f,k_j^f,e_j^f)} \qquad pc_j^f - rk_j^f - we_j^f$$

subject to: $c_j^f \leq F(k_j^f, e_j^f) = A\left[k_j^f\right]^{\alpha}\left[e_j^f\right]^{1-\alpha}$.

AND

$$\int c_i di = \int c_j^f dj;$$

$$\int k_i di = \int k_j^f dj;$$

$$\int e_i di = \int e_j^f dj;$$

$$\Pi_i^{HH} = \frac{1}{I}\int_j \Pi_j^f dj;$$

$$\Pi_j^f = pc_j^f - rk_j^f - we_j^f;$$

$$\bar{n}^h = \int n_i^h di.$$

ASSUME: g^1, g^2, F, and G are all CRS in the private choices, but may be IRS in the public ones. In this case, it follows that $\Pi_j^f = 0$ for all j and so $\Pi_i^{HH} = 0$ for all i.

$$Max_{(n_i^h,n_i^y)}U(\tfrac{1}{p}\left[rk_0 + w\left[n_i^y\right]^{\delta}\left[b\left(n_i^h\right)^{\nu}\left(\bar{n}^h\right)^{\eta}\right]^{1-\delta}\right], 1 - n_i^h - n_i^y, b\left(n_i^h\right)^{\nu}\left(\bar{n}^h\right)^{\eta})$$

Assume $U(c,\ell,h) = \phi_c \log(c) + \phi_\ell \log(\ell) + \phi_h \log(h)$.

$$Max_{(n_i^h, n_i^y)} \qquad \phi_c \log\left[\frac{1}{p}\left[rk_0 + w\left[n_i^y\right]^\delta \left[b\left(n_i^h\right)^\nu \left(\bar{n}^h\right)^\eta\right]^{1-\delta}\right]\right] + \phi_\ell \log\left[1 - n_i^h - n_i^y\right] + \phi_h \log\left[b\left(n_i^h\right)^\nu \left(\bar{n}^h\right)^\eta\right]$$

Equivalently:

$$Max_{(n_i^h, n_i^y)} \qquad \phi_c \log\left[rk_0 + w\left[n_i^y\right]^\delta \left[b\left(n_i^h\right)^\nu \left(\bar{n}^h\right)^\eta\right]^{1-\delta}\right] + \phi_\ell \log\left[1 - n_i^h - n_i^y\right] + \phi_h \nu \log\left[n_i^h\right]$$

Assume $k_0 = 0,\ F_k = 0$ to start to get:

$$Max_{(n_i^h, n_i^y)} \qquad \phi_c \log\left[w\left[n_i^y\right]^\delta \left[b\left(n_i^h\right)^\nu \left(\bar{n}^h\right)^\eta\right]^{1-\delta}\right] + \phi_\ell \log\left[1 - n_i^h - n_i^y\right] + \phi_h \nu \log\left[n_i^h\right]$$

Or,

$$Max_{(n_i^h, n_i^y)} \qquad \phi_c \left[\log(w) + \delta \log(n_i^y) + (1-\delta)\log(b) + (1-\delta)\nu \log(n_i^h) + (1-\delta)\eta \log\left(\bar{r}\right.\right.$$

$$+\phi_\ell \log\left[1 - n_i^h - n_i^y\right] + \phi_h \nu \log\left[n_i^h\right]$$

Or,

$$Max_{(n_i^h, n_i^y)} \qquad \phi_c \left[\delta \log(n_i^y) + (1-\delta)\nu \log(n_i^h)\right] + \phi_\ell \log\left[1 - n_i^h - n_i^y\right] + \phi_h \nu \log\left[n_i^h\right].$$

FOC's:

$n_i^h: \qquad \left[\phi_c(1-\delta)\nu + \phi_h \nu\right]\frac{1}{n_i^h} = \phi_\ell \frac{1}{1-n_i^h-n_i^y};$

$n_i^y: \qquad \phi_c \delta \frac{1}{n_i^y} = \phi_\ell \frac{1}{1-n_i^h-n_i^y}.$

NOTE THE PUBLIC GOODS PROBLEM: If $\nu = 0$, then $n_i^h = 0$, no matter what others pick!

So,

$$\left[\phi_c(1-\delta)\nu + \phi_h \nu\right]\frac{1}{n_i^h} = \phi_c \delta \frac{1}{n_i^y},$$

$$n_i^y = D n_i^h,$$

where

$$D = \frac{\phi_c \delta}{\left[\phi_c(1-\delta)\nu + \phi_h \nu\right]}.$$

From FOCn_i^y :

$$n_i^y : \qquad \phi_c\delta\left[1-n_i^h-n_i^y\right]=\phi_\ell n_i^y;$$

$$n_i^y : \qquad \phi_c\delta\left[1-n_i^h-Dn_i^h\right]=\phi_\ell Dn_i^h;$$

$$n_i^y : \qquad \phi_c\delta\times 1=\left[\phi_\ell D+\phi_c\delta(1+D)\right]n_i^h;$$

$$n_i^y : \qquad n_i^h=\tfrac{\phi_c\delta}{[\phi_\ell D+\phi_c\delta(1+D)]}\times 1;$$

and,

$$n_i^y=Dn_i^h=\tfrac{\phi_c\delta D}{[\phi_\ell D+\phi_c\delta(1+D)]}\times 1.$$

Thus,

$$\bar{n}^h=\tfrac{\phi_c\delta}{[\phi_\ell D+\phi_c\delta(1+D)]}I;$$

and so,

$$h_i=b\left(n_i^h\right)^\nu\left(\bar{n}^h\right)^\eta=b\left(\tfrac{\phi_c\delta}{[\phi_\ell D+\phi_c\delta(1+D)]}\right)^\nu\left(\tfrac{\phi_c\delta}{[\phi_\ell D+\phi_c\delta(1+D)]}I\right)^\eta=b\left(\tfrac{\phi_c\delta}{[\phi_\ell D+\phi_c\delta(1+D)]}\right)^{\nu+\eta}I^\eta.$$

Note that the size of h_i depends on the size of the external effect which in turn depends on the size of the population, I. If we had used the other form of the externality, the $I's$ would have cancelled out. Part of this came about because of the log/Cobb-Douglas formulation. It implied that the optimal choice of n_i^h didn't depend on what others chose, or the size of the population, etc. Other (more realistic?) formulations would not have this property.

From this, we can see:

$$e_i^{HH}=\left[n_i^y\right]^\delta\left[h_i\right]^{1-\delta}=\left[\tfrac{\phi_c\delta D}{[\phi_\ell D+\phi_c\delta(1+D)]}\right]^\delta\left[b\left(\tfrac{\phi_c\delta}{[\phi_\ell D+\phi_c\delta(1+D)]}\right)^{\nu+\eta}I^\eta\right]^{1-\delta};$$

$$e_i^{HH}=D^\delta b^{\nu+\eta}\left[\tfrac{\phi_c\delta}{[\phi_\ell D+\phi_c\delta(1+D)]}\right]^{\delta+(1-\delta)(\nu+\eta)}I^{\eta(1-\delta)}.$$

And so, give that we killed capital meaning that:

$$c=we,$$

we have:

$$c=wD^\delta b^{\nu+\eta}\left[\tfrac{\phi_c\delta}{[\phi_\ell D+\phi_c\delta(1+D)]}\right]^{\delta+(1-\delta)(\nu+\eta)}I^{\eta(1-\delta)}.$$

This implies that larger 'countries' will have higher c, other things equal.

2.3 Planner's Problem Version

This equilibrium allocation is typically not efficient. What would the Planner's Solution be?

$$Max_{\{c_i, n_i^h, n_i^y, e_i, \ell_i\}_{i \in I}} \qquad \int_I U(c_i, \ell_i, h_i) di$$

subject to:

$$\int c_i di = w \int e_i di;$$

$$n_i^h + n_i^y + \ell_i \leq 1;$$

$$h_i = b \left(n_i^h\right)^{\nu} \left(\bar{n}^h\right)^{\eta};$$

$$e_i^{HH} = [n_i^y]^{\delta} [h_i]^{1-\delta};$$

$$\bar{n}^h = \int n_i^h di.$$

Impose symmetry and substitute:

$$Max_{\{c, n^h, n^y, e, \ell\}} \qquad IU(c, \ell, h)$$

subject to:

$$I \times c = I \times we;$$

$$n^h + n^y + \ell \leq 1;$$

$$h = b \left(n_i^h\right)^{\nu} \left(In^h\right)^{\eta} = bI^{\eta} \left(n^h\right)^{\nu+\eta};$$

$$e = [n^y]^{\delta} [h]^{1-\delta}.$$

More substitution and drop $I's$ where not needed:

$$Max_{\{n^h, n^y\}} \qquad U(w [n^y]^{\delta} \left[bI^{\eta} \left(n^h\right)^{\nu+\eta}\right]^{1-\delta}, 1 - n^h - n^y, bI^{\eta} \left(n^h\right)^{\nu+\eta}).$$

Assume $U(c, \ell, h) = \phi_c \log(c) + \phi_\ell \log(\ell) + \phi_h \log(h)$ and substitute to get:

$$Max_{\{n^h, n^y\}} \qquad \phi_c \log(w [n^y]^{\delta} \left[bI^{\eta} \left(n^h\right)^{\nu+\eta}\right]^{1-\delta}) + \phi_\ell \log(1 - n^h - n^y) + \phi_h \log(bI^{\eta} \left(n^h\right)^{\nu+\eta})$$

Equivalent to:

$$Max_{\{n^h,n^y\}} \qquad \phi_c\left[\log(w)+(1-\delta)\log(bI^{\eta})+\delta\log(n^y)+(1-\delta)(\nu+\eta)\log(n^h)\right]$$

$$+\phi_\ell\log(1-n^h-n^y)+\phi_h\log(bI^{\eta})+\phi_h(\nu+\eta)\log(n^h).$$

Simplify:

$$Max_{\{n^h,n^y\}} \qquad \phi_c\left[\delta\log(n^y)+(1-\delta)(\nu+\eta)\log(n^h)\right]+\phi_\ell\log(1-n^h-n^y)+\phi_h(\nu+\eta)\log(n^h).$$

FOC's:

$$n^h: \qquad \left[\phi_c(1-\delta)(\nu+\eta)+\phi_h(\nu+\eta)\right]\tfrac{1}{n^h}=\phi_\ell\tfrac{1}{1-n^h-n^y};$$

$$n^y: \qquad \phi_c\delta\tfrac{1}{n^y}=\phi_\ell\tfrac{1}{1-n^h-n^y}.$$

So,

$$\phi_c\delta\tfrac{1}{n^y}=\left[\phi_c(1-\delta)(\nu+\eta)+\phi_h(\nu+\eta)\right]\tfrac{1}{n^h},$$

$$n_i^y=D_{PP}n_i^h,$$

where,

$$D_{PP}=\tfrac{\phi_c\delta}{[\phi_c(1-\delta)(\nu+\eta)+\phi_h(\nu+\eta)]}.$$

I think that:

$$D=\tfrac{\phi_c\delta}{[\phi_c(1-\delta)\nu+\phi_h\nu]}>D_{PP}=\tfrac{\phi_c\delta}{[\phi_c(1-\delta)(\nu+\eta)+\phi_h(\nu+\eta)]}.$$

That is, less n goes into output production and more n goes to h in the planner's problem than in the equilibrium allocation.

True for sure if $\phi_\ell=0$. And true for sure as a fraction of total work too. Not sure if there is leisure too.

From the FOC for n^y :

$$n^y: \qquad \phi_c\delta\left[1-n^h-n^y\right]=\phi_\ell n^y;$$

$$n^y: \qquad \phi_c\delta\left[1-n^h-D_{PP}n^h\right]=\phi_\ell D_{PP}n^h;$$

$$n^y: \qquad \phi_c\delta\times 1=\left[\phi_\ell D_{PP}+\phi_c\delta(1+D_{PP})\right]n^h;$$

$$n^y: \qquad n^h=\tfrac{\phi_c\delta}{[\phi_\ell D_{PP}+\phi_c\delta(1+D_{PP})]}\times 1;$$

and so,

$$n^y = D_{PP}n^h = \frac{\phi_c\delta D_{PP}}{[\phi_\ell D_{PP}+\phi_c\delta(1+D_{PP})]} \times 1.$$

Thus,

$$\bar{n}^h = \frac{\phi_c\delta}{[\phi_\ell D_{PP}+\phi_c\delta(1+D_{PP})]} I;$$

and so,

$$h = b\left(\frac{\phi_c\delta}{[\phi_\ell D_{PP}+\phi_c\delta(1+D_{PP})]}\right)^{\nu+\eta} I^\eta.$$

Note that the size of h_i depends on the size of the external effect which in turn depends on the size of the population, I. If we had used the other form of the externality, the $I's$ would have cancelled out. Part of this came about because of the log/Cobb-Douglas formulation. It implied that the optimal choice of n_i^h didn't depend on what others chose, or the size of the population, etc. Other (more realistic?) formulations would not have this property.

From this, we can see:

$$e = [n^y]^\delta [h]^{1-\delta} = \left[\frac{\phi_c\delta D_{PP}}{[\phi_\ell D_{PP}+\phi_c\delta(1+D_{PP})]}\right]^\delta \left[b\left(\frac{\phi_c\delta}{[\phi_\ell D_{PP}+\phi_c\delta(1+D_{PP})]}\right)^{\nu+\eta} I^\eta\right]^{1-\delta};$$

$$e = D_{PP}^\delta b^{\nu+\eta}\left[\frac{\phi_c\delta}{[\phi_\ell D_{PP}+\phi_c\delta(1+D_{PP})]}\right]^{\delta+(1-\delta)(\nu+\eta)} I^{\eta(1-\delta)}.$$

And so, give that we killed capital meaning that:

$$c = we,$$

we have:

$$c = wD_{PP}^\delta b^{\nu+\eta}\left[\frac{\phi_c\delta}{[\phi_\ell D_{PP}+\phi_c\delta(1+D_{PP})]}\right]^{\delta+(1-\delta)(\nu+\eta)} I^{\eta(1-\delta)}.$$

This implies that larger 'countries' will have higher c, other things equal. And also they will have higher c than in countries that are 'less centralized.'

3 The Romer Model

3.1 Equilibrium in the Romer Model

There are a continuum of households indexed by $i \in [0,1]$ and a continuum of firms indexed by $j \in [0,1]$. They are all identical, the households have the same utility functions, initial endowments and labor supplies. The firms all have the same technology. For simplicity, we will assume that each consumer has an equal share in each of the firms.

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_{it}, k_{it}, x_{ikt}, \ell_{it}, n_{it})\}_{t=0}^{\infty} = z_i^{HH}$

Quantity decisions for the output firms: $\{(c_{jt}^f, x_{jkt}^f, k_{jt}^f, n_{jt}^f)\}_{t=0}^{\infty} = z_j^f$,

SUCH THAT:

1) For each $i \in [0,1]$, z_i^{HH} is the solution to:

$Max_{\{(c_{it}, k_{it}, x_{ikt}, n_{it}, \ell_{it})\}_{t=0}^{\infty}} \sum_t \beta^t u(c_{it}, \ell_{it})$

subject to:

$\sum_{t=0}^{\infty} p_t \left[c_{it} + x_{ikt}\right] \leq \sum_{t=0}^{\infty} \left[r_t k_{it} + w_t n_{it}\right] + \Pi_i$

$k_{it+1} \leq (1-\delta_k) k_{it} + x_{ikt}$

$n_{it} + \ell_{it} \leq 1,$

k_{i0} fixed.

2) For each $j \in [0,1]$, z_j^f is the solution to:

$Max_{\{(c_{jt}^f, x_{jkt}^f, k_{jt}^f, n_{jt}^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_{jt}^f + x_{jkt}^f) - r_t k_{jt}^f - w_t n_{jt}^f\right]$

subject to: $c_{jt}^f + x_{jkt}^f \leq F(k_{jt}^f, n_{jt}^f; K_t)$.

AND

$\int_0^1 c_{it} di = \int_0^1 c_{jt}^f dj$

$\int_0^1 x_{ikt} di = \int_0^1 x_{jkt}^f dj$

$\int_0^1 k_{it} di = \int_0^1 k_{jt}^f dj$

$\int_0^1 n_{it} di = \int_0^1 n_{jt}^f dj$

$\int_0^1 \Pi_i di = \int_0^1 \sum_{t=0}^{\infty} \left[p_t (c_{jt}^f + x_{jkt}^f) - r_t k_{jt}^f - w_t n_{jt}^f \right] dj$

$K_t = \int_0^1 k_{jt}^f dj$

NOTE: K_t affects the period t output of firm j, but it is NOT one of $j's$ choice variables. Thus, it is an external effect!

We will assume that F is CRS in (k_{jt}^f, n_{jt}^f) that is, $F(\lambda k_{jt}^f, \lambda_{jt}^f; K_t) = \lambda F(k_{jt}^f, n_{jt}^f; K_t)$

Assuming that $u(c, \ell) = c^{1-\sigma}/(1-\sigma)$, i.e., CES and inelastic labor supply, and since each individual firm has a CRS production function (in own inputs), the problem for the representative household can be rewritten as:

$Max_{\{(c_{it}, k_{it})\}_{t=0}^{\infty}} \sum_t \beta^t u(c_{it})$

subject to:

$\sum_{t=0}^{\infty} p_t \left[c_{it} + k_{it+1} - (1-\delta_k) k_{it} \right] \leq \sum_{t=0}^{\infty} \left[r_t k_{it} + w_t \right]$

k_{i0} fixed.

The FOC's are standard:

$c_t: \qquad \beta^t u'(c_{it}) = \lambda p_t$

$k_{t+1}: \qquad p_t = (1-\delta_k) p_{t+1} + r_{t+1}$

Or,

$$u'(c_t) = \beta u'(c_{t+1}) \left[1 - \delta_k + \frac{r_{t+1}}{p_{t+1}} \right],$$

or

$$\gamma_c^{\sigma} = \beta \left[1 - \delta_k + \frac{r_{t+1}}{p_{t+1}} \right].$$

This is the standard relationship.

The FOC's from the firms problem give:

$$\frac{w_t}{p_t} = F_n(k_t, n_t; K_t)$$

and,

$$\frac{r_t}{p_t} = F_k(k_t, n_t; K_t).$$

Using this in the EE of the household, we get:

$$\gamma_{ct}^{\sigma} = \beta\left[1 - \delta_k + F_k(k_{t+1}, n_{t+1}; K_{t+1})\right]$$

So far, everything is just like normal.

Now, assume that for each firm, the production function is a time-invariant Cobb-Douglas form:

$$F(k, n; K) = Ak^{\alpha}n^{1-\alpha}K^{\eta}$$

In this case, it follows as usual that

$$F_n = \frac{(1-\alpha)F}{n} \text{ and}$$

$$F_k = \frac{\alpha F}{k}$$

Thus, in any SYMMETRIC equilibrium we have that:

$$F_k = \frac{\alpha F}{k} = \frac{\alpha Ak^{\alpha}n^{1-\alpha}K^{\eta}}{k} = \frac{\alpha AK^{\alpha}n^{1-\alpha}K^{\eta}}{K} = \alpha AK^{\alpha+\eta-1}n^{1-\alpha}$$

since in symmetric equilbrium, $k = K$.

Since $n_t = 1$ for all t we have that

$$F_k(t+1) = \alpha AK_{t+1}^{\alpha+\eta-1}, \text{ and hence,}$$

$$\gamma_{ct}^{\sigma} = \beta\left[1 - \delta_k + \alpha AK_{t+1}^{\alpha+\eta-1}\right]$$

3.2 Romer Equilibrium: $\alpha + \eta < 1$

Assume that $\alpha + \eta < 1$. In this case, the arguments given above imply that it is not feasible for the economy to grow.

3.3 Romer Equilibrium: $\alpha + \eta = 1$

Assume that $\alpha + \eta = 1$.

In this case, the Euler Equation becomes:

$$\gamma_{ct}^{\sigma} = \beta \left[1 - \delta_k + \alpha A\right]$$

This looks just like the EE from an Ak model, but it is αA that appears rather than A.

Can we guess and verify that the equilibrium is the same as the relevant Ak model, but, with αA in place of A?

This can't be completely right because the feasibility constraints for this artificial αAk economy is:

$$c_t + k_{t+1} = \alpha A k_t + (1 - \delta_k) k_t$$

and, for the true economy it is:

$$c_t + k_{t+1} = A k_t + (1 - \delta_k) k_t$$

What is the equilibrium?

Let's conjecture that there is a BGP for this economy with $c_t = \gamma^t c_0$, etc.

We know for sure that if it is symmetric and interior this growth rate is give by:

$$\gamma^{\sigma} = \beta \left[1 - \delta_k + \alpha A\right]$$

Feasibility for this economy is given by:

$$c_t + k_{t+1} = A k_t + (1 - \delta_k) k_t = \{A + 1 - \delta_k\} k_t$$

Under our conjecture, $k_{t+1} = \gamma k_t$, and hence:

$c_t + \gamma k_t = \{A + 1 - \delta_k\} k_t$, or,

$c_t = \{A + 1 - \delta_k - \gamma\} k_t$, or,

$\frac{c_t}{k_t} = \{A + 1 - \delta_k - \gamma\}$, and so,

$\frac{c_t}{y_t} = \frac{c_t}{Ak_t} = \frac{1}{A}\{A + 1 - \delta_k - \gamma\}$

And,

$\frac{x_t}{y_t} = \frac{x_t}{Ak_t} = 1 - \frac{1}{A}\{A + 1 - \delta_k - \gamma\}$

I think this completely determines the equilibrium.

Check conditions!!!!

Thus, comparing it with the normal Ak solution, consumption is higher, and growth is lower.

*********Write down completely what the conjectured equilibrium path is!

Note that it is quite possible that:

$$\gamma = [\beta [1 - \delta_k + \alpha A]]^{1/\sigma} < 1,$$

i.e., the economy shrinks toward zero in equilibrium even though

$$\beta [1 - \delta_k + \alpha A] > 1!$$

****

THE ROLE OF POPULATION SIZE AND THE FORM OF THE EXTERNAL EFFECT

NOTE NICE TRICK– IRS BY CRS AND EXTERNALITY

3.4 Romer: The Planner's Problem Version, $\alpha + \eta = 1$

The Planner's Problem for the Romer Economy is:

max $\quad \int_0^1 \lambda_i \sum_t \beta^t u(c_{it}, \ell_{it}) di$

s.t.

$$\int [c_{it}]\, di + x_{kt} \leq F(K_t, 1; K_t)$$

$$K_{t+1} \leq (1 - \delta_k) K_t + x_{kt}$$

That is, the planner recognizes that $k = K$.

Under symmetry, under the assumption that $\alpha + \eta = 1$, this is simply an Ak model, so that the Planner's solution grows at the rate:

$$\gamma_{PP} = [\beta [1 - \delta_k + A]]^{1/\sigma}$$

Note that since $\alpha < 1$, $\gamma_{PP} > \gamma$.

The CE of this model has inefficiently too low growth. This follows directly from the fact that it has inefficiently too low an interest rate:

$$1 + R_{CE} = 1 - \delta_k + \alpha A \text{ vs. } 1 + R_{PP} = 1 - \delta_k + A$$

This is because the individual firms fail to take into account the extra benefits that other firms obtain when an individual firm increases its capital stock.

3.5 Romer when $\alpha + \eta > 1$

This one is more difficult and less is known about it as a result. The only things that are known (as far as I know) are partial statements of the form:

IF $k_t \to \infty$, THEN a bunch of other things must be true too.

As above, we have that:

$$\gamma_{ct}^{\sigma} = \beta \left[1 - \delta_k + \alpha A K_{t+1}^{\alpha+\eta-1} \right]$$

But, here notice that since $\alpha + \eta > 1$, if K_t grows over time it follows that $\alpha A K_{t+1}^{\alpha+\eta-1}$ does as well, and hence, so do γ_{ct} and R_t. Indeed they both go to infinity.

Thus, if there is an equilibrium with growth, it occurs at an accelerating rate. This was actually one of the things that Romer thought was a plus about the effort. He thought that it matched up well with a slow growth in growth rates, and an acceleration in knowledge accumulation that many people have said is a part of the data over long history. But, it seems to have been lost over the years. Probably because there seems to be no indication that $\gamma's$ are increasing over the last 50 years, which is the only time good data has been available.

3.5.1 Planner's Problem Again

The Planner's Problem for the Romer Economy is:

$$\max \qquad \int_0^1 \lambda_i \sum_t \beta^t u(c_{it}, \ell_{it}) di$$

s.t.

$$\int [c_{it}]\, di + x_{kt} \le F(K_t, 1; K_t)$$

$$K_{t+1} \le (1-\delta_k)K_t + x_{kt}$$

That is, the planner recognizes that $k = K$.

Under symmetry, this becomes:

$$\max \quad \sum_t \beta^t u(c_t, \ell_t)$$

s.t.

$$c_t + x_{kt} \le F(K_t, 1; K_t)$$

$$K_{t+1} \le (1-\delta_k)K_t + x_{kt}$$

Letting $G(K) = F(K, 1; K) + (1-\delta_k)K$, we can write this problem as:

$$\max \quad \sum_t \beta^t u(c_t, \ell_t)$$

s.t.

$$c_t + K_{t+1} \le G(K)$$

Thus, this is a DP in standard form. But, $G(K)$ may not be convex.

However, it does follow that $V(K)$, the value function for the problem is strictly increasing.

Assuming that labor supply is inelastic and that the solution is interior we still get a version of the standard Euler Equation for the model:

$$\gamma_t^\sigma = \beta\left[1 - \delta_k + F_k(t+1)\right] = \beta\left[1 - \delta_k + \alpha A k_{t+1}^{\alpha+\eta-1}\right]$$

3.5.2 Log preferences with $\delta_k = 1$

The special case of $u(c) = \log(c)$ coupled with the assumption that $\delta_k = 1$ is special (as always). In this case, a closed form solution for the problem can be found. It is of the usual variety for problems of this form with $V(k) = D_0 + D_1 \log(k)$ and $g(k) = \theta k$. You should go through the guess and verify routine for yourselves with Bellman's equation to find that the usual expressions for D_0, D_1 and θ apply in this case.

In particular, it still holds that:

$$g(k) = (\alpha + \eta)\beta G(k)$$

where $G(k)$ is as defined above, $G(k) = Ak^{\alpha+\eta}$.

Note that in this case, $G'(0) = 0$ and $G'(\infty) = \infty$ however. It follows from this that there is a unique interior steady state satisfying $G'(k_{ss}) = \frac{1}{\beta}$.

This steady state is globally unstable however as can be seen immediately from the description of g give above. In contrast, both 0 and ∞ are stable steady states. Thus, we have the following:

1) If $k_0 = k_{ss}$, $k_t = k_{ss}$ for all t,

$\gamma = 0$ for all t.

$1 + R_t = 1 - \delta_k + \alpha A k_{t+1}^{\alpha+\eta-1} = 1 - \delta_k + \alpha A k_{ss}^{\alpha+\eta-1}$ for all t.

2) If $k_0 < k_{ss}$, k_t converges monotonically to 0.

$$\gamma_t = \frac{y_{t+1}}{y_t} = \frac{Ak_{t+1}^{\alpha+\eta}}{Ak_t^{\alpha+\eta}} = \frac{A\left[(\alpha+\eta)\beta A k_t^{\alpha+\eta}\right]^{\alpha+\eta}}{Ak_t^{\alpha+\eta}} = [(\alpha+\eta)\beta A]^{\alpha+\eta}\, k_t^{(\alpha+\eta)[\alpha+\eta-1]} \to 0$$

$$1 + R_t = 1 - \delta_k + \alpha A k_{t+1}^{\alpha+\eta-1} \to 1 - \delta_k$$

3) If $k_0 > k_{ss}$, k_t converges monotonically to ∞.

$$\gamma_t = \frac{y_{t+1}}{y_t} = [(\alpha+\eta)\beta A]^{\alpha+\eta}\, k_t^{(\alpha+\eta)[\alpha+\eta-1]} \to \infty$$

$$1 + R_t = 1 - \delta_k + \alpha A k_{t+1}^{\alpha+\eta-1} \to \infty$$

In all three cases,

$$TFP_t^i = \frac{y_t^i}{\left(k_t^i\right)^{.33}\left(n_t^i\right)^{.67}} = \frac{A\left(k_t^i\right)^{\alpha+\eta}}{\left(k_t^i\right)^{.33}\left(n_t^i\right)^{.67}} = A\left(k_t^i\right)^{\alpha+\eta-.33}$$

3.6 Alternative Methods for Including the Externality?

e.g.,

firms have k, not households and k'=(1-d)k+x*X?

etc

3.7 Multiplicity in the Romer Model

There is a small literature on the relationship between externalities and the presence of multiple equilibria. Boldrin has a series of papers on this topic with a series of co-authors. The one with Rustichini deals with the Romer model in the case that $\alpha + \eta < 1$ (so that output is bounded). Check out his web page, or look at the work by Benhabib from NYU.

The easiest way to see the kind of thing that can happen is to consider the full depreciation case, $\delta_k = 1$.

4 Heterogeneity Across Countries in the Romer Model with $\alpha + \eta = 1$

4.0.1 Romer Model, $\alpha + \eta = 1$, Equilibrium: Initial Conditions

Suppose that the only differences in countries is in k_0. What would the Heston-Summers data set look like?

$$y_i = F(k_t^i, 1; k_t^i) = Ak_t^i = A\gamma^t k_0^i.$$

$$1 + R_t^i = 1 - \delta_k + \alpha A$$

$$TFP_t^i = \frac{y_t^i}{\left(k_t^i\right)^{.33}\left(n_t^i\right)^{.67}} = \frac{Ak_t^i}{\left(k_t^i\right)^{.33}\left(n_t^i\right)^{.67}} = A\left(k_t^i\right)^{.67}$$

Thus, this looks a lot like that $A(k, h)$ model.

5 The Lucas Version

An alternative model based on the same ideas is the one developed in Lucas. Here, there are formally two types of capital, h and k. The external effect is just like that in Romer, i.e., it is multiplicative in output that depends on total or average h in the economy. As an added tweak, it has a separate sector for producing h. It is an unusual modeling assumption, which could be called 'learning by thinking' in that only time and old h enter into the production of new h.

A quick and dirty outline of the details is:

There are a continuum of households indexed by $i \in [0,1]$ and a continuum of firms indexed by $j \in [0,1]$. They are all identical, the households have the same utility functions, initial endowments and labor supplies. The firms all have the same technology. For simplicity, we will assume that each consumer has an equal share in each of the firms.

An equilibrium is:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_{it}, k_{it}, x_{ikt}, \ell_{it}, n_{it}^y, n_{it}^h, h_{it})\}_{t=0}^{\infty} = z_i^{HH}$

Quantity decisions for the output firms: $\{(c_{jt}^f, x_{jkt}^f, k_{jt}^f, n_{jt}^f)\}_{t=0}^{\infty} = z_j^f$,

SUCH THAT:

1) For each $i \in [0,1]$, z_i^{HH} is the solution to:

$Max_{\{(c_{it}, k_{it}, x_{ikt}, \ell_{it}, n_{it}^y, n_{it}^h, h_{it}, z_{it}^y)\}_{t=0}^{\infty}} \sum_t \beta^t u(c_{it}, \ell_{it})$

subject to:

$\sum_{t=0}^{\infty} p_t \left[c_{it} + x_{ikt}\right] \leq \sum_{t=0}^{\infty} \left[r_t k_{it} + w_t z_{it}^y\right] + \Pi_i$

$k_{it+1} \leq (1-\delta_k)k_{it} + x_{ikt}$

$n_{it}^y + n_{it}^h + \ell_{it} \leq 1,$

$z_{it}^y = n_{it}^y h_{it}$

$h_{it+1} \leq \xi(n_{it}^h) h_t$

k_{i0} fixed.

2) For each $j \in [0,1]$, z_j^f is the solution to:

$Max_{\{(c_{jt}^f, x_{jkt}^f, k_{jt}^f, n_{jt}^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t(c_{jt}^f + x_{jkt}^f) - r_t k_{jt}^f - w_t z_{jt}^f\right]$

subject to: $c_{jt}^f + x_{jkt}^f \leq F(k_{jt}^f, z_{jt}^f; H_t)$.

AND

$\int_0^1 c_{it} di = \int_0^1 c^f_{jt} dj$

$\int_0^1 x_{ikt} di = \int_0^1 x^f_{jkt} dj$

$\int_0^1 k_{it} di = \int_0^1 k^f_{jt} dj$

$\int_0^1 z^y_{it} di = \int_0^1 z^f_{jt} dj$

$\int_0^1 \Pi_i di = \int_0^1 \sum_{t=0}^{\infty} \left[p_t(c^f_{jt} + x^f_{jkt}) - r_t k^f_{jt} - w_t z^f_{jt} \right] dj$

$H_t = \int_0^1 h_{it} di$

NOTE: H_t affects the period t output of firm j, but it is NOT one of $j's$ choice variables. Thus, it is an external effect!

We will assume that F is CRS in (k^f_{jt}, z^f_{jt}) that is, $F(\lambda k^f_{jt}, \lambda z^f_{jt}; H_t) = \lambda F(k^f_{jt}, z^f_{jt}; H_t)$

Indeed, let's assume straight out that $u(c, \ell) = c^{1-\sigma}/(1-\sigma)$ and $F(k, z; H) = Ak^\alpha z^{1-\alpha} H^\theta$

If $\alpha + \theta < 1$ there is no growth.

If $\alpha + \theta = 1$ there is balanced growth, but it is inefficiently low.

If $\alpha + \theta > 1$ but $\alpha < 1$, there is also balanced growth and it is inefficiently low. Thus, unlike the Romer model, this version of the external effect does not exhibit explosive growth dynamics in the case where there are IRS in the two reproducible factors. Why is this?

6 Productive Government Spending: The Barro Model

A second and related point is to consider the possibility of having some of the goods for which there are external effects be produced directly by the government. This is at the heart of what I will call the 'Barro Model.' This is of course, an old idea, Barro is the person that introduced it explicitly into Endogenous Growth models though.

The basic modelling change is to include the idea that government spending is productive. There are two ways to do this, directly into the utility function and as a productive input in the firm side.

One of the main motivations for this is that, empirically, it is NOT true that countries with low growth rates, or low levels of output are those that have high levels of taxation and spending. Indeed Walker's Law says that it is the opposite. This connection has been explored extensively empirically with important distinctions drawn between government 'consumption' and 'investment.'

We begin with a couple of simple examples.

6.1 First Example, g in the Utility Function

A continuum of identical households, $i \in [0, I]$, one representative firm producing output, no k.

An equilibrium given g is:

Prices: (p, w)

Quantity decisions for the households: $(c_i, \ell_i, n_i) = z_i^{HH}$;

Quantity decisions for the firm: $(c^f, g^f, n^f) = z^f$;

Specifications of profits for HH, output firm: (Π_i^{HH}, Π^f);

SUCH THAT:

1) z_i^{HH} is the solution to:

$$Max_{(c_i, \ell_i, n_i)} U(c_i, \ell_i; g)$$

subject to:

$$pc_i \leq (1-\tau)wn_i + \Pi_i^{HH};$$

$$n_i + \ell_i \leq 1;$$

2) z^f is the solution to:

$$Max_{(c^f, g^f, n^f)} \qquad p(c^f + g^f) - wn^f$$

subject to: $c^f + g^f \leq F(n^f) = \psi n^f$.

AND

$$\int c_i di = c_j^f;$$

$\int n_i di = n^f;$

$\Pi_i^{HH} = \frac{1}{I}\Pi^f;$

$\Pi^f = p(c^f + g^f) - wn^f;$

$g^f = g;$

$\tau w \int n_i^f di = pg.$

ASSUMED: F is CRS. It follows that $\Pi^f = 0$, and so $\Pi_i^{HH} = 0$ for all i.

Assume symmetry across $i's$.

From Firm's problem:

$Max_{(n^f)} \qquad p(\psi n^f) - -wn^f;$

$\frac{w}{p} = \psi.$

From Government BC:

$\tau wn^f = pg;$

$g = \tau \frac{w}{p} n^f;$

$g = \tau\psi I(1-\ell);$

From Household Problem:

$Max_{n_i} U(\frac{1}{p}(1-\tau)wn_i, 1-n_i; g)$

$\frac{w}{p}(1-\tau)U_c(c,\ell;g) = U_\ell(c,\ell;g);$

$\psi(1-\tau)U_c(c,\ell;g) = U_\ell(c,\ell;g);$

$\psi(1-\tau)U_c(c,\ell;\tau\psi I(1-\ell)) = U_\ell(c,\ell;\tau\psi I(1-\ell));$

$LHS(\tau) = RHS(\tau).$

Note dependence on I.

Picture?

What is $U_{\ell g}, U_{cg}$? How do $LHS(\tau)$ and $RHS(\tau)$ change with τ? What is optimal choice of τ?

Assume $U(c, \ell; g) = \alpha_c \log(c) + \alpha_\ell \log(\ell) + \alpha_g \log(g)$.

$LHS(\tau) = \psi(1-\tau)U_c(c, \ell; \tau\psi I(1-\ell)) = \psi(1-\tau)\frac{\alpha_c}{c}$

$= \psi(1-\tau)\frac{\alpha_c}{(1-\tau)\psi n} = \frac{\alpha_c}{n}$.

$RHS(\tau) = U_\ell(c, \ell; \tau\psi I(1-\ell)) = \frac{\alpha_\ell}{(1-n)}$.

So:

$\frac{\alpha_c}{n} = \frac{\alpha_\ell}{(1-n)}$;

$\frac{\alpha_c}{\alpha_\ell}(1-n) = n$;

$\frac{\alpha_c}{\alpha_\ell} = n(1 + \frac{\alpha_c}{\alpha_\ell})$;

$n = \frac{\frac{\alpha_c}{\alpha_\ell}}{(1+\frac{\alpha_c}{\alpha_\ell})} = \frac{\alpha_c}{\alpha_c+\alpha_\ell}$;

$\ell = \frac{\alpha_\ell}{\alpha_c+\alpha_\ell}$;

$c = \psi(1-\tau)n = \psi(1-\tau)\frac{\alpha_c}{\alpha_c+\alpha_\ell}$;

$g = I\tau\psi n = I\tau\psi\frac{\alpha_c}{\alpha_c+\alpha_\ell}$.

This gives:

$U(\tau) = \alpha_c \log(\psi(1-\tau)\frac{\alpha_c}{\alpha_c+\alpha_\ell}) + \alpha_\ell \log(\frac{\alpha_\ell}{\alpha_c+\alpha_\ell}) + \alpha_g \log(I\tau\psi\frac{\alpha_c}{\alpha_c+\alpha_\ell})$;

$U(\tau) = (\alpha_c + \alpha_g)\log(\psi\frac{\alpha_c}{\alpha_c+\alpha_\ell}) + \alpha_\ell \log(\frac{\alpha_\ell}{\alpha_c+\alpha_\ell}) + \alpha_g \log(I) + \alpha_c \log(1-\tau) + \alpha_g \log(\tau)$;

$U(\tau) = D + \alpha_g \log(I) + \alpha_c \log(1-\tau) + \alpha_g \log(\tau)$.

NOTE: Increasing in I. Should we have modelled this differently? Is there 'congestion'? etc.

Optimal τ:

$U'(\tau) = -\alpha_c\frac{1}{1-\tau} + \alpha_g\frac{1}{\tau} = 0$;

$\alpha_c\frac{1}{1-\tau} = \alpha_g\frac{1}{\tau}$;

$\frac{\alpha_c}{\alpha_g}\tau = 1 - \tau;$

$\left[1 + \frac{\alpha_c}{\alpha_g}\right]\tau = 1;$

$\tau^* = \frac{1}{1+\frac{\alpha_c}{\alpha_g}} = \frac{\alpha_g}{\alpha_c+\alpha_g};$

$\ell = \frac{\alpha_\ell}{\alpha_c+\alpha_\ell};$

$c = \psi(1 - \frac{\alpha_g}{\alpha_c+\alpha_g})\frac{\alpha_c}{\alpha_c+\alpha_\ell} = \psi\frac{\alpha_c}{\alpha_c+\alpha_g}\frac{\alpha_c}{\alpha_c+\alpha_\ell};$

$g = I\psi\frac{\alpha_g}{\alpha_c+\alpha_g}\frac{\alpha_c}{\alpha_c+\alpha_\ell}.$

Comparative Statics?

First Best?

$Max_{(c,\ell,n,g)}U(c, \ell; g)$

subject to:

$Ic + g \leq I\psi n;$

$n + \ell \leq 1.$

OR,

$Max_{(c,\ell,n,g)}U(c, \ell; g)$

subject to:

$c + \frac{g}{I} \leq \psi n;$

$n + \ell \leq 1.$

NOTE: 'price' of g is $\frac{1}{I}$ times the 'price' of c.

OR,

$Max_{(c,\ell,g)}U(c, \ell; g)$

subject to:

$c + \frac{g}{I} + \psi\ell = \psi \times 1.$

FOC's:

$c \qquad \frac{U_c}{1} = \frac{U_\ell}{\psi};$

$g \qquad IU_g = \frac{U_g}{\frac{1}{I}} = \frac{U_\ell}{\psi};$

$\ell \qquad c + \frac{g}{I} + \psi\ell = \psi \times 1.$

Assume $U(c, \ell; g) = \alpha_c \log(c) + \alpha_\ell \log(\ell) + \alpha_g \log(g)$.

$c \qquad \frac{\alpha_c}{c} = \frac{\frac{\alpha_\ell}{\ell}}{\psi};$

$g \qquad I\frac{\alpha_g}{g} = \frac{\frac{\alpha_\ell}{\ell}}{\psi};$

$\ell \qquad c + \frac{g}{I} + \psi\ell = \psi \times 1.$

OR

$c \qquad \frac{\ell\psi}{c} = \frac{\alpha_\ell}{\alpha_c};$

$g \qquad \frac{\ell\psi}{g/I} = \frac{\alpha_\ell}{\alpha_g};$

$\ell \qquad c + \frac{g}{I} + \psi\ell = \psi \times 1.$

OR

$c \qquad c = \frac{\alpha_c}{\alpha_\ell}\ell\psi;$

$g \qquad \frac{g}{I} = \frac{\alpha_g}{\alpha_\ell}\ell\psi;$

$\ell \qquad \frac{\alpha_c}{\alpha_\ell}\ell\psi + \frac{\alpha_g}{\alpha_\ell}\ell\psi + \psi\ell = \psi \times 1.$

So,

$\ell \qquad \ell\left[\frac{\alpha_c}{\alpha_\ell} + \frac{\alpha_g}{\alpha_\ell} + 1\right] = 1;$

$\ell \qquad \ell = \frac{1}{\left[\frac{\alpha_c}{\alpha_\ell} + \frac{\alpha_g}{\alpha_\ell} + 1\right]} = \frac{\alpha_\ell}{\alpha_c + \alpha_g + \alpha_\ell};$

$c \qquad c = \frac{\alpha_c}{\alpha_c + \alpha_g + \alpha_\ell}\psi;$

$g \qquad g = I\frac{\alpha_g}{\alpha_c + \alpha_g + \alpha_\ell}\psi.$

How do these compare with those obtained above?

How would you implement this?

When do you use LS taxes and when do you distort margins? I.e., compare this with Human Capital, R&D solution from above?

6.2 What if it is in Production?

How would we even do this? One way:

A continuum of identical households, $i \in [0, I]$, two sectors, one to produce g, one representative firm producing there, one sector to produce output, one representative firm there, no k.

An equilibrium given g is:

Prices: (p, p_g, w)

Quantity decisions for the households: $(c_i, \ell_i, n_i) = z_i^{HH}$;

Quantity decisions for the g firm: $(g^g, n^g) = z^g$;

Quantity decisions for the c firm: $(c^f, n^f) = z^f$;

Specifications of profits for HH, output firm: $(\Pi_i^{HH}, \Pi^g, \Pi^f)$;

SUCH THAT:

1) z_i^{HH} is the solution to:

$$Max_{(c_i,\ell_i,n_i)} U(c_i, \ell_i)$$

subject to:

$$pc_i \leq (1 - \tau)wn_i + \Pi_i^{HH};$$

$$n_i + \ell_i \leq 1;$$

2) z^g is the solution to:

$$Max_{(g^g,n^f)} \qquad p_g g^g - wn^g$$

subject to: $g^g \leq F^g(n^g) = \psi n^g$.

3) z^f is the solution to:

$$Max_{(c^f,n^f)} \qquad pc^f - wn^f$$

subject to: $c^f \leq F(n^f; g) = \phi(g)n^f$.

NOTE: This is IRS in (g, n^f).

AND

$\int c_i di = c^f;$

$\int n_i di = n^f + n^g;$

$\Pi_i^{HH} = \frac{1}{I}\left[\Pi^f + \Pi^g\right];$

$\Pi^g = p_g g^g - wn^g;$

$\Pi^f = pc^f - wn^f;$

$g^g = g;$

$\int \tau w n_i di = p_g g.$

ASSUMED: F is CRS. It follows that $\Pi^f = \Pi^g = 0$, and so $\Pi_i^{HH} = 0$ for all i.

SOLVING:

Government firm's problem gives:

$\psi = \frac{w}{p_g}.$

Output firm's problem gives:

$\phi(g) = \frac{w}{p}.$

So,

$\frac{p_g}{p} = \frac{\frac{w}{p}}{\frac{w}{p_g}} = \frac{\phi(g)}{\psi}.$

Consumer's Problem is:

$Max_{n_i} U(\frac{1}{p}(1-\tau)wn_i, 1-n_i)$

FOC:

ℓ $\quad \frac{(1-\tau)w}{p}U_c = U_\ell;$

c $\quad pc + w(1-\tau)\ell = (1-\tau)w;$

ℓ $\quad (1-\tau)\phi(g)U_c = U_\ell;$

$c \qquad c + \phi(g)(1-\tau)\ell = (1-\tau)\phi(g).$

Assume that $U(c,\ell) = \alpha_c \log(c) + \alpha_\ell \log(\ell)$.

$\ell \qquad (1-\tau)\phi(g)\frac{\alpha_c}{c} = \frac{\alpha_\ell}{\ell};$

$\ell \qquad c = (1-\tau)\phi(g)\frac{\alpha_c}{\alpha_\ell}\ell;$

$c \qquad (1-\tau)\phi(g)\frac{\alpha_c}{\alpha_\ell}\ell + \phi(g)(1-\tau)\ell = (1-\tau)\phi(g);$

$c \qquad \left[\frac{\alpha_c}{\alpha_\ell}+1\right]\ell = 1;$

$c \qquad \ell(\tau) = \frac{1}{\left[\frac{\alpha_c}{\alpha_\ell}+1\right]} = \frac{\alpha_\ell}{\alpha_c+\alpha_\ell};$

$\ell \qquad c(\tau) = \frac{\alpha_c}{\alpha_c+\alpha_\ell}(1-\tau)\phi(g(\tau)).$

From the government budget constraint, we have that:

$$p_g g = \tau w \int n_i di;$$

assuming symmetry, we get:

$$g(\tau) = \tau\frac{w}{p_g}n(\tau)I = \tau\psi I\left[1-\ell(\tau)\right] = \tau\psi I\frac{\alpha_c}{\alpha_c+\alpha_\ell}.$$

Ramsey Problem:

$$Max_\tau \qquad U(c(\tau),\ell(\tau)) = \alpha_c\log(c(\tau)) + \alpha_\ell\log(\ell(\tau));$$

$$Max_\tau \qquad \alpha_c\log(\tfrac{\alpha_c}{\alpha_c+\alpha_\ell}(1-\tau)\phi(g(\tau))) + \alpha_\ell\log(\tfrac{\alpha_\ell}{\alpha_c+\alpha_\ell});$$

$$Max_\tau \qquad \alpha_c\log(\tfrac{\alpha_c}{\alpha_c+\alpha_\ell}(1-\tau)\phi(\tau\psi I\tfrac{\alpha_c}{\alpha_c+\alpha_\ell})) + \alpha_\ell\log(\tfrac{\alpha_\ell}{\alpha_c+\alpha_\ell});$$

$$Max_\tau \qquad \alpha_c\log(\tfrac{\alpha_c}{\alpha_c+\alpha_\ell}) + \alpha_\ell\log(\tfrac{\alpha_\ell}{\alpha_c+\alpha_\ell}) + \alpha_c\left[\log(1-\tau) + \log\left[\phi(\tau\psi I\tfrac{\alpha_c}{\alpha_c+\alpha_\ell}))\right]\right];$$

$$Max_\tau \qquad \left[\log(1-\tau) + \log\left[\phi(\tau\psi I\tfrac{\alpha_c}{\alpha_c+\alpha_\ell}))\right]\right].$$

Assume $\phi(g) = Ag^\eta$ to get:

$$Max_\tau \qquad \log(1-\tau) + \log\left[A\left[\tau\psi I\tfrac{\alpha_c}{\alpha_c+\alpha_\ell}\right]^\eta\right];$$

$$Max_\tau \qquad \log(1-\tau) + \eta\log\left[\tau\right].$$

FOC:

$$\frac{1}{1-\tau} = \frac{\eta}{\tau};$$

$\tau = \eta(1-\tau)$;

$\tau^* = \frac{\eta}{1+\eta}$;

$g^* = \psi I \frac{\eta}{1+\eta}\frac{\alpha_c}{\alpha_c+\alpha_\ell}$;

$\ell^* = \frac{\alpha_\ell}{\alpha_c+\alpha_\ell}$;

$c^* = \frac{\alpha_c}{\alpha_c+\alpha_\ell}\frac{1}{1+\eta}A\left[\psi I \frac{\eta}{1+\eta}\frac{\alpha_c}{\alpha_c+\alpha_\ell}\right]^\eta$;

$n^{f*} = \frac{c^*}{\phi(g^*)} = \frac{\frac{\alpha_c}{\alpha_c+\alpha_\ell}\frac{1}{1+\eta}\phi(g^*)}{\phi(g^*)} = \frac{\alpha_c}{\alpha_c+\alpha_\ell}\frac{1}{1+\eta}$;

$n^{g*} = 1 - \ell^* - n^{f*} = 1 - \frac{\alpha_\ell}{\alpha_c+\alpha_\ell} - \frac{\alpha_c}{\alpha_c+\alpha_\ell}\frac{1}{1+\eta} = 1 - \frac{\alpha_\ell}{\alpha_c+\alpha_\ell}\left[1+\frac{1}{1+\eta}\right]$

$= 1 - \frac{\alpha_\ell}{\alpha_c+\alpha_\ell}\frac{2+\eta}{1+\eta}$.

Note that τ^* and ℓ^* are independent of pretty much everything, but that g^* is increasing in I and c^* is increasing in both A and I.

What is the first best?

$$Max_{(c,\ell,n^f,n^g)} U(c,\ell)$$

subject to:

$c \leq \phi(g)n^f$;

$g \leq I\psi n^g$;

$n^f + n^g + \ell \leq 1$.

OR,

$$Max_{(n^f,n^g)} U(\phi(I\psi n^g)n^f, 1-n^f-n^g).$$

Assume $U = \alpha_c \log(c) + \alpha_\ell \log(\ell)$ to get:

$$Max_{(n^f,n^g)} \qquad \alpha_c \log\left[\phi(I\psi n^g)n^f\right] + \alpha_\ell \log\left[1-n^f-n^g\right].$$

Assume $\phi(g) = Ag^\eta$ to get:

$$Max_{(n^f,n^g)} \qquad \alpha_c \log\left[A\left(I\psi n^g\right)^\eta n^f\right] + \alpha_\ell \log\left[1-n^f-n^g\right];$$

$$Max_{(n^f,n^g)} \qquad \alpha_c\left[\log\left[A\left(I\psi\right)^\eta\right] + \eta\log\left(n^g\right) + \log(n^f)\right] + \alpha_\ell \log\left[1-n^f-n^g\right];$$

$$Max_{(n^f,n^g)} \qquad \alpha_c\eta\log\left(n^g\right)+\alpha_c\log(n^f)+\alpha_\ell\log\left[1-n^f-n^g\right].$$

FOC's:

$n^f \qquad \frac{\alpha_c}{n^f}=\frac{\alpha_\ell}{1-n^f-n^g}$;

$n^g \qquad \frac{\alpha_c\eta}{n^g}=\frac{\alpha_\ell}{1-n^f-n^g}$.

This gives:

$\frac{\alpha_c}{n^f}=\frac{\alpha_c\eta}{n^g}$;

$n^g=\eta n^f$;

$\alpha_c\eta\left(1-n^f-n^g\right)=\alpha_\ell n^g$;

$\alpha_c\eta\left(1-n^f-\eta n^f\right)=\alpha_\ell\eta n^f$;

$\alpha_c\left(1-n^f-\eta n^f\right)=\alpha_\ell n^f$;

$\alpha_c-\alpha_c(1+\eta)n^f=\alpha_\ell n^f$;

$\alpha_c=\alpha_c(1+\eta)n^f+\alpha_\ell n^f=n^f\left[\alpha_c(1+\eta)+\alpha_\ell\right]$;

$n^f_{PP}=\frac{\alpha_c}{[\alpha_c(1+\eta)+\alpha_\ell]}$;

$n^g_{PP}=\eta n^f_{PP}=\frac{\eta\alpha_c}{[\alpha_c(1+\eta)+\alpha_\ell]}$;

$\ell_{PP}=1-n^g_{PP}-n^f_{PP}=1-(1+\eta)n^f_{PP}=1-\frac{\alpha_c(1+\eta)}{[\alpha_c(1+\eta)+\alpha_\ell]}$;

$\ell_{PP}=1-\frac{\alpha_c(1+\eta)}{[\alpha_c(1+\eta)+\alpha_\ell]}<\frac{\alpha_\ell}{\alpha_c+\alpha_\ell}=\ell^*$

if and only if

$1<\frac{\alpha_\ell}{\alpha_c+\alpha_\ell}+\frac{\alpha_c(1+\eta)}{[\alpha_c(1+\eta)+\alpha_\ell]}$;

$[\alpha_c+\alpha_\ell]\left[\alpha_c(1+\eta)+\alpha_\ell\right]<\alpha_\ell\left[\alpha_c(1+\eta)+\alpha_\ell\right]+[\alpha_c+\alpha_\ell]\,\alpha_c(1+\eta)$;

$\alpha_\ell\left[\alpha_c+\alpha_\ell\right]<\alpha_\ell\left[\alpha_c+\alpha_\ell+\alpha_c\eta\right]$;

This is always true– there is always less leisure under the PP.

Other comparisons?

Implementing the Optimum? LS taxes... etc....

6.3 A Dynamic Version

Here, the public good is provided by the government:

An equilibrium given a sequence of fiscal policy choices $\{(g_t, \tau_t)\}_{t=0}^{\infty}$:

a sequence of prices: $\{(p_t, r_t, w_t)\}_{t=0}^{\infty}$

Quantity decisions for the households: $\{(c_{it}, k_{it}, x_{ikt}, \ell_{it}, n_{it})\}_{t=0}^{\infty} = z_i^{HH}$

Quantity decisions for the output firms: $\{(c_{jt}^f, x_{jkt}^f, k_{jt}^f, n_{jt}^f)\}_{t=0}^{\infty} = z_j^f$,

SUCH THAT:

1) For each $i \in [0,1]$, z_i^{HH} is the solution to:

$Max_{\{(c_{it}, k_{it}, x_{ikt}, n_{it}, \ell_{it})\}_{t=0}^{\infty}} \sum_t \beta^t u(c_{it}, \ell_{it})$

subject to:

$\sum_{t=0}^{\infty} p_t \left[c_{it} + x_{ikt}\right] \leq \sum_{t=0}^{\infty} \left[(1-\tau_{kt}) r_t k_{it} + (1-\tau_{nt}) w_t n_{it}\right] + \Pi_i$

$k_{it+1} \leq (1-\delta_k) k_{it} + x_{ikt}$

$n_{it} + \ell_{it} \leq 1,$

k_{i0} fixed.

2) For each $j \in [0,1]$, z_j^f is the solution to:

$Max_{\{(c_{jt}^f, x_{jkt}^f, k_{jt}^f, n_{jt}^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t (c_{jt}^f + x_{jkt}^f + g_{jt}^f) - r_t k_{jt}^f - w_t n_{jt}^f\right]$

subject to: $c_{jt}^f + x_{jkt}^f + g_{jt}^f \leq F(k_{jt}^f, n_{jt}^f; G_t)$.

AND

$\int_0^1 c_{it} di = \int_0^1 c_{jt}^f dj$

$\int_0^1 x_{ikt} di = \int_0^1 x_{jkt}^f dj$

$\int_0^1 g_{jt}^f di = g_t$

$\int_0^1 k_{it} di = \int_0^1 k_{jt}^f dj$

$\int_0^1 n_{it} di = \int_0^1 n_{jt}^f dj$

$\int_0^1 \Pi_i di = \int_0^1 \sum_{t=0}^{\infty} \left[p_t (c_{jt}^f + x_{jkt}^f) - r_t k_{jt}^f - w_t n_{jt}^f \right] dj$

$\sum_t p_t g_t = \sum_t \left[\tau_{kt} r_t \int_I k_{it} di + \tau_{nt} w_t \int_I n_{it} di \right]$

$G_{t+1} = (1 - \delta_G) G_t + g_t$

NOTE: g_t is produced in period $t-1$ and effects the period t output of ALL firms j, but it is NOT one of $j's$ choice variables. Thus, it is an external effect, but one that is 'resolved' by the government. Accordingly, we take g_0 as given to the government to start out and not a part of the GBC.

We will assume that F is CRS in (k_{jt}^f, n_{jt}^f) that is, $F(\lambda k_{jt}^f, \lambda_{jt}^f; G_t) = \lambda F(k_{jt}^f, n_{jt}^f; G_t)$.

Assume further that $\tau_{kt} = \tau_{nt} = \tau_t$ for all t and that $p_t g_t = \tau_t \left[r_t \int_I k_{it} di + w_t \int_I n_{it} di \right]$, i.e., there is a balanced budget for each period. Finally, assume that all households are identical as are all firms and that $\tau_t = \tau$ for all t.

Assuming that $u(c, \ell) = c^{1-\sigma}/(1-\sigma)$, i.e., CES and inelastic labor supply, the problem for the representative household can be rewritten as:

$Max_{\{(c_{it}, k_{it})\}_{t=0}^{\infty}} \sum_t \beta^t u(c_{it})$

subject to:

$\sum_{t=0}^{\infty} p_t \left[c_{it} + k_{it+1} - (1 - \delta_k) k_{it} \right] \leq \sum_{t=0}^{\infty} (1 - \tau) \left[r_t k_{it} + w_t \right]$

k_{i0} fixed.

This gives the standard results:

$c_t \qquad \lambda p_t = \beta^t u'(t)$

$k_{t+1} \qquad p_t = p_{t+1} \left[1 - \delta_k + (1 - \tau) \frac{r_{t+1}}{p_{t+1}} \right]$

Using the FOC's for the firm we get the usual expression:

$$\left[\frac{c_{t+1}}{c_t} \right]^{\sigma} = \beta \left[1 - \delta_k + (1 - \tau) F_k(t+1) \right].$$

Assuming that $F(k, n; G) = A k^{\alpha} n^{1-\alpha} G^{\eta}$, we see that $F_k = \frac{\alpha F}{k} = \frac{\alpha A n^{1-\alpha} G^{\eta}}{k^{1-\alpha}}$. Since labor is inelastically supplied, we have that $F_k = \frac{\alpha A G^{\eta}}{k^{1-\alpha}}$.

In equilibrium, since the government is balancing it's budget period by period $g_t = \tau y_{t-1}$. Substituting, we obtain:

$$\left[\frac{c_{t+1}}{c_t}\right]^{\sigma} = \beta\left[1-\delta_k+(1-\tau)\frac{\alpha A G_t^{\eta}}{k_t^{1-\alpha}}\right]$$

From here, things depend on the size of $\alpha+\eta$ as in the Romer model. If $\alpha+\eta<1$ growth is not feasible. If $\alpha+\eta>1$, there are increasing returns, even in the planner's problem version of the model, and not much is known about the solution (unless probably $\delta_k=1$ and preferences are logarithmic).

The one simple case is that when $\alpha+\eta=1$. Further, let's assume that $\delta_G=1$ so that $G_t=g_{t-1}$, and conjecture that $\frac{G}{k}$, $\frac{c}{k}$, $\frac{x_k}{k}$, $\frac{y}{k}$ and γ are constant along the equilibrium path (is this even possible for all $\tau's$?????)

First, note that, from feasibility, we have:

$$c_t + x_{kt} + g_t = A k_t^{\alpha} g_{t-1}^{1-\alpha}$$

$$\frac{c_t}{k_t} + \frac{x_{kt}}{k_t} + \frac{g_t}{k_t} = A\left[\frac{g_{t-1}}{k_t}\right]^{1-\alpha}$$

$$\frac{c_t}{k_t} + \frac{k_{t+1}-(1-\delta_k)k_t}{k_t} + \frac{g_t}{k_t} = A\left[\frac{g_{t-1}}{k_t}\right]^{1-\alpha}$$

$$\frac{c_t}{k_t} + \gamma + 1 - \delta_k + \frac{g_t}{k_t} = A\left[\frac{g_{t-1}}{k_t}\right]^{1-\alpha}$$

$$\frac{c_t}{k_t} + \gamma + 1 - \delta_k + \gamma\frac{g_{t-1}}{k_t} = A\left[\frac{g_{t-1}}{k_t}\right]^{1-\alpha}$$

This will determines $\frac{c}{k}$ given γ and $\frac{g_{t-1}}{k_t}$ after these are determined below (but this will never be used here).

From the EE, we have:

$$\gamma^{\sigma} = \beta\left[1-\delta_k+(1-\tau)\alpha A\left[\frac{G_t}{k_t}\right]^{1-\alpha}\right] = \beta\left[1-\delta_k+(1-\tau)\alpha A\left[\frac{g_{t-1}}{k_t}\right]^{1-\alpha}\right]$$

From the Government Budget Constraint, we have that:

$$g_t = \tau y_t = \tau A k_t^{\alpha} g_{t-1}^{1-\alpha},$$

$$\frac{g_t}{k_t} = \tau A\left[\frac{g_{t-1}}{k_t}\right]^{1-\alpha}, \text{ or,}$$

$$\gamma\frac{g_{t-1}}{k_t} = \tau A\left[\frac{g_{t-1}}{k_t}\right]^{1-\alpha}, \text{ or}$$

$$\gamma\left[\frac{g_{t-1}}{k_t}\right]^{\alpha} = A\tau, \text{ or,}$$

$$\frac{g_{t-1}}{k_t} = \left[\frac{\tau A}{\gamma}\right]^{1/\alpha}.$$

Substituting this into EE, we obtain:

$$\gamma^{\sigma} = \beta\left[1 - \delta_k + (1-\tau)\alpha A \left[\frac{q_{t-1}}{k_t}\right]^{1-\alpha}\right]$$

$$= \beta\left[1 - \delta_k + (1-\tau)\alpha A \left[\left[\frac{\tau A}{\gamma}\right]^{1/\alpha}\right]^{1-\alpha}\right]$$

$$= \beta\left[1 - \delta_k + (1-\tau)\alpha A \left[\frac{\tau A}{\gamma}\right]^{(1-\alpha)/\alpha}\right]$$

$$= \beta\left[1 - \delta_k + \alpha A^{1/\alpha}(1-\tau)\tau^{(1-\alpha)/\alpha}\gamma^{(\alpha-1)/\alpha}\right]$$

Or, $LHS(\gamma) = RHS(\gamma)$.

$LHS(\gamma)$ is an increasing function of γ, while $RHS(\gamma)$ is a decreasing function of γ. Thus, if there is an intersection, it is unique.

Further, $LHS(\gamma)$ does not depend on τ, while $RHS(\gamma)$ does, but in a non-monotone way. Any change in τ that shifts up $RHS(\gamma)$ will increase γ while the opposite is true for any change in τ that causes $RHS(\gamma)$ to shift down.

Thus, the issue is how does the function $w(\tau) = (1-\tau)\tau^{\zeta}$, where $\zeta = \frac{1-\alpha}{\alpha}$ depend on τ?

First, note that $w(0) = w(1) = 0$. Second, $w(\tau) \geq 0$ for all τ and finally,

$$w'(\tau) = -\tau^{\zeta} + \zeta(1-\tau)\tau^{\zeta-1}.$$

Thus, w is maximized where:

$$\tau^{\zeta} = \zeta(1-\tau)\tau^{\zeta-1}$$

$$1 = \zeta(1-\tau)\tau^{-1}$$

$$\tau = \zeta(1-\tau)$$

$$\tau(1+\zeta) = \zeta$$

$$\tau = \frac{\zeta}{1+\zeta} = \frac{\frac{1-\alpha}{\alpha}}{1+\frac{1-\alpha}{\alpha}} = \frac{\frac{1-\alpha}{\alpha}}{\frac{\alpha+1-\alpha}{\alpha}} = \frac{1-\alpha}{1} = 1-\alpha$$

Thus, for $\tau \in [0, 1-\alpha]$, increases in τ cause $RHS(\gamma)$ to shift up, and so γ is increasing in τ in this range. For $\tau \in [1-\alpha, 1]$, increases in τ cause $RHS(\gamma)$ to shift down and so γ is decreasing in this range.

Note that growth maximization is NOT NECESSARILY equivalent to welfare maximization even with this restricted set of instruments.

Note that here, we have:

$$TFP_t = \frac{y_t}{k_t^{.33} n_t^{.67}} = \frac{Ak_t^{\alpha} n_t^{1-\alpha} G_t^{1-\alpha}}{k_t^{.33} n_t^{.67}} = AG_t^{.67}$$

6.4 The Barro Model – Planner's Version

The Planner's Problem Version of the Barro Model with Productive Government Spending is:

$Max_{\{(c_t,k_t,x_{kt},n_t,\ell_t,g_t,G_t)\}_{t=0}^{\infty}} \sum_t \beta^t u(c_{it},\ell_{it})$

subject to:

$c_t + x_{kt} + g_t \leq F(k_t, n_t; G_t);$

$k_{t+1} \leq (1-\delta_k)k_t + x_{kt};$

$G_{t+1} \leq (1-\delta_G)G_t + g_t;$

$n_t + \ell_t \leq 1;$

k_0, G_0 fixed.

This is a standard one sector, two capital good model of growth. Assume that labor is inelastically supplied so that $n_t = 1$ for all t and further assume that $\delta_G = \delta_k = 1$ for simplicity.

Then we have the familiar results:

$(EEK) \qquad \gamma^{\sigma} = \beta F_k(t+1);$

$(EEG) \qquad \gamma^{\sigma} = \beta F_G(t+1);$

Assuming that $F(k, n; G) = Ak^{\alpha}n^{1-\alpha}G^{1-\alpha}$ we have:

$$F_k(t+1) = F_G(t+1);$$

which becomes:

$$\alpha F(t+1)/k_{t+1} = (1-\alpha)F(t+1)/G_{t+1};$$

or,

$\frac{G}{k} = \frac{1-\alpha}{\alpha}$ for all t.

Substituting, this gives:

$$\gamma^{\sigma}_{PP} = \beta A\alpha^{\alpha}(1-\alpha)^{1-\alpha}.$$

How does this compare to that from the Tax Implemenation above? From above, we have:

$$\gamma^{\sigma} = \beta\left[1 - \delta_k + \alpha A^{1/\alpha}(1-\tau)\tau^{(1-\alpha)/\alpha}\gamma^{(\alpha-1)/\alpha}\right];$$

or,

$$LHS_{\tau}(\gamma) = RHS_{\tau}(\gamma).$$

And from above, we have that:

$$\gamma^{\sigma}_{PP} = \beta A\alpha^{\alpha}(1-\alpha)^{1-\alpha};$$

or,

$$LHS_{PP}(\gamma) = RHS_{PP}(\gamma).$$

And note that $LHS_{PP}(\gamma) = LHS_{\tau}(\gamma)$ for all γ.

Also note that

1) $0 < (1-\tau) < 1$;

2) $0 < \tau < 1 \Longrightarrow 0 < \tau^{(1-\alpha)/\alpha} < 1$ since $\frac{1-a}{\alpha} > 0$;

3) $0 < \gamma^{(\alpha-1)/\alpha} \leq 1$ if $\gamma \geq 1$ since $\frac{a-1}{\alpha} > 0$.

Thus, for any γ such that $\gamma \geq 1$, it follows that $RHS_{PP}(\gamma) > RHS_{\tau}(\gamma)$.

Draw picture.

From this it follows that IF $\gamma_{\tau} \geq 1$, $\gamma_{PP} > \gamma_{\tau}$.

6.4.1 Variations

1. $\delta_G < 1$
2. For a given τ, is the growth rate increasing in I? It seems like it should be.
3. Crowding of G – in response to 2.
4. Two kinds of G one that is productive, one that is purely consumption?
5. High taxes are here associated with HIGH growth rates, at least up to a point (given by $1-\alpha$) and LOW growth rates beyond that. Assuming we are always on the left hand side of this relationship, we see that increases in τ are associated with increases in γ.
6. Implementation of PP solution?

7 Models with R&D or Innovation in Their Title

There are a lot of these and so I'll just give you a short list of the best known ones:

1. Romer
2. Grossman and Helpman
3. Aghion and Howitt
4. Stokey
5. Boldrin and Levine

Mostly these papers are about trying to fill in the gaps between our loose discussion about how one would try to model Knowledge, or R&D, or Innovation as a public good, and what actually appeared in both the Romer and the Lucas models. Because of this, they make a variety of special assumptions about how knowledge creation is carried out. They typically assume that this is done at the firm level, so that 'firms own knowledge,' it is NOT embodied in people. The competitive structure usually adopted is some sort of monopolistic competition, often along the lines of Dixit-Stiglitz aggregators. Grossman and Helpman are a noted exception in that they developed a 'quality ladder' approach to the problem, so that new goods are strictly 'better' than old, already existing goods.

The emphasis is on the drive to make improvements as an attempt to monopolize a market and make profits in this way, either through 'inventing' a new good, or 'innovating' in the production process and lowering costs, etc. However, these are models with large numbers of firms, and they always (?) have as an equilibrium condition that profits are, in fact, equal to zero. (Perhaps only in an ex ante, expected sense.) Thus, in the end, they share that feature with perfectly competitive models (even the ones with external effects), i.e., firms strive to make profits as high as possible, but, because of competition, they end up earning 0.

Many of the papers deal with the publicness of knowledge in a fairly ad hoc way. Because of the results above, privately maximizing agents have little incentive to produce knowledge that is a public good. Thus, they will do something like new processes are private for τ periods, after which they become publically available.

Because of this emphasis, the discussion in the literature centers around questions of patent length, copyrights, etc., as methods for making these goods 'less public' and increasing efficiency. (The equilibria are almost always inefficient, and sometimes there is too much 'innovation' sometimes too little.')

It is not clear what this line of research adds to our set of tables. For example, should we try and make a table in which country level heterogeneity is based on the degree of enforcement of patent rights in different countries and then compare that with the Heston-Summers data? In the end, the literature seems to suffer from a lack of strong empirical foundation and direction. Rather, it's more a theoretical discussion about what the 'right' description of the production side of the economy should look like with special attention paid to 'innovation.' Maybe some empirical basis could be gotten out of some of them.

The Boldrin and Levine series of papers are quite different from this, but they are primarily focused on the policy recommendations part of the debate. That is, what should patent law be, etc. Equilibria in the above described set of models is inefficient because it is assumed to be inefficient. This does not come from some deep, precise set of factual observations that are being matched. It is more loose, and qualitative– innovation seems to have a public nature to it, etc. The B&L papers attack this head on. They provide a series of models of innovation which is perfectly competitive in it's nature. They do this in such a way that they reproduce much of the loose qualitative discussion of the literature above, but, because it is A-D in nature, the policy conclusions are exactly the opposite. The equilibrium is efficient, patents etc might redistribute, but they do not increase welfare, etc.

I personally find the details of the papers a bit confusing. So I made up my own example of a similar kind of thing, which is what's next in the notes.

8 An Ak Model of Innovation

These notes outline a framework for a simple example on 'Innovation and Growth'. The idea is to build something after the ideas of the Boldrin and Levine papers, but which has a tighter connection to the standard growth literature.

We want to preserve two features of their model

1) that there is a 'minimum effective scale' for innovation, and,

2) Past that scale, the production set is a cone.

8.1 Notation, and etc.

Frontier Knowledge at date t: H_t this is supposed to represent the absolute frontier of what 'society' knows at date t.

Average 'worker' knowledge at date t: h_t this is supposed to represent the average knowledge of those workers that work in the final goods sector.

Final Goods at date t: c_t this represents the production of final goods at date t.

Workers: There are two types of workers, 'researchers,' and 'workers.' For now, I will assume that these are assigned once and for all. Assume that there are L_1 researchers, and L_2 workers, where $L = L_1 + L_2$ is total labor supply.

We will assume that the level of knowledge of researchers is given by H_t, and that of workers is given by h_t.

8.2 Production Functions

We will assume that:

$$H_{t+1} = (1 - \delta_H)H_t + I_{Ht},$$

$$h_{t+1} = (1 - \delta_h)h_t + I_{ht},$$

$$I_{Ht} = F^H(Z^H_{Ht}, Z^H_{ht}),$$

$I_{ht} = F^h(Z^h_{Ht}, Z^h_{ht}),$

$c_t = F^c(Z^c_{Ht}, Z^c_{ht})$

where Z^i_{jt} is quality adjusted labor of worker of type j going into the production in activity i.

We will assume that $Z = LH$, (resp. $Z = Lh$) everywhere following Rosen and that the $F's$ are CRS (typically Cobb-Douglas).

The idea here is that the I_H technology is Research and Development, or Innovation.... it moves out the frontier of knowledge.

Similarly, we think of the I_h technology as Education/Schooling. This is where the people at the frontier spend part of their time educating the production line workers on new techniques. The more time the frontier workers spend in I_h, the less time they have to spend in I_H, and hence workers are better prepared and more productive, but the frontier moves out more slowly. Note that increasing $L^h_{2t}h_t$, holding $L^h_{1t}H_t$ constant increases total output of worker productive knowledge (new h) but lowers the average product of frontier knowledge workers in educating (bigger classes give more total new training, but less output per student). ***** check that this is right... ******

8.3 Preferences

The standard thing,

$U(c) = \sum_t \beta^t u(c_t)$, with $u(c) = c^{1-\sigma}/(1-\sigma)$.

This is the inelastic labor supply version. This is what Boldrin and Levine assume too.

8.4 Conjectures

This model looks like a fancy version of a $(A_H, A_h, A_c) \cdot (H, h)$ model whatever that means... but so, it should behave a lot like an Ak model in some form.

Conjecture 4 *There is a balanced growth rate with the endogenously determined growth rate depending on* $A_H, A_h, A_c, \delta_H, \delta_h, \beta, \sigma, L_1, L_2$. *On that BGP,* $L^h_{1t}, L^H_{1t}, L^h_{2t}, L^c_{2t}$ *are constant. Whether or not the economy is on this BGP for every t, may depend on initial conditions,* H_0 *and* h_0.

Conjecture 5 *The BGP of this model has the same properties with respect to taxation as do the standrard Ak, models, increases in taxes decrease growth rates, etc.*

Suppose that the model is just like above except for the innovation sector which is instead given by:

$I_{Ht} = F^H(Z^H_{Ht}, Z^H_{ht})$ if $Z^H_{Ht} \geq \hat{Z}^H_{Ht}$, and $Z^H_{ht} \geq \hat{Z}^H_{ht}$ and $I_{Ht} = 0$ if $Z^H_{Ht} < \hat{Z}^H_{Ht}$, and $Z^H_{ht} < \hat{Z}^H_{ht}$. (Or other similar kinds of restrictions.)

Conjecture 6 *For choices of the $\hat{Z}'s$ that are 'low' the equilibrium described above is still an equilibrium. How low the $\hat{Z}'s$ must be depends on the parameters, $A_H, A_h, A_c, \delta_H, \delta_h, \beta, \sigma$.*

This would give a model in spirit like that in B&L, non-convexity, growth in some cases, etc.....

Also, different countries with different $(L_1, L_2)'s$ might have different growth experiences. With some of them (optimally) having no growth and others (optimally) having positive growth.

What other things would we like to get out of this?

8.5 Simple Example 1

8.5.1 Functional Forms

Let's look at something simple as a first step. An idea would be to look at:

$$\delta_H = \delta_h = \delta,$$

$$F^H(z_H, z_h) = F^h(z_H, z_h) = F^c(z_H, z_h) = Az_H^\alpha z_h^{1-\alpha}.$$

In this case, the sectors aggregate and we can write one overall feasibility constraint as:

$$c_t + H_{t+1} + h_{t+1} \leq Az_H^\alpha z_h^{1-\alpha} + (1-\delta)H_t + (1-\delta)h_t.$$

Thus, the equilibrium solves the following maximization problem:

$$\max_{c,H,h,L_1^H,L_1^h,L_2^h,L_2^c} \quad \sum_t \beta^t u(c_t)$$

$$\text{st} \qquad c_t + H_{t+1} + h_{t+1} \leq AL_1^\alpha L_2^{1-\alpha} H_t^\alpha h_t^{1-\alpha} + (1-\delta)H_t + (1-\delta)h_t.$$

H_0 and h_0 fixed.

We have ignored non-negativity constraints.

8.5.2 FOC's and Solution

In this case, it can be shown that if $H_0/h_0 = \alpha/(1-\alpha)$, the solution to the problem has constant growth and shares all along the path. That is, $H_t/h_t = \alpha/(1-\alpha)$ for all t, etc.

This model behaves just like an Ak model, in which the effective A is given by $A^* = AL_1^{\alpha}L_2^{1-\alpha}\alpha^a(1-\alpha)^{(1-\alpha)}$.

One way to alter this example would be to require that

$$H_{t+1}/H_t \geq \gamma^*$$

or,

$$H_{t+1}/\left[AL_1^{\alpha}L_2^{1-\alpha}H_t^{\alpha}h_t^{1-\alpha} + (1-\delta)H_t + (1-\delta)h_t\right]$$

or something, with zero change in H if this is not true. Then try and characterize those settings such that this is not binding and find out what happens if they are.

8.6 Simple Example 2

The following example would be better if we could get it to work probably. Like the one above, it involves some very special assumptions, but this set is probably closer to what people would like to see.

8.6.1 Functional Forms

An idea would be to look at:

$\delta_H = 0$, that is, 'society's knowledge' never disappears... or the frontier never moves backword.

$I_{Ht} = A_H L_{1t}^H H_t$,

$I_{ht} = A_h (L_{1t}^h H_t)^{\alpha} (L_{2t}^h h_t)^{1-\alpha}$,

$c_t = A_c L_{2t}^c h_t,$

$L_{1t}^H + L_{1t}^h = L_1,$

$L_{2t}^h + L_{2t}^c = L_2.$

The idea here is that the I_H technology is Research and Development, or Innovation.... it moves out the frontier of knowledge. For simplicity we have assumed that it only uses Frontier Knowledge (along with time) to produce.

Finally, the c sector is just standard Ak stuff.

NOTE: There are implicitly a couple of "0" assumptions. h isn't useful in producing new H, and H isn't useful in producing c. These could easily be changed.

8.6.2 Finding the BGP

Do some simplification first:

Let $\varphi_t^c = L_{2t}^c / L_2$, then $1 - \varphi_t^c = L_{2t}^h / L_2$.

Let $\psi_t^H = L_{1t}^H / L_1$, then $1 - \psi_t^H = L_{1t}^h / L_1$.

Using this, we can rewrite the constraints as:

$$c_t \leq A_c L_{2t}^c h_t = A_c L_2 \varphi_t^c h_t = A_c^* \varphi_t^c h_t,$$

$$h_{t+1} \leq (1-\delta_h) h_t + A_h L_1^\alpha L_2^{1-\alpha} (1-\psi_t^H)^\alpha {H_t}^\alpha (1-\varphi_t^c)^{1-\alpha} {h_t}^{1-\alpha}$$

$$= (1-\delta_h) h_t + A_h^* (1-\psi_t^H)^\alpha {H_t}^\alpha (1-\varphi_t^c)^{1-\alpha} {h_t}^{1-\alpha},$$

$$H_{t+1} \leq H_t + A_H L_1 \psi_t^H H_t = H_t + A_H^* \psi_t^H H_t$$

Thus, the maximization problem is:

$$\max_{c,H,h,\psi^H,\varphi^c} \quad \sum_t \beta^t u(c_t)$$

$$\text{s.t.} \quad c_t \leq A_c^* \varphi_t^c h_t, \qquad (\beta^t \lambda_t)$$

$$h_{t+1} \leq (1-\delta_h) h_t + A_h^* (1-\psi_t^H)^\alpha {H_t}^\alpha (1-\varphi_t^c)^{1-\alpha} {h_t}^{1-\alpha}, \qquad (\beta^t \mu_t)$$

$$H_{t+1} \leq H_t + A_H^* \psi_t^H H_t, \qquad (\beta^t \nu_t)$$

where $A_H^* = A_H L_1$, $A_h^* = A_h L_1^\alpha L_2^{1-\alpha}$, and, $A_c^* = A_c L_2$.

The FOC's for this problem are:

$$c_t: \qquad \beta^t \lambda_t = \beta^t u_t'$$

$$\varphi_t^c: \qquad \beta^t \lambda_t A_c^* h_t = \beta^t \mu_t (1-\alpha) F_t^h / (1-\varphi_t^c)$$

$$\psi_t^H: \qquad \beta^t \nu_t A_H^* H_t = \beta^t \mu_t \alpha F_t^h / (1-\psi_t^H)$$

$$h_{t+1}: \qquad \beta^{t+1} \lambda_{t+1} A_c^* \varphi_{t+1}^c + \beta^{t+1} \mu_{t+1} \left[(1-\alpha) F_{t+1}^h / h_{t+1} + (1-\delta_h)\right] = \beta^t \mu_t$$

$$H_{t+1}: \qquad \beta^{t+1} \nu_{t+1} \left[1 + A_H^* \psi_{t+1}^H\right] + \beta^{t+1} \mu_{t+1} \alpha F_{t+1}^h / H_{t+1} = \beta^t \nu_t$$

where we have used the notation, $F_t^h = A_h^* (1-\psi_t^H)^\alpha {H_t}^\alpha (1-\varphi_t^c)^{1-\alpha} {h_t}^{1-\alpha}$.

Thus,

$$\beta^t \lambda_t \tfrac{A_c^* h_t (1-\varphi_t^c)}{(1-\alpha) F_t^h} = \beta^t \mu_t, \text{ and,}$$

$$\beta^t \nu_t = \beta^t \mu_t \tfrac{\alpha F_t^h}{A_H^* H_t (1-\psi_t^H)} = \beta^t \lambda_t \tfrac{A_c^* h_t (1-\varphi_t^c)}{(1-\alpha) F_t^h} \tfrac{\alpha F_t^h}{A_H^* H_t (1-\psi_t^H)} = \beta^t \lambda_t \tfrac{\alpha A_c^* h_t (1-\varphi_t^c)}{(1-\alpha) A_H^* H_t (1-\psi_t^H)}.$$

Using these and substituting give:

$$\beta^{t+1} \lambda_{t+1} A_c^* \varphi_{t+1}^c + \beta^{t+1} \lambda_{t+1} \tfrac{A_c^* h_{t+1} (1-\varphi_{t+1}^c)}{(1-\alpha) F_{t+1}^h} \left[(1-\alpha) F_{t+1}^h / h_{t+1} + (1-\delta_h)\right] = \beta^t \lambda_t \tfrac{A_c^* h_t (1-\varphi_t^c)}{(1-\alpha) F_t^h}$$

$$\tfrac{u_t'}{u_{t+1}'} \tfrac{A_c^* h_t (1-\varphi_t^c)}{(1-\alpha) F_t^h} = \beta \left[A_c^* \varphi_{t+1}^c + \tfrac{A_c^* h_{t+1} (1-\varphi_{t+1}^c)}{(1-\alpha) F_{t+1}^h} \left[(1-\alpha) F_{t+1}^h / h_{t+1} + (1-\delta_h)\right]\right]$$

$$\tfrac{u_t'}{u_{t+1}'} \tfrac{A_c^* h_t (1-\varphi_t^c)}{(1-\alpha) F_t^h} = \beta \left[A_c^* \varphi_{t+1}^c + A_c^* (1-\varphi_{t+1}^c) + \tfrac{A_c^* h_{t+1} (1-\varphi_{t+1}^c)}{(1-\alpha) F_{t+1}^h} (1-\delta_h)\right]$$

$$\tfrac{u_t'}{u_{t+1}'} \tfrac{A_c^* h_t (1-\varphi_t^c)}{(1-\alpha) F_t^h} = \beta \left[A_c^* + \tfrac{A_c^* h_{t+1} (1-\varphi_{t+1}^c)}{(1-\alpha) F_{t+1}^h} (1-\delta_h)\right]$$

$$\left(\tfrac{c_{t+1}}{c_t}\right)^\sigma = \beta \tfrac{(1-\alpha) F_t^h}{A_c^* h_t (1-\varphi_t^c)} \left[A_c^* + \tfrac{A_c^* h_{t+1} (1-\varphi_{t+1}^c)}{(1-\alpha) F_{t+1}^h} (1-\delta_h)\right]$$

We will assume that the system follows a BGP, on which:

$$h_{t+1}/h_t = c_{t+1}/c_t = H_{t+1}/H_t = F_{t+1}^h / F_t^h = \gamma$$

and,

$$\varphi_t^c = \varphi^c, \ \psi_t^H = \psi^H, \ F_t^h / h_t = F^h / h, \text{ etc.}$$

This will probably require some assumption about initial conditions, H_0/h_0 is some constant.

Under this assumption, it follows that:

$$\gamma^\sigma = \beta \left[\tfrac{(1-\alpha)F^h}{(1-\varphi^c)h} + (1 - \delta_h) \right]$$

On the other...

$$\beta^{t+1}\lambda_{t+1} \tfrac{\alpha A_c^* h_{t+1}(1-\varphi_{t+1}^c)}{(1-\alpha)A_H^* H_{t+1}(1-\psi_{t+1}^H)} \left[1 + A_H^* \psi_{t+1}^H\right] + \beta^{t+1}\lambda_{t+1} \tfrac{A_c^* h_{t+1}(1-\varphi_{t+1}^c)\alpha F_{t+1}^h}{(1-\alpha)F_{t+1}^h H_{t+1}} = \beta^t \lambda_t \tfrac{\alpha A_c^* h_t(1-\varphi_t^c)}{(1-\alpha)A_H^* H_t(1-\psi_t^H)}$$

$$\tfrac{u_t'}{u_{t+1}'} \tfrac{\alpha A_c^* h_t(1-\varphi_t^c)}{(1-\alpha)A_H^* H_t(1-\psi_t^H)} = \beta \tfrac{\alpha A_c^* h_{t+1}(1-\varphi_{t+1}^c)}{(1-\alpha)A_H^* H_{t+1}(1-\psi_{t+1}^H)} \left[1 + A_H^* \psi_{t+1}^H\right] + \beta \tfrac{A_c^* \alpha(1-\varphi_{t+1}^c)h_{t+1}}{(1-\alpha)H_{t+1}}$$

$$\left(\tfrac{c_{t+1}}{c_t}\right)^\sigma = \beta \left[\tfrac{(1-\alpha)A_H^* H_t(1-\psi_t^H)}{\alpha A_c^* h_t(1-\varphi_t^c)} \tfrac{\alpha A_c^* h_{t+1}(1-\varphi_{t+1}^c)}{(1-\alpha)A_H^* H_{t+1}(1-\psi_{t+1}^H)} \left[1 + A_H^* \psi_{t+1}^H\right] + \tfrac{(1-\alpha)A_H^* H_t(1-\psi_t^H)}{\alpha A_c^* h_t(1-\varphi_t^c)} \tfrac{A_c^* \alpha(1-\varphi_{t+1}^c)h_{t+}}{(1-\alpha)H_{t+1}} \right.$$

$$\left(\tfrac{c_{t+1}}{c_t}\right)^\sigma = \beta \left[\tfrac{H_t(1-\psi_t^H)}{H_{t+1}(1-\psi_{t+1}^H)} \tfrac{h_{t+1}(1-\varphi_{t+1}^c)}{h_t(1-\varphi_t^c)} \left[1 + A_H^* \psi_{t+1}^H\right] + \tfrac{A_H^* H_t(1-\psi_t^H)}{H_{t+1}} \tfrac{(1-\varphi_{t+1}^c)h_{t+1}}{(1-\varphi_t^c)h_t} \right]$$

Or, on a BGP,

$$\gamma^\sigma = \beta \left[1 + A_H^* \psi^H + A_H^*(1 - \psi^H)\right], \text{ or}$$

$$\gamma^\sigma = \beta \left[1 + A_H^*\right]$$

This equation gives us γ as a function of the basic parameters σ, β, and $A_H^* = A_H L_1$. In particular, note that it depends on L_1.

In particular, γ is increasing in L_1.

(NOTE: This would have been $\gamma^\sigma = \beta \left[1 - \delta_H + A_H^*\right]$, if we had allowed for $\delta_H > 0$.)

Since γ does not depend on the other parameters, it will probably turn out that the only way income taxes affect growth rates here is through their effect on the R&D sector. That is, if we have an income tax that is linear on income generated in the H sector, and it is used in a Balanced Budget way to buy output from the H sector, the growth rate will fall to:

$$\gamma_\tau^\sigma = \beta[1 + (1 - \tau)A_H^*].$$

Presumably something similar happens if the tax is uniform across sectors and purchases are uniform across sectors. Since lump-sum rebates don't affect FOC's, the same will probably be true if the revenue is rebated.

In particular, if income from the h and/or c sectors are taxed, but that from the H sector is not, there will probably be no effects on growth. This is reminiscent of the Lucas model and the 2-sector model in Rebelo.

8.6.3 Finding the other Endogenous Variables on the BGP

From above, we have that:

$$\left[\frac{(1-\alpha)F^h}{(1-\varphi^c)h} + (1-\delta_h)\right] = [1 + A_H^*]$$

$$F^h/h = \frac{A_H^*+\delta_h}{(1-\alpha)}(1-\varphi^c)$$

The feasibility restrictions can be rewritten on the BGP as:

$$c/h = A_c^*\varphi^c,$$

$$\gamma = (1-\delta_h) + F^h/h,$$

$$\gamma = 1 + A_H^*\psi^H,$$

From this we get,

$$\psi^H = \left[(\beta\,[1+A_H^*])^{1/\sigma} - 1\right]/A_H^*$$

$$\frac{A_H^*+\delta_h}{(1-\alpha)}(1-\varphi^c) = \gamma + \delta_h - 1$$

$$\varphi^c = -\frac{(1-\alpha)}{A_H^*+\delta_h}\left[\gamma + \delta_h - 1\right] + 1 = -\frac{(1-\alpha)}{A_H^*+\delta_h}\left[(\beta\,[1+A_H^*])^{1/\sigma} + \delta_h - 1\right] + 1$$

$$F^h/h = \gamma + \delta_h - 1 = (\beta\,[1+A_H^*])^{1/\sigma} + \delta_h - 1$$

etc.

Notice that since the amount of time spent in R&D is given by $\psi^H L_1$, we have that

$$L_1^H = \psi^H L_1 = \frac{\left[(\beta[1+A_H^*])^{1/\sigma} - 1\right]}{A_H^*}L_1 = \frac{\left[(\beta[1+A_H L_1])^{1/\sigma} - 1\right]}{A_H L_1}L_1 = \left[(\beta\,[1+A_H L_1])^{1/\sigma} - 1\right]/A_H.$$

This is an increasing function of L_1.

8.6.4 Adding Back in the Non-Convexity

Now, turn to the problem of the alternative form of the technology in which the non-convexity is present. We will assume that output in the R&D sector is given by:

$I_t^H = A_H L_{1t}^H$ if $L_{1t}^H \geq \hat{L}$, and $I_t^H = 0$ if $L_{1t}^H < \hat{L}$.

In this case, the planner's problem is given by:

Thus, the maximization problem is:

$$\max_{c,H,h,\psi^H,\varphi^c} \quad \sum_t \beta^t u(c_t)$$

$$\text{s.t.} \quad c_t \leq A_c^* \varphi_t^c h_t,$$

$$h_{t+1} \leq (1-\delta_h)h_t + A_h^*(1-\psi_t^H)^\alpha {H_t}^\alpha (1-\varphi_t^c)^{1-\alpha} {h_t}^{1-\alpha},$$

$$H_{t+1} \leq H_t + A_H^* \psi_t^H H_t \qquad if \qquad \psi_t^H \geq \hat{L}/L_1,$$

$$H_{t+1} \leq H_t \qquad if \qquad \psi_t^H < \hat{L}/L_1.$$

The solution to this problem is the same as above as long as:

$$L_1^H = \left[(\beta \left[1 + A_H L_1 \right])^{1/\sigma} - 1 \right] / A_H \geq \hat{L}.$$

Thus, in this region of the parameter space, it behaves just like above.

What happens in other cases is much more difficult to guess. One possibility would be that there is no R&D investment ever. But it does not follow from the above that this is the equilibrium. Since that assumed non-negativity everywhere, the alternative is 0 SOMEWHERE, not 0 EVERYWHERE.

Thus, there might be situations in which the solution has R&D spending part of the time, and no R&D spending other periods.